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Quantum mechanical analogy for unstable particle collisions

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In Chiral Perturbation Theory instability appears due to interaction, for example, $\rho(770)$ instability due to coupling to pions:

$$L = -\frac{1}{4}(\partial_\mu\rho_\nu - \partial_\nu\rho_\mu)(\partial^\mu\rho^\nu - \partial^\nu\rho^\mu) + \frac{1}{2}m_\rho^2\partial_\mu\rho_\nu\partial^\mu\rho^\nu +$$

$$ig_{\rho\pi\pi}(\pi^+\partial_\mu\pi^- - \pi^-\partial_\mu\pi^+)\rho^\mu + L_\pi$$

In terms of Quantum Mechanics ρ is an eigenvector of unperturbed Hamiltonian \hat{H}_0 and becomes unstable because of perturbation \hat{V} .

The direct way to consider quantum mechanical analogy of two particles scattering, where one particle is unstable:

$$\hat{H} = \hat{H}_0 + \hat{V}$$

$$\hat{H}_0 = \hat{H}_{01}(\vec{r}_1) + \hat{H}_{02}(\vec{r}_2) = \frac{\hat{p}_1^2}{2m_1} + \frac{\hat{p}_2^2}{2m_2}$$

$$\hat{V} = \hat{V}_1(\vec{r}_1) + \hat{V}_{12}(\vec{r}_1 - \vec{r}_2)$$

$$\psi(\vec{r}_1, \vec{r}_2, t = 0) = \psi_1(\vec{r}_1)\psi_2(\vec{r}_2)$$

where ψ_1 и ψ_2 are eigenfunctions of \hat{H}_{01} and \hat{H}_{02} , and $\hat{V}_1(\vec{r}_1)$ is responsible for first particle instability.

Let's make a trick: include $\hat{V}_1(\vec{r}_1)$ in \hat{H}_0 and redefine \hat{V} :

$$\hat{H}_0 = \hat{H}_{01}(\vec{r}_1) + \hat{H}_{02}(\vec{r}_2) = \frac{\hat{p}_1^2}{2m_1} + \hat{V}_1(\vec{r}_1) + \frac{\hat{p}_2^2}{2m_2}$$

$$\hat{V} = -\hat{V}_1(\vec{r}_1) + \hat{V}_{12}(\vec{r}_1 - \vec{r}_2)$$

$$\hat{H} = \frac{\hat{p}_1^2}{2m_1} + \frac{\hat{p}_2^2}{2m_2} + \hat{V}_{12}(\vec{r}_1 - \vec{r}_2)$$

$$\psi(\vec{r}_1, \vec{r}_2, t = 0) = \psi_1(\vec{r}_1)\psi_2(\vec{r}_2)$$

• \hat{H}_0 becomes not free, but the scheme is the same. ψ_1 and ψ_2 are still eigenfunctions of \hat{H}_{01} and \hat{H}_{02} , and first particle state is also unstable "in itself".

Let's consider an example:

$$\hat{H}_0 = \frac{\hat{p}_1^2}{2m_1} + G_1\delta(x_1) + \frac{\hat{p}_2^2}{2m_2}$$

$$\hat{V} = -G_1\delta(x_1) + G_2\delta(x_1 - x_2)$$

$$\hat{H} = \frac{\hat{p}_1^2}{2m_1} + \frac{\hat{p}_2^2}{2m_2} + G_2\delta(x_1 - x_2)$$

$$G_1, G_2 > 0$$

Well-known eigenfunctions of the Hamiltonian $\frac{\hat{p}^2}{2m} + G\delta(x)$:

$$\psi_k(x) = \frac{1}{\sqrt{\pi}} \cos(k|x| + \phi(k)) \text{ (even) and } \psi_k(x) = \frac{1}{\sqrt{\pi}} \sin(kx) \text{ (odd).}$$

$$\text{Here } k \geq 0, \text{ tg}(\phi(k)) = -\frac{\varkappa}{k}, \text{ } \varkappa = \frac{mG}{\hbar^2}$$

For calculation of integrals usual regularization is applied: integrand is multiplied by $e^{-\varepsilon|x|}$ and at the end $\varepsilon \rightarrow 0$. Then

$$\frac{1}{\sqrt{\pi}} \cos(k|x| + \phi) = \cos(\phi) \frac{1}{\sqrt{\pi}} \cos(kx) + \sin(\phi) \int_0^\infty \frac{dk'}{2\pi} \frac{1}{\sqrt{\pi}} \cos(k'x) \times$$

$$\times \left(\frac{1}{k' - k - i\varepsilon} + \frac{1}{k' - k + i\varepsilon} - \frac{1}{k' + k - i\varepsilon} - \frac{1}{k' + k + i\varepsilon} \right)$$

For $k' \neq k$ the expression in brackets gives $\frac{2k}{k'^2 - k^2}$. The distribution $P(k') = \frac{4k^2}{(k'^2 - k^2)^2}$ has a non-integrable singularity at $k' = k$, which it is natural for continuous spectrum. It means that infinite norm is concentrated in the vicinity of $k' = k$, so $\cos(k|x| + \phi)$ is not purely an eigenvalue of the free Hamiltonian $\hat{p}^2/2m$, but very close to it in some sense.

In particular, unit-standardized package

$$\psi(x) = \int_{k_0}^{k_0+\Delta k} \frac{dk}{\sqrt{\Delta k}} \frac{1}{\sqrt{\pi}} \cos(k|x| + \phi)$$

does not spread out with free Hamiltonian at $\Delta k \rightarrow 0$:

$$\langle \psi(x, 0) | \psi(x, t) \rangle \rightarrow e^{-i\hbar t k_0^2 / 2m}$$

At the same time the state $\psi(x) = \frac{\cos(k|x|+\phi)+i\sin(kx)}{\sqrt{\pi}}$ contains a part of $e^{-ikx}/\sqrt{2\pi}$, it is proportional to $(1 - \cos \phi)^2$:

$$\int_{-\infty}^{\infty} \frac{1}{2\pi} dx (\cos(k|x| + \phi) + i \sin(kx)) e^{ik'x} e^{-\varepsilon|x|} = -\frac{1}{2} \delta(k' - k) +$$

$$+ \frac{i}{4\pi} \left(\frac{e^{i\phi}}{k + k' + i\varepsilon} - \frac{e^{-i\phi}}{k + k' - i\varepsilon} + \frac{e^{i\phi}}{k - k' + i\varepsilon} - \frac{e^{-i\phi}}{k - k' - i\varepsilon} \right) =$$

$$-\frac{1}{2} \delta(k' - k) + \frac{1}{2} \cos \phi \delta(k' - k) + \dots$$

So we arrive to something like mixing effect.

Returning to the scattering, to obtain its characteristics we need to decompose the state

$$(e^{i\phi_1(k_1)} \cos(k_1|x_1| + \phi_1(k_1)) + i \sin(k_1|x_1|))e^{ik_2x_2}$$

into eigenfunctions of the Hamiltonian $\frac{\hat{p}_1^2}{2m_1} + \frac{\hat{p}_2^2}{2m_2} + G_2\delta(x_1 - x_2)$. Denoting $m_s = m_1 + m_2$, we have

$$\begin{aligned}
& \frac{1}{\pi\sqrt{2}} \cos\left(k_1|x_1| + \phi_1(k_1)\right) e^{ik_2x_2} = \int_0^\infty dk_3 \int_{-\infty}^\infty dk_4 \frac{1}{\pi\sqrt{2}} \cos\left(k_3|x_1 - x_2| + \phi_2(k_3)\right) e^{ik_4 \frac{m_1x_1 + m_2x_2}{m_s}} \\
& \quad \left(\frac{e^{i\phi_1(k)}}{k_1 + k_2 - k_4 + i\varepsilon} + \frac{e^{-i\phi_1(k)}}{k_1 + k_2 - k_4 - i\varepsilon} + \frac{e^{-i\phi_1(k)}}{-k_1 + k_2 - k_4 + i\varepsilon} - \frac{e^{i\phi_1(k)}}{-k_1 + k_2 - k_4 - i\varepsilon} \right) \\
& \quad \times \left(\frac{e^{i\phi_2(k_3)}}{k_2 - k_4 \frac{m_2}{m_s} - k_3 - i\varepsilon} - \frac{e^{-i\phi_2(k_3)}}{k_2 - k_4 \frac{m_2}{m_s} - k_3 + i\varepsilon} + \frac{e^{-i\phi_2(k_3)}}{k_2 - k_4 \frac{m_2}{m_s} + k_3 - i\varepsilon} - \frac{e^{i\phi_2(k_3)}}{k_2 - k_4 \frac{m_2}{m_s} + k_3 + i\varepsilon} \right) \\
& + \int_0^\infty dk_3 \int_{-\infty}^\infty dk_4 \frac{1}{\pi\sqrt{2}} \sin(k_3(x_1 - x_2)) e^{ik_4 \frac{m_1x_1 + m_2x_2}{m_s}} \frac{1}{4\pi} \left(\delta\left(k_2 - k_4 \frac{m_2}{m_s} - k_3\right) - \delta\left(k_2 - k_4 \frac{m_2}{m_s} + k_3\right) \right) \\
& \quad \times \left(\frac{e^{i\phi_1(k)}}{k_1 + k_2 - k_4 + i\varepsilon} + \frac{e^{-i\phi_1(k)}}{k_1 + k_2 - k_4 - i\varepsilon} + \frac{e^{-i\phi_1(k)}}{-k_1 + k_2 - k_4 + i\varepsilon} - \frac{e^{i\phi_1(k)}}{-k_1 + k_2 - k_4 - i\varepsilon} \right)
\end{aligned}$$

$$\begin{aligned}
\frac{1}{\pi\sqrt{2}} \sin(k_1 x_1) e^{ik_2 x_2} &= \int_0^\infty dk_3 \int_{-\infty}^\infty dk_4 \frac{1}{\pi\sqrt{2}} \sin(k_3(x_1 - x_2)) e^{ik_4 \frac{m_1 x_1 + m_2 x_2}{m_s}} \frac{1}{2} \left(\delta(k_4 - k_1 - k_2) \delta(k_3 - \frac{m_2}{m_s} k_1 + \frac{m_1}{m_s} k_2) \right. \\
&+ \delta(k_4 + k_1 - k_2) \delta(k_3 - \frac{m_1 k_2 + m_2 k_1}{m_s}) - \delta(k_4 - k_1 - k_2) \delta(k_3 + \frac{m_2}{m_s} k_1 - \frac{m_1}{m_s} k_2) - \delta(k_4 + k_1 - k_2) \delta(k_3 + \frac{m_1 k_2 + m_2 k_1}{m_s}) \Big) + \\
&\int_0^\infty dk_3 \int_{-\infty}^\infty dk_4 \frac{1}{\pi\sqrt{2}} \cos\left(k_3 |x_1 - x_2| + \phi_2(k_3)\right) e^{ik_4 \frac{m_1 x_1 + m_2 x_2}{m_s}} \frac{1}{4\pi} \left(\delta(k_4 - k_1 - k_2) \left(\frac{e^{i\phi_2(k_3)}}{k_3 + \frac{m_1}{m_s} k_2 - \frac{m_2}{m_s} k_1 + i\varepsilon} - \frac{e^{-i\phi_2(k_3)}}{k_3 + \frac{m_1}{m_s} k_2 - \frac{m_2}{m_s} k_1 - i\varepsilon} \right) \right. \\
&\quad \left. + \delta(k_4 - k_1 - k_2) \left(\frac{e^{-i\phi_2(k_3)}}{-k_3 + \frac{m_1}{m_s} k_2 - \frac{m_2}{m_s} k_1 + i\varepsilon} - \frac{e^{i\phi_2(k_3)}}{-k_3 + \frac{m_1}{m_s} k_2 - \frac{m_2}{m_s} k_1 - i\varepsilon} \right) \right. \\
&\quad \left. - \delta(k_4 + k_1 - k_2) \left(\frac{e^{i\phi_2(k_3)}}{k_3 + \frac{m_1 k_2 + m_2 k_1}{m_s} + i\varepsilon} - \frac{e^{-i\phi_2(k_3)}}{k_3 + \frac{m_1 k_2 + m_2 k_1}{m_s} - i\varepsilon} \right) - \delta(k_4 + k_1 - k_2) \left(\frac{e^{-i\phi_2(k_3)}}{-k_3 + \frac{m_1 k_2 + m_2 k_1}{m_s} + i\varepsilon} - \frac{e^{i\phi_2(k_3)}}{-k_3 + \frac{m_1 k_2 + m_2 k_1}{m_s} - i\varepsilon} \right) \right)
\end{aligned}$$

As a characteristic of the scattering let's calculate an average reflection coefficient, that is obtained by decomposition of the package

$$\int_{k_1}^{k_1+\Delta k_1} \int_{k_2}^{k_2+\Delta k_2} \frac{dk'_1 dk'_2}{\sqrt{\Delta k_1 \Delta k_2}} \frac{1}{2\pi} (e^{i\phi_1(k'_1)} \cos(k'_1|x|+\phi_1(k_1)+i \sin(k'_1 x_1))) e^{ik'_2 x_2}$$

by scattering functions in the potential $G_2\delta(x_1 - x_2)$

$$\psi_{k_1, k_2}^{\pm}(x_1, x_2) = (e^{i\phi(k_1)} \cos(k_1|x_1-x_2|+\phi(k_1)) \pm \sin(k_1(x_1-x_2))) e^{ik_2 \frac{m_1 x_1 + m_2 x_2}{m_s}}$$

Then we tend Δk_1 and Δk_2 to zero.

The result is

$$R = \frac{\varkappa_2^2}{\varkappa_2^2 + \frac{(m_2 k_1 - m_1 k_2)^2}{m_s^2}} \left(1 - \frac{\varkappa_1^2}{2k_1^2 + 2\varkappa_1^2}\right) + \frac{\varkappa_2^2}{\varkappa_2^2 + \frac{(m_2 k_1 + m_1 k_2)^2}{m_s^2}} \frac{\varkappa_1^2}{2k_1^2 + 2\varkappa_1^2}$$

When $m_2 k_1 + m_1 k_2$ is close to zero the correction receives amplification. Note in this region and when $m_2 k_1 - m_1 k_2$ is close to zero the result is not an analytical function of the coupling: its Taylor series diverges. This situation is like in Breit-Wigner formula case and goes back to scattering amplitudes in delta potential:

$$A = \frac{-i\frac{\varkappa}{k}}{1+i\frac{\varkappa}{k}}, \quad B = \frac{1}{1+i\frac{\varkappa}{k}}$$

It means that standard perturbation theory is not applicable in these regions.

The considered example is

1. one-dimensional;
2. with local potential;
3. with absence of bound states;
4. with practically stable initial state.

It is interesting what happens after changing these features.

Thank you for your attention!