Solutions in Quadratures of Einstein-Dilaton-Maxwell Models with 3-form Fields

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Holographic Model of Anisotropic Plasma in Magnetic Field at Nonzero Chemical Potential

$$S = \int d^5x \, \sqrt{-g} \left[R - \frac{f_0(\phi)}{4} F_{(0)}^2 - \frac{f_1(\phi)}{4} F_{(1)}^2 - \frac{f_3(\phi)}{4} F_{(3)}^2 - \frac{1}{2} (\partial\phi)^2 - V(\phi) \right]$$

$$ds^2 = \frac{L^2}{z^2} \, \mathfrak{b}(z) \left[-g(z) \, dt^2 + dx^2 + \left(\frac{z}{L}\right)^{2-\frac{2}{\nu}} dy_1^2 + e^{c_B z^2} \left(\frac{z}{L}\right)^{2-\frac{2}{\nu}} dy_2^2 + \frac{dz^2}{g(z)} \right]$$

$$A_{(0)} = A_t(z) \, dt, \;\; A_t(0) = \mu \qquad F_{(1)} = q_{(1)} \, dy^1 \wedge dy^2 \qquad F_{(3)} = q_{(3)} \, dx \wedge dy^1$$

Li, Yang, Yuan'17 *Aref'eva, Rannu JHEP'18* *Aref'eva, Rannu, Slepov JHEP '21*

Thermodynamics: black brane solution $g(0) = 1, \; g(z_h) = 0$

$$T = \left. \frac{|g'|}{4\pi} \right|_{z=z_h} \longrightarrow \text{QGP temperature (Maldacena)}$$

$$s = \left. \frac{\sqrt{g_{yy} g_{x_1x_1} g_{x_2x_2}}}{4} \right|_{z=z_h} \longrightarrow \text{multiplicity of process (Landau)}$$

Overview

1. Class of Models
2. Observations
   - Pattern in Einstein Equations
   - Formulas for Diagonal Metric $g_{\mu\nu}(z)$
3. Solutions in Quadratures
   - Simple Example
   - Four-Maxwell Example
   - Arbitrary Model
4. Applications

Class of Models

Einstein-dilaton-Maxwell action with 2-forms and 3-forms

$$S = \int d^D x \sqrt{-g_D} \times$$
$$\times \left[R - \frac{f_{(0)}(\phi)}{2 \cdot 2!} F_{(0)}^2 - \sum_{\mathcal{M}} \frac{f_{(\mathcal{M})}(\phi)}{2 \cdot 2!} F_{(\mathcal{M})}^2 - \sum_{\mathcal{N}} \frac{h_{(\mathcal{N})}(\phi)}{2 \cdot 3!} H_{(\mathcal{N})}^2 - \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - V(\phi) \right]$$

$$ds^2 = \frac{L^2 \, \mathfrak{b}(z)}{z^2} \left(-g(z) dt^2 + \sum_{i=1}^{D-2} \mathfrak{g}_i(z) dx_i^2 + \frac{dz^2}{g(z)} \right)$$

Material fields ansatses

- electric ansatz $F_{(0)} = A_t'(z) \, dz \wedge dt$: $\quad A_t(0) = \mu, \; A_t(z_h) = 0$
- magnetic ansatz $F_{(\mathcal{M})} = q_{(\mathcal{M})} \, dx^p \wedge dx^k$, $H_{(\mathcal{N})} = Q_{(\mathcal{N})} \, dx^p \wedge dx^k \wedge dx^l$ for some $p, k, l = \overline{1, D-2}$
- $\phi = \phi(z)$

Thermodynamics

$$\begin{aligned} g(0) &= 1 \\ g(z_h) &= 0 \end{aligned} \quad \Longrightarrow \quad T(z_h) = \frac{|g'(z_h)|}{4\pi}, \quad s(z_h) = \frac{1}{4} \sqrt{\prod_{i=1}^{D-2} g_{ii}(z_h)}$$

Equations of Motions

$$G_{\mu\nu} = T_{\mu\nu} \quad \longrightarrow \quad \text{D EOM}$$

$$\partial_\mu\left(\sqrt{-g_D}\, f_{(0)}\, F^{\mu\nu}_{(0)}\right) = 0 \quad \longrightarrow \quad \text{1 EOM}$$

$$\partial_\mu\left(\sqrt{-g_D}\, f_{(\mathcal{M})}\, F^{\mu\nu}_{(\mathcal{M})}\right) = 0 \quad \longrightarrow \quad \text{0 EOM}$$

$$\partial_\mu\left(\sqrt{-g_D}\, h_{(\mathcal{N})}\, H^{\mu\nu\rho}_{(\mathcal{N})}\right) = 0 \quad \longrightarrow \quad \text{0 EOM}$$

$$\begin{aligned} D_\mu D^\mu \phi - \partial_\phi V(\phi) - \frac{1}{4}\partial_\phi f_{(0)}(\phi)\, F^2_{(0)} - \\ - \sum_{\mathcal{M}} \frac{1}{4}\partial_\phi f_{(\mathcal{M})}(\phi)\, F^2_{(\mathcal{M})} - \sum_{\mathcal{N}} \frac{1}{12}\partial_\phi h_{(\mathcal{N})}(\phi)\, H^2_{(\mathcal{N})} = 0 \end{aligned} \quad \longrightarrow \quad \text{1 EOM}$$

Equations of Motions

$$G_{\mu\nu} = T_{\mu\nu} \quad \longrightarrow \quad \text{D EOM}$$

$$\partial_\mu\left(\sqrt{-g_D}\, f_{(0)}\, F^{\mu\nu}_{(0)}\right) = 0 \quad \longrightarrow \quad 1 \text{ EOM}$$

$$\partial_\mu\left(\sqrt{-g_D}\, f_{(\mathcal{M})}\, F^{\mu\nu}_{(\mathcal{M})}\right) = 0 \quad \longrightarrow \quad 0 \text{ EOM}$$

$$\partial_\mu\left(\sqrt{-g_D}\, h_{(\mathcal{N})}\, H^{\mu\nu\rho}_{(\mathcal{N})}\right) = 0 \quad \longrightarrow \quad 0 \text{ EOM}$$

$$D_\mu D^\mu \phi - \partial_\phi V(\phi) - \frac{1}{4}\partial_\phi f_{(0)}(\phi)\, F^2_{(0)} - \quad \longrightarrow \quad 1 \text{ EOM}$$

$$-\sum_{\mathcal{M}} \frac{1}{4}\partial_\phi f_{(\mathcal{M})}(\phi)\, F^2_{(\mathcal{M})} - \sum_{\mathcal{N}} \frac{1}{12}\partial_\phi h_{(\mathcal{N})}(\phi)\, H^2_{(\mathcal{N})} = 0$$

Not independent:

$$0 = \nabla_\mu T^{\mu\nu}_{total} = \frac{1}{2g_{zz}\sqrt{-g_D}}\left(A_t' \frac{\delta \mathcal{L}}{\delta A_t} + \phi' \frac{\delta \mathcal{L}}{\delta \phi}\right) \implies \frac{\delta \mathcal{L}}{\delta \phi} = 0 \quad \text{if} \quad \frac{\delta \mathcal{L}}{\delta A_t} = 0$$

I. Ya. Aref'eva, K. Rannu, and P. Slepov arXiv:2409.12131v1 [hep-th]

Pattern in $T_{\mu\mu}/g_{\mu\mu}$

$$\phi = \phi(z): \qquad \frac{T^{(\phi)}_{\mu\mu}}{g_{\mu\mu}} = \frac{V(\phi)}{2}\Big(-,\,\ldots,\,-,\,\ldots,\,-\Big)^T + \\ + \frac{\phi'^2}{4g_{zz}}\Big(-,\,\ldots,\,-,\,+\Big)^T$$

Pattern in $T_{\mu\mu}/g_{\mu\mu}$

$$\phi = \phi(z): \qquad \frac{T^{(\phi)}_{\mu\mu}}{g_{\mu\mu}} = \frac{V(\phi)}{2}\Big(-, \ldots, -, \ldots, -\Big)^T + \\ + \frac{\phi'^2}{4g_{zz}}\Big(-, \ \ldots, \ -, \ +\Big)^T$$

$$F_{(0)} = A'_t(z)\, dt \wedge dz: \qquad \frac{T^{(0)}_{\mu\mu}}{g_{\mu\mu}} = \frac{f_{(0)}(\phi)\, A'^2_t(z)}{4g_{00}g_{zz}}\Big(\underbrace{+, \ldots, -, \ldots, +}_{\text{plus sign before } t,\ z}\Big)^T$$

$$F_{(\mathcal{M})} = q_{(\mathcal{M})}\, dx^p \wedge dx^k: \qquad \frac{T^{(\mathcal{M})}_{\mu\mu}}{g_{\mu\mu}} = \frac{f_{(\mathcal{M})}(\phi)\, q^2_{(\mathcal{M})}}{4g_{pp}g_{kk}}\Big(-, \ldots, \underbrace{+, \ldots, +}_{\text{plus sign before } p,\ k}, \ldots, -\Big)^T$$

Pattern in $T_{\mu\mu}/g_{\mu\mu}$

$$\phi = \phi(z): \qquad \frac{T^{(\phi)}_{\mu\mu}}{g_{\mu\mu}} = \frac{V(\phi)}{2}\Big(-, \ldots, -, \ldots, -\Big)^T + \frac{\phi'^2}{4g_{zz}}\Big(-, \ldots, -, +\Big)^T$$

$$F_{(0)} = A_t'(z)\,dt \wedge dz: \qquad \frac{T^{(0)}_{\mu\mu}}{g_{\mu\mu}} = \frac{f_{(0)}(\phi)\,A_t'^2(z)}{4g_{00}g_{zz}}\Big(+, \ldots, -, \ldots, +\Big)^T$$

plus sign before t, z

$$F_{(\mathcal{M})} = q_{(\mathcal{M})}\,dx^p \wedge dx^k: \qquad \frac{T^{(\mathcal{M})}_{\mu\mu}}{g_{\mu\mu}} = \frac{f_{(\mathcal{M})}(\phi)\,q^2_{(\mathcal{M})}}{4g_{pp}g_{kk}}\Big(-, \ldots, +, \ldots, +, \ldots, -\Big)^T$$

plus sign before p, k

$$H_{(\mathcal{N})} = Q_{(\mathcal{N})}\,dx^p \wedge dx^k \wedge dx^l: \qquad \frac{T^{(\mathcal{N})}_{\mu\mu}}{g_{\mu\mu}} = \frac{h_{(\mathcal{N})}(\phi)\,Q^2_{(\mathcal{N})}}{4g_{pp}g_{kk}g_{ll}}\Big(-, \ldots, +, \ldots, +, \ldots, +, \ldots, -\Big)^T$$

plus sign before p, k, l

Pattern in $T_{\mu\mu}/g_{\mu\mu}$

$$\phi = \phi(z): \qquad \frac{T^{(\phi)}_{\mu\mu}}{g_{\mu\mu}} = \frac{V(\phi)}{2}\Big(-,\,\dots,\,-,\,\dots,\,-\Big)^T + \frac{\phi'^2}{4g_{zz}}\Big(-,\,\dots,\,-,\,+\Big)^T$$

$$F_{(0)} = A'_t(z)\,dt\wedge dz: \qquad \frac{T^{(0)}_{\mu\mu}}{g_{\mu\mu}} = \frac{f_{(0)}(\phi)\,A'^2_t(z)}{4g_{00}g_{zz}}\Big(+,\,\dots,\,-,\,\dots,\,+\Big)^T$$

plus sign before t, z

$$F_{(\mathcal{M})} = q_{(\mathcal{M})}\,dx^p\wedge dx^k: \qquad \frac{T^{(\mathcal{M})}_{\mu\mu}}{g_{\mu\mu}} = \frac{f_{(\mathcal{M})}(\phi)\,q^2_{(\mathcal{M})}}{4g_{pp}g_{kk}}\Big(-,\,\dots,\,+,\,\dots,\,+,\,\dots,\,-\Big)^T$$

plus sign before p, k

$$H_{(\mathcal{N})} = Q_{(\mathcal{N})}\,dx^p\wedge dx^k\wedge dx^l: \qquad \frac{T^{(\mathcal{N})}_{\mu\mu}}{g_{\mu\mu}} = \frac{h_{(\mathcal{N})}(\phi)\,Q^2_{(\mathcal{N})}}{4g_{pp}g_{kk}g_{ll}}\Big(-,\,\dots,\,+,\,\dots,\,+,\,\dots,\,+,\,\dots,\,-\Big)^T$$

plus sign before p, k, l

$$\mu\neq\nu: \qquad T^{(\phi)}_{\mu\nu} = T^{(0)}_{\mu\nu} = T^{(\mathcal{M})}_{\mu\nu} = T^{(\mathcal{N})}_{\mu\nu} = 0, \quad g_{\mu\nu} = 0$$

Pattern in $T_{\mu\mu}/g_{\mu\mu}$: Sign Tables

Model: $D=6, \quad \phi=\phi(z), \quad F = q\, dx^3 \wedge dx^4, \quad H = Q\, dx^2 \wedge dx^3 \wedge dx^4$

		$\frac{\phi'^2}{4g_{zz}}$	$\frac{V(\phi)}{2}$	$\frac{f(\phi)\, q^2}{4g_{x_3x_3}g_{x_4x_4}}$	$\frac{h(\phi)\, Q^2}{4g_{x_2x_2}g_{x_3x_3}g_{x_4x_4}}$	
(z)	$\frac{T_{zz}}{g_{zz}}$	$+$	$-$	$-$	$-$	$=\frac{G_{zz}}{g_{zz}}$
(x_4)	$\frac{T_{x_4x_4}}{g_{x_4x_4}}$	$-$	$-$	$+$	$+$	$=\frac{G_{x_4x_4}}{g_{x_4x_4}}$
(x_3)	$\frac{T_{x_3x_3}}{g_{x_3x_3}}$	$-$	$-$	$+$	$+$	$=\frac{G_{x_3x_3}}{g_{x_3x_3}}$
(x_2)	$\frac{T_{x_2x_2}}{g_{x_2x_2}}$	$-$	$-$	$-$	$+$	$=\frac{G_{x_2x_2}}{g_{x_2x_2}}$
(x_1)	$\frac{T_{x_1x_1}}{g_{x_1x_1}}$	$-$	$-$	$-$	$-$	$=\frac{G_{x_1x_1}}{g_{x_1x_1}}$
(t)	$-\frac{T_{tt}}{g_{tt}}$	$-$	$-$	$-$	$-$	$=-\frac{G_{tt}}{g_{tt}}$

$$\frac{T_{x_3x_3}}{g_{x_3x_3}} \equiv -\frac{\phi'^2}{4g_{zz}} - \frac{V(\phi)}{2} + \frac{f(\phi)\, q^2}{4g_{x_3x_3}g_{x_4x_4}} + \frac{h(\phi)\, Q^2}{4g_{x_2x_2}g_{x_3x_3}g_{x_4x_4}} = \frac{G_{x_3x_3}}{g_{x_3x_3}}$$

Pattern in $T_{\mu\mu}/g_{\mu\mu}$: Sign Tables

Model: $D=6, \quad \phi=\phi(z), \quad F = q\, dx^3 \wedge dx^4, \quad H = Q\, dx^2 \wedge dx^3 \wedge dx^4$

		$\frac{\phi'^2}{4g_{zz}}$	$\frac{V(\phi)}{2}$	$\frac{f(\phi)\, q^2}{4g_{x_3x_3}g_{x_4x_4}}$	$\frac{h(\phi)\, Q^2}{4g_{x_2x_2}g_{x_3x_3}g_{x_4x_4}}$	
(z)	$\frac{T_{zz}}{g_{zz}}$	$+$	$-$	$-$	$-$	$=\frac{G_{zz}}{g_{zz}}$
(x_4)	$\frac{T_{x_4x_4}}{g_{x_4x_4}}$	$-$	$-$	$+$	$+$	$=\frac{G_{x_4x_4}}{g_{x_4x_4}}$
(x_3)	$\frac{T_{x_3x_3}}{g_{x_3x_3}}$	$-$	$-$	$+$	$+$	$=\frac{G_{x_3x_3}}{g_{x_3x_3}}$
(x_2)	$\frac{T_{x_2x_2}}{g_{x_2x_2}}$	$-$	$-$	$-$	$+$	$=\frac{G_{x_2x_2}}{g_{x_2x_2}}$
(x_1)	$\frac{T_{x_1x_1}}{g_{x_1x_1}}$	$-$	$-$	$-$	$-$	$=\frac{G_{x_1x_1}}{g_{x_1x_1}}$
(t)	$-\frac{T_{tt}}{g_{tt}}$	$-$	$-$	$-$	$-$	$=-\frac{G_{tt}}{g_{tt}}$

$$(\boldsymbol{x_1}) - (\boldsymbol{t}) \qquad \frac{G_{x_1x_1}}{g_{x_1x_1}} + \frac{G_{tt}}{g_{tt}} = 0$$

Transformed Einstein Equations

Model: $D = 6, \quad \phi = \phi(z), \quad F = q\, dx^3 \wedge dx^4, \quad H = Q\, dx^2 \wedge dx^3 \wedge dx^4$

$$(z) - (t) \qquad \frac{G_{zz}}{g_{zz}} + \frac{G_{tt}}{g_{tt}} = \frac{\phi'^2}{2\, g_{zz}}$$

$$(z) + (x_4) \qquad \frac{G_{zz}}{g_{zz}} + \frac{G_{x_4x_4}}{g_{x_4x_4}} = -V(\phi)$$

$$(x_3) - (x_2) \qquad \frac{G_{x_3x_3}}{g_{x_3x_3}} - \frac{G_{x_2x_2}}{g_{x_2x_2}} = \frac{f(\phi)\, q^2}{2\, g_{x_3x_3} g_{x_4x_4}}$$

$$(x_2) - (x_1) \qquad \frac{G_{x_2x_2}}{g_{x_2x_2}} - \frac{G_{x_1x_1}}{g_{x_1x_1}} = \frac{h(\phi)\, Q^2}{2\, g_{x_2x_2} g_{x_3x_3} g_{x_4x_4}}$$

$$(x_1) - (t) \qquad \frac{G_{x_1x_1}}{g_{x_1x_1}} + \frac{G_{tt}}{g_{tt}} = 0$$

$$(x_4) - (x_3) \qquad \frac{G_{x_4x_4}}{g_{x_4x_4}} - \frac{G_{x_3x_3}}{g_{x_3x_3}} = 0$$

Main Formula(s) for $G_{\mu\mu}/g_{\mu\mu}$

$$ds^2 = g_{00}(z)\,dt^2 + \sum_{i=1}^{D-2} g_{ii}(z)\,dx_i^2 + g_{zz}\,dz^2$$

$$[\boldsymbol{\alpha},\,\boldsymbol{\beta} \neq \boldsymbol{z}]: \quad \frac{G_{\alpha\alpha}}{g_{\alpha\alpha}} - \frac{G_{\beta\beta}}{g_{\beta\beta}} = \frac{1}{2\sqrt{g_D}}\left(\frac{\sqrt{g_D}}{g_{zz}}\frac{d}{dz}\ln\frac{g_{\beta\beta}}{g_{\alpha\alpha}}\right)'$$

$$[\boldsymbol{\alpha} \neq \boldsymbol{z}]: \quad \frac{G_{\alpha\alpha}}{g_{\alpha\alpha}} + \frac{G_{zz}}{g_{zz}} = \frac{1}{2\sqrt{g_D}}\left(\frac{\sqrt{g_D}}{g_{zz}}\frac{d}{dz}\ln\frac{g_D}{g_{\alpha\alpha}g_{zz}}\right)'$$

$$\text{Euclidean} \to \text{Lorentz} \qquad g_{00}(z) \to -\,g_{tt}(z)$$

$$[\boldsymbol{\alpha_{1,2}},\,\boldsymbol{\beta_{1,2}} \neq \boldsymbol{z}]: \quad \left(\frac{G_{\alpha_1\alpha_1}}{g_{\alpha_1\alpha_1}} - \frac{G_{\beta_1\beta_1}}{g_{\beta_1\beta_1}}\right) + \left(\frac{G_{\alpha_2\alpha_2}}{g_{\alpha_2\alpha_2}} - \frac{G_{\beta_2\beta_2}}{g_{\beta_2\beta_2}}\right) =$$

$$= \frac{1}{2\sqrt{g_D}}\left(\frac{\sqrt{g_D}}{g_{zz}}\frac{d}{dz}\ln\frac{g_{\beta_1\beta_1}g_{\beta_2\beta_2}}{g_{\alpha_1\alpha_1}g_{\alpha_2\alpha_2}}\right)'$$

Solution in Quadratures: Simplest Example

$$\text{Euclidean } [\boldsymbol{\alpha \neq z}]: \qquad \frac{G_{\alpha\alpha}}{g_{\alpha\alpha}} - \frac{G_{00}}{g_{00}} = \frac{1}{2\sqrt{g_D}} \left(\frac{\sqrt{g_D}}{g_{zz}} \frac{d}{dz} \ln \frac{g_{00}}{g_{\alpha\alpha}} \right)'$$

$$\text{Lorentz } [\boldsymbol{\alpha \neq z}]: \qquad \frac{G_{\alpha\alpha}}{g_{\alpha\alpha}} + \frac{G_{tt}}{g_{tt}} = \frac{1}{2\sqrt{-g_D}} \left(\frac{\sqrt{-g_D}}{g_{zz}} \frac{d}{dz} \ln \frac{g_{tt}}{g_{\alpha\alpha}} \right)'$$

Model: $\quad D = 6, \quad \phi = \phi(z), \quad F = q\, dx^3 \wedge dx^4, \quad H = Q\, dx^2 \wedge dx^3 \wedge dx^4$

$$\boldsymbol{(x_1) - (t)} \qquad 0 = \frac{G_{x_1 x_1}}{g_{x_1 x_1}} + \frac{G_{tt}}{g_{tt}} \equiv \frac{1}{2\sqrt{-g_D}} \Bigg(\underbrace{\mathfrak{g}_1(z) \prod_{\gamma \neq z, t} \sqrt{g_{\gamma\gamma}(z)}\, \frac{d}{dz} \frac{g(z)}{\mathfrak{g}_1(z)}}_{\equiv C_1} \Bigg)'$$

Solution in Quadratures: Simplest Example

$$\text{Euclidean } [\boldsymbol{\alpha \neq z}]: \qquad \frac{G_{\alpha\alpha}}{g_{\alpha\alpha}} - \frac{G_{00}}{g_{00}} = \frac{1}{2\sqrt{g_D}} \left(\frac{\sqrt{g_D}}{g_{zz}} \frac{d}{dz} \ln \frac{g_{00}}{g_{\alpha\alpha}} \right)'$$

$$\text{Lorentz } [\boldsymbol{\alpha \neq z}]: \qquad \frac{G_{\alpha\alpha}}{g_{\alpha\alpha}} + \frac{G_{tt}}{g_{tt}} = \frac{1}{2\sqrt{-g_D}} \left(\frac{\sqrt{-g_D}}{g_{zz}} \frac{d}{dz} \ln \frac{g_{tt}}{g_{\alpha\alpha}} \right)'$$

Model: $\quad D = 6, \quad \phi = \phi(z), \quad F = q\, dx^3 \wedge dx^4, \quad H = Q\, dx^2 \wedge dx^3 \wedge dx^4$

$$\boldsymbol{(x_1) - (t)} \qquad 0 = \frac{G_{x_1x_1}}{g_{x_1x_1}} + \frac{G_{tt}}{g_{tt}} \equiv \frac{1}{2\sqrt{-g_D}} \Bigg(\underbrace{\mathfrak{g}_1(z) \prod_{\gamma\neq z,t} \sqrt{g_{\gamma\gamma}(z)} \, \frac{d}{dz} \frac{g(z)}{\mathfrak{g}_1(z)}}_{\equiv C_1} \Bigg)'$$

$$\mathfrak{s}(z) = \prod_{\gamma\neq z,t} \sqrt{g_{\gamma\gamma}(z)} \quad \Longrightarrow \quad g(z) = \mathfrak{g}_1(z) \left[C_1 \int_{z_0}^{z} \frac{d\xi}{\mathfrak{g}_1(\xi)\mathfrak{s}(\xi)} + C_2 \right]$$

Solution in Quadratures: Four-Maxwell Example

$$D = 5, \quad \phi = \phi(z), \quad F_{(0)} = A_t'(z)\, dz \wedge dt$$

Model: $F_{(1)} = q_{(1)}\, dx^2 \wedge dx^3, \quad F_{(2)} = q_{(2)}\, dx^1 \wedge dx^3, \quad F_{(3)} = q_{(3)}\, dx^1 \wedge dx^2$

$$ds^2 = \mathfrak{B}^2(z)\left(-g(z)dt^2 + \mathfrak{g}_1(z)dx_1^2 + \mathfrak{g}_2(z)dx_2^2 + \mathfrak{g}_3(z)dx_3^2 + \frac{dz^2}{g(z)}\right)$$

$$g'' + g'\left(\frac{3\mathfrak{B}'}{\mathfrak{B}} + \frac{3\mathfrak{g}_1'}{2\mathfrak{g}_1} - \frac{\mathfrak{g}_2'}{2\mathfrak{g}_2} - \frac{\mathfrak{g}_3'}{2\mathfrak{g}_3}\right) -$$

$$- g\left[-\frac{\mathfrak{g}_1''}{\mathfrak{g}_1} + \frac{\mathfrak{g}_1'^2}{2\mathfrak{g}_1} + \frac{\mathfrak{g}_2''}{\mathfrak{g}_2} - \frac{\mathfrak{g}_2'^2}{2\mathfrak{g}_2} + \frac{\mathfrak{g}_3''}{\mathfrak{g}_3} - \frac{\mathfrak{g}_3'^2}{2\mathfrak{g}_3} + \frac{3\mathfrak{B}'}{\mathfrak{B}}\left(-\frac{\mathfrak{g}_1'}{\mathfrak{g}_1} + \frac{\mathfrak{g}_2'}{\mathfrak{g}_2} + \frac{\mathfrak{g}_3'}{\mathfrak{g}_3}\right) + \frac{\mathfrak{g}_2'}{\mathfrak{g}_2}\frac{\mathfrak{g}_3'}{\mathfrak{g}_3}\right] -$$

$$- \frac{f_{(0)} A_t'^2}{\mathfrak{B}^2} - \frac{2 f_{(1)} q_{(1)}^2}{\mathfrak{B}^2 \mathfrak{g}_2 \mathfrak{g}_3} = 0$$

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Sign Table $\longrightarrow$ $$\frac{G_{x_2x_2}}{g_{x_2x_2}} + \frac{G_{x_3x_3}}{g_{x_3x_3}} + \frac{G_{tt}}{g_{tt}} - \frac{G_{x_1x_1}}{g_{x_1x_1}} = \frac{f_{(1)}(\phi)\, q_{(1)}^2}{g_{x_2x_2}\, g_{x_3x_3}} + \frac{f_{(0)}(\phi)\, A_t'^2(z)}{2\, g_{tt}\, g_{zz}}$$

Solution in Quadratures: Four-Maxwell Example

$$D = 5, \quad \phi = \phi(z), \quad F_{(0)} = A_t'(z)\, dz \wedge dt$$

Model: $F_{(1)} = q_{(1)}\, dx^2 \wedge dx^3, \quad F_{(2)} = q_{(2)}\, dx^1 \wedge dx^3, \quad F_{(3)} = q_{(3)}\, dx^1 \wedge dx^2$

$$ds^2 = \mathfrak{B}^2(z) \left(-g(z)dt^2 + \mathfrak{g}_1(z)dx_1^2 + \mathfrak{g}_2(z)dx_2^2 + \mathfrak{g}_3(z)dx_3^2 + \frac{dz^2}{g(z)} \right)$$

$$g'' + g' \left(\frac{3\mathfrak{B}'}{\mathfrak{B}} + \frac{3\mathfrak{g}_1'}{2\mathfrak{g}_1} - \frac{\mathfrak{g}_2'}{2\mathfrak{g}_2} - \frac{\mathfrak{g}_3'}{2\mathfrak{g}_3} \right) -$$

$$- g \left[-\frac{\mathfrak{g}_1''}{\mathfrak{g}_1} + \frac{\mathfrak{g}_1'^2}{2\mathfrak{g}_1} + \frac{\mathfrak{g}_2''}{\mathfrak{g}_2} - \frac{\mathfrak{g}_2'^2}{2\mathfrak{g}_2} + \frac{\mathfrak{g}_3''}{\mathfrak{g}_3} - \frac{\mathfrak{g}_3'^2}{2\mathfrak{g}_3} + \frac{3\mathfrak{B}'}{\mathfrak{B}} \left(-\frac{\mathfrak{g}_1'}{\mathfrak{g}_1} + \frac{\mathfrak{g}_2'}{\mathfrak{g}_2} + \frac{\mathfrak{g}_3'}{\mathfrak{g}_3} \right) + \frac{\mathfrak{g}_2'}{\mathfrak{g}_2}\frac{\mathfrak{g}_3'}{\mathfrak{g}_3} \right] -$$

$$- \frac{f_{(0)} A_t'^2}{\mathfrak{B}^2} - \frac{2 f_{(1)} q_{(1)}^2}{\mathfrak{B}^2 \mathfrak{g}_2 \mathfrak{g}_3} = 0$$

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Sign Table $\longrightarrow$ $\dfrac{G_{x_2x_2}}{g_{x_2x_2}} + \dfrac{G_{x_3x_3}}{g_{x_3x_3}} + \dfrac{G_{tt}}{g_{tt}} - \dfrac{G_{x_1x_1}}{g_{x_1x_1}} = \dfrac{f_{(1)}(\phi)\, q_{(1)}^2}{g_{x_2x_2}\, g_{x_3x_3}} + \dfrac{f_{(0)}(\phi)\, A_t'^2(z)}{2\, g_{tt}\, g_{zz}}$

$$\mathfrak{G} = \frac{\mathfrak{g}_2\mathfrak{g}_3}{\mathfrak{g}_1} \quad \Longrightarrow \quad \frac{1}{2}\frac{1}{\mathfrak{B}^2\mathfrak{s}} \left(\mathfrak{s}\mathfrak{G} \left(\frac{g}{\mathfrak{G}} \right)' \right)' = \frac{f_{(1)} q_{(1)}^2}{\mathfrak{B}^4 \mathfrak{g}_2\mathfrak{g}_3} + \frac{f_{(0)} A_t'^2}{2\mathfrak{B}^4}$$

Solution in Quadratures: Four-Maxwell Example

EOM for $g(z)$

$$\frac{1}{2}\frac{1}{\mathfrak{B}^2\mathfrak{s}}\left(\mathfrak{s}\mathfrak{G}\left(\frac{g}{\mathfrak{G}}\right)'\right)' = \frac{f_{(1)}q_{(1)}^2}{\mathfrak{B}^4\mathfrak{g}_2\mathfrak{g}_3} + \frac{f_{(0)}A_t'^2}{2\mathfrak{B}^4}$$

Step 1: No RHS

$$\left(\mathfrak{s}\mathfrak{G}\left(\frac{g}{\mathfrak{G}}\right)'\right)' = 0 \quad \Longrightarrow \quad g(z) = \mathfrak{G}(z)\left[C_1\int_{z_0}^{z}\frac{d\xi}{\mathfrak{s}(\xi)\,\mathfrak{G}(\xi)} + C_2\right]$$

Solution in Quadratures: Four-Maxwell Example

EOM for $g(z)$

$$\frac{1}{2}\frac{1}{\mathfrak{B}^2\mathfrak{s}}\left(\mathfrak{s}\mathfrak{G}\left(\frac{g}{\mathfrak{G}}\right)'\right)' = \frac{f_{(1)}q_{(1)}^2}{\mathfrak{B}^4\mathfrak{g}_2\mathfrak{g}_3} + \frac{f_{(0)}A_t'^2}{2\mathfrak{B}^4}$$

Step 1: No RHS

$$\left(\mathfrak{s}\mathfrak{G}\left(\frac{g}{\mathfrak{G}}\right)'\right)' = 0 \quad \Longrightarrow \quad g(z) = \mathfrak{G}(z)\left[C_1\int_{z_0}^{z}\frac{d\xi}{\mathfrak{s}(\xi)\,\mathfrak{G}(\xi)} + C_2\right]$$

Step 2:
EOM for $A_t(z)$

$$0 = \frac{d}{dz}\left(\frac{\mathfrak{s}(z)\,f_{(0)}(\phi(z))}{\mathfrak{B}^2(z)}\,A_t'(z)\right),\quad A_t(0)=\mu,\quad A_t(z_h)=0$$

$$I(z) = \int_0^z \frac{\mathfrak{B}^2(\xi)}{\mathfrak{s}(\xi)\,f_{(0)}(\phi(\xi))}\,d\xi \quad \Longrightarrow \quad A_t(z) = \mu\left(1-\frac{I(z)}{I(z_h)}\right)$$

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Step 2:
EOM for $A_t(z)$
$$0 = \frac{d}{dz}\left(\frac{\mathfrak{s}(z)\,f_{(0)}(\phi(z))}{\mathfrak{B}^2(z)}\,A_t'(z)\right), \quad A_t(0) = \mu, \quad A_t(z_h) = 0$$
$$I(z) = \int_0^z \frac{\mathfrak{B}^2(\xi)}{\mathfrak{s}(\xi)\,f_{(0)}(\phi(\xi))}\,d\xi \quad \Longrightarrow \quad A_t(z) = \mu\left(1 - \frac{I(z)}{I(z_h)}\right)$$

Step 3: With RHS
$$g(z) = \mathfrak{G}(z)\left[C\int_{z_h}^{z}\frac{d\xi}{\mathfrak{s}(\xi)\,\mathfrak{G}(\xi)} + 2\,q_{(1)}^2\int_{z_h}^{z}\frac{d\xi}{\mathfrak{s}(\xi)\,\mathfrak{G}(\xi)}\int_{z_h}^{\xi}\frac{\mathfrak{s}(\eta)}{\mathfrak{B}^2(\eta)}\,\frac{f_{(1)}(\phi(\eta))}{\mathfrak{g}_2(\eta)\mathfrak{g}_3(\eta)}\,d\eta + \right.$$
$$\left. + \left(\frac{\mu}{I(z_h)}\right)^2\int_{z_h}^{z}\frac{d\xi}{\mathfrak{s}(\xi)\,\mathfrak{G}(\xi)}\int_{z_h}^{\xi}\frac{\mathfrak{B}^2(\eta)}{\mathfrak{s}(\eta)\,f_{(0)}(\phi(\eta))}\,d\eta\right]$$

Solution in Quadratures for Arbitrary Model

Arbitrary Model:
D is arbitrary, $\phi(z)$ is always present
only one electric $F_{(0)}$ is possible
any magnetic $F_{(\mathcal{M})}$ and $H_{(\mathcal{N})}$ are possible

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Case I: EOM with no RHS $$\sum_{\mu=0}^{D-1} k_\mu \frac{G_{\mu\mu}}{g_{\mu\mu}} = [G_{\mu\nu} = T_{\mu\nu}] = \sum_{\mu=0}^{D-1} k_\mu \frac{T_{\mu\mu}}{g_{\mu\mu}} = 0, \quad k_\mu \in \mathbb{R}$$

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$$\sum_{\mu=0}^{D-1} k_\mu \frac{T_{\mu\mu}}{g_{\mu\mu}} = \frac{V(\phi)}{2}\left(-\sum_{\mu=0}^{D-1} k_\mu\right) + \frac{\phi'^2}{4\,g_{zz}}\left(k_{D-1} - \sum_{\mu=0}^{D-2} k_\mu\right) +$$

$$+ \frac{f_{(0)}(\phi)\,A_t'^2(z)}{4\,g_{00}\,g_{zz}}\left(k_0 + k_z - \sum_{\mu=1}^{D-2} k_\mu\right) + \frac{f_{(\mathcal{M})}(\phi)\,q^2_{(\mathcal{M})}}{4\,g_{ii}\,g_{jj}}\left(k_i + k_j - \sum_{\mu=0,\,\mu\neq i,j}^{D-1} k_\mu\right)$$

$$+ \frac{h_{(\mathcal{N})}(\phi)\,Q^2_{(\mathcal{N})}}{4\,g_{mm}\,g_{nn}\,g_{pp}}\left(k_m + k_n + k_p - \sum_{\mu=0,\,\mu\neq m,n,p}^{D-1} k_\mu\right)$$

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$$\Downarrow$$

$$\phi(z): \qquad k_{D-1} \equiv k_z = 0, \qquad \sum_{\mu=0}^{D-2} k_\mu = 0$$

$$[\phi(z)+]\ \ F_{(0)} = A_t'(z)\ dt \wedge dz: \qquad k_0 + k_{D-1} = 0 \to k_0 = k_{D-1} \equiv k_z = 0$$

$$[\phi(z)+]\ \ F_{(\mathcal{M})} = q_{(\mathcal{M})}\ dx^i \wedge dx^j: \qquad k_i + k_j = 0$$

$$[\phi(z)+]\ \ H_{(\mathcal{N})} = Q_{(\mathcal{N})}\ dx^m \wedge dx^n \wedge dx^p: \qquad k_m + k_n + k_p = 0$$

Solution in Quadratures for Arbitrary Model

$\phi(z):\quad k_{D-1} \equiv k_z = 0,\quad \sum_{\mu=0}^{D-2} k_\mu = 0$	$[\boldsymbol{\alpha},\, \boldsymbol{\beta} \neq \boldsymbol{z}]:\quad \frac{G_{\alpha\alpha}}{g_{\alpha\alpha}} - \frac{G_{\beta\beta}}{g_{\beta\beta}} = \frac{1}{2\sqrt{g_D}}\left(\frac{\sqrt{g_D}}{g_{zz}}\frac{d}{dz}\ln\frac{g_{\beta\beta}}{g_{\alpha\alpha}}\right)'$

Solution in Quadratures for Arbitrary Model

$$\phi(z): \quad k_{D-1} \equiv k_z = 0, \quad \sum_{\mu=0}^{D-2} k_\mu = 0 \quad \Big| \quad [\boldsymbol{\alpha}, \boldsymbol{\beta} \neq \boldsymbol{z}]: \quad \frac{G_{\alpha\alpha}}{g_{\alpha\alpha}} - \frac{G_{\beta\beta}}{g_{\beta\beta}} = \frac{1}{2\sqrt{g_D}} \left(\frac{\sqrt{g_D}}{g_{zz}} \frac{d}{dz} \ln \frac{g_{\beta\beta}}{g_{\alpha\alpha}} \right)'$$

Step 1: simplify

$$k_{D-1} \equiv k_z = 0 \quad \Longrightarrow \quad \sum_{\mu=0}^{D-1} k_\mu \frac{G_{\mu\mu}}{g_{\mu\mu}} \equiv k_0 \frac{G_{00}}{g_{00}} + \sum_{i=0}^{D-2} k_i \frac{G_{x_i x_i}}{g_{x_i x_i}} + \cancel{k_z \frac{G_{zz}}{g_{zz}}} = 0$$

Solution in Quadratures for Arbitrary Model

$$\phi(z): \quad k_{D-1} \equiv k_z = 0, \quad \sum_{\mu=0}^{D-2} k_\mu = 0 \quad \Bigg| \quad [\boldsymbol{\alpha}, \boldsymbol{\beta} \neq \boldsymbol{z}]: \quad \frac{G_{\alpha\alpha}}{g_{\alpha\alpha}} - \frac{G_{\beta\beta}}{g_{\beta\beta}} = \frac{1}{2\sqrt{g_D}} \left(\frac{\sqrt{g_D}}{g_{zz}} \frac{d}{dz} \ln \frac{g_{\beta\beta}}{g_{\alpha\alpha}} \right)'$$

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Step 2: observe

$$\sum_{\mu=0}^{D-2} k_\mu = 0, \quad k_\mu \in \mathbb{Z} \quad \Longrightarrow \quad \sum_{k_\mu > 0} |k_\mu| = \sum_{k_\nu < 0} |k_\nu| \Longrightarrow \sum_{k_\mu > 0} |k_\mu| \frac{G_{\mu\mu}}{g_{\mu\mu}} = \sum_{k_\nu < 0} |k_\nu| \frac{G_{\nu\nu}}{g_{\nu\nu}}$$

the same number of $G_{\rho\rho}/g_{\rho\rho}$

$$[\boldsymbol{\alpha_{1,2}}, \boldsymbol{\beta_{1,2}} \neq \boldsymbol{z}]: \quad \left(\frac{G_{\alpha_1\alpha_1}}{g_{\alpha_1\alpha_1}} - \frac{G_{\beta_1\beta_1}}{g_{\beta_1\beta_1}} \right) + \left(\frac{G_{\alpha_2\alpha_2}}{g_{\alpha_2\alpha_2}} - \frac{G_{\beta_2\beta_2}}{g_{\beta_2\beta_2}} \right) =$$

$$= \frac{1}{2\sqrt{g_D}} \left(\frac{\sqrt{g_D}}{g_{zz}} \frac{d}{dz} \ln \frac{g_{\beta_1\beta_1} g_{\beta_2\beta_2}}{g_{\alpha_1\alpha_1} g_{\alpha_2\alpha_2}} \right)'$$

Solution in Quadratures for Arbitrary Model

$$\phi(z): \quad k_{D-1} \equiv k_z = 0, \quad \sum_{\mu=0}^{D-2} k_\mu = 0 \qquad [\boldsymbol{\alpha}, \boldsymbol{\beta} \neq \boldsymbol{z}]: \quad \frac{G_{\alpha\alpha}}{g_{\alpha\alpha}} - \frac{G_{\beta\beta}}{g_{\beta\beta}} = \frac{1}{2\sqrt{g_D}} \left(\frac{\sqrt{g_D}}{g_{zz}} \frac{d}{dz} \ln \frac{g_{\beta\beta}}{g_{\alpha\alpha}} \right)'$$

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$$k_{D-1} \equiv k_z = 0 \quad \Longrightarrow \quad \sum_{\mu=0}^{D-1} k_\mu \frac{G_{\mu\mu}}{g_{\mu\mu}} \equiv k_0 \frac{G_{00}}{g_{00}} + \sum_{i=0}^{D-2} k_i \frac{G_{x_i x_i}}{g_{x_i x_i}} + \cancel{k_z \frac{G_{zz}}{g_{zz}}} = 0$$

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the same number of $G_{\rho\rho}/g_{\rho\rho}$

Step 3: apply formula

$$\sum_{\mu=0}^{D-1} k_\mu \frac{G_{\mu\mu}}{g_{\mu\mu}} = \frac{1}{2\sqrt{g_D}} \left(\frac{\sqrt{g_D}}{g_{zz}} \frac{d}{dz} \ln \frac{\prod_{k_\nu < 0} g_{\nu\nu}^{|k_\nu|}}{\prod_{k_\mu > 0} g_{\mu\mu}^{|k_\mu|}} \right)' = 0$$

Step 4: further simplify

$$\left(g\, \mathfrak{s}_D \frac{d}{dz} \ln \left(g^{k_0} \prod_{\rho=1}^{D-2} \mathfrak{g}_\rho^{k_\rho} \right) \right)' = 0$$

Solution in Quadratures for Arbitrary Model

EOM with no RHS: $$\sum_{\mu=0}^{D-1} k_\mu \frac{G_{\mu\mu}}{g_{\mu\mu}} = 0 \quad \Longrightarrow \quad \left(g\,\mathfrak{s}_D \frac{d}{dz} \ln\left(g^{k_0} \prod_{\rho=1}^{D-2} \mathfrak{g}_\rho^{k_\rho} \right) \right)' = 0$$

- $k_0 = 0$, or $\dfrac{G_{tt}}{g_{tt}}$ is absent:

1st–order: $$\left(g\,\mathfrak{s}_D \frac{d}{dz} \ln\left(\prod_{\rho=1}^{D-2} \mathfrak{g}_\rho^{k_\rho} \right) \right)' = 0 \quad \Longrightarrow \quad g(z) = \frac{C}{\mathfrak{s}_D(z) \dfrac{d}{dz} \ln \prod_{\rho=1}^{D-2} \mathfrak{g}_\rho^{k_\rho}(z)}$$

Solution in Quadratures for Arbitrary Model

EOM with no RHS: $\displaystyle\sum_{\mu=0}^{D-1} k_\mu \frac{G_{\mu\mu}}{g_{\mu\mu}} = 0 \implies \left(g\, \mathfrak{s}_D \frac{d}{dz} \ln \left(g^{k_0} \prod_{\rho=1}^{D-2} \mathfrak{g}_\rho^{k_\rho} \right) \right)' = 0$

- $k_0 = 0$, or $\dfrac{G_{tt}}{g_{tt}}$ is absent:

$$1^{\text{st}}\text{–order:} \quad \left(g\, \mathfrak{s}_D \frac{d}{dz} \ln \left(\prod_{\rho=1}^{D-2} \mathfrak{g}_\rho^{k_\rho} \right) \right)' = 0 \implies g(z) = \frac{C}{\mathfrak{s}_D(z) \dfrac{d}{dz} \ln \displaystyle\prod_{\rho=1}^{D-2} \mathfrak{g}_\rho^{k_\rho}(z)}$$

- $k_0 \neq 0$, or $\dfrac{G_{tt}}{g_{tt}}$ is present:

$$2^{\text{nd}}\text{–order:} \quad k_0 \cdot \left(g\, \mathfrak{s}_D \frac{d}{dz} \ln \left(g \prod_{\rho=1}^{D-2} \mathfrak{g}_\rho^{k_\rho/k_0} \right) \right)' = 0$$

$$\mathfrak{G}(z) = \prod_{\rho=1}^{D-2} \mathfrak{g}_\rho^{-k_\rho/k_0}(z) \implies g(z) = \mathfrak{G}(z) \left[C_1 \int_{z_0}^{z} \frac{d\xi}{\mathfrak{s}_D(\xi)\mathfrak{G}(\xi)} + C_2 \right]$$

Remark: 2^{nd}–order EOM without the RHS is impossible when $F_{(0)}$ is present

$$F_{(0)} = A_t'(z)\; dt \wedge dz : \quad k_0 = k_{D-1} \equiv k_z = 0$$

Solution in Quadratures for Arbitrary Model

Case II: EOM with the RHS $\sum_{\mu=0}^{D-1} k_\mu \frac{G_{\mu\mu}}{g_{\mu\mu}} = [G_{\mu\nu} = T_{\mu\nu}] = \sum_{\mu=0}^{D-1} k_\mu \frac{T_{\mu\mu}}{g_{\mu\mu}}, \quad k_\mu \in \mathbb{R}$

Step 1: force $\phi(z)$ to not contribute

$$\sum_{\mu=0}^{D-1} k_\mu \frac{T_{\mu\mu}}{g_{\mu\mu}} = \frac{V(\phi)}{2}\left(-\sum_{\mu=0}^{D-1} k_\mu\right) + \frac{\phi'^2}{4\,g_{zz}}\left(k_{D-1} - \sum_{\mu=0}^{D-2} k_\mu\right) + \cdots$$

$$\phi(z): \qquad k_{D-1} = 0, \qquad \sum_{\mu=0}^{D-2} k_\mu = 0$$

Solution in Quadratures for Arbitrary Model

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$$\phi(z): \qquad k_{D-1} = 0, \qquad \sum_{\mu=0}^{D-2} k_\mu = 0$$

Step 2: apply the formula

$$-\frac{k_0}{2\,L^2}\,\frac{z^2}{\mathfrak{b}\,\mathfrak{s}_D}\,\frac{d}{dz}\left(\mathfrak{s}_D\,\mathfrak{G}\,\frac{d}{dz}\frac{g}{\mathfrak{G}}\right) = \sum_{\mu=0}^{D-2} k_\mu \frac{T_{\mu\mu}}{g_{\mu\mu}}, \qquad \mathfrak{G}(z) = \prod_{\rho=1}^{D-2} \mathfrak{g}_\rho^{-k_\rho/k_0}(z)$$

Step 3: solve without the RHS

$$g(z) = \mathfrak{G}(z)\,\left[C_1 \int_{z_0}^{z} \frac{d\xi}{\mathfrak{s}_D(\xi)\mathfrak{G}(\xi)} + C_2\right]$$

Solution in Quadratures for Arbitrary Model

Case II: EOM with the RHS $\displaystyle\sum_{\mu=0}^{D-1} k_\mu \frac{G_{\mu\mu}}{g_{\mu\mu}} = [G_{\mu\nu} = T_{\mu\nu}] = \sum_{\mu=0}^{D-1} k_\mu \frac{T_{\mu\mu}}{g_{\mu\mu}}, \quad k_\mu \in \mathbb{R}$

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Step 3: solve without the RHS

$$g(z) = \mathfrak{G}(z)\,\left[C_1 \int_{z_0}^{z} \frac{d\xi}{\mathfrak{s}_D(\xi)\mathfrak{G}(\xi)} + C_2\right]$$

Step 4: vary the parameters C_1, C_2

On the RHS: $\displaystyle\frac{z^4}{L^4}\frac{f_{(0)}(\phi)A_t'^2}{4\,\mathfrak{b}^2}, \quad \frac{f_{(\mathcal{M})}(\phi)q^2_{(\mathcal{M})}}{4g_{ii}g_{jj}}, \quad \frac{h_{(\mathcal{N})}(\phi)Q^2_{(\mathcal{N})}}{4g_{pp}g_{kk}g_{ll}}$

Quadratures: Result

All models, regardless of D and the collection of 2- and 3-form fields, **are solved by quadratures**, meaning

1. $g(z)$ is expressed in terms of integrals of the other metric functions and *some* of the coupling functions (after solving for $A_t(z)$):

$$g(z) = \text{Function}\Big(\mathfrak{b}(z),\ \mathfrak{g}_i(z),\ f_p(\phi(z)),\ h_r(\phi(z))\Big), \quad i = \overline{1,\ D-2},$$

where the procedure yields an **explicit** expression for Function$(\cdots)$;

2. all the other functions — $\phi(z),\ V(\phi(z)),\ f_q(\phi(z)),\ h_s(\phi(z))$ — are then found as functions of z $(q \neq p,\ s \neq r)$.

Remark: $\phi(z) = \pm \int \sqrt{2g_{zz}\left(\dfrac{G_{zz}}{g_{zz}} + \dfrac{G_{tt}}{g_{tt}}\right)}\, dz$

Null Energy Condition

$$T_{\mu\nu}\,\xi^\mu\xi^\nu \geq 0, \quad \forall\,\xi^\mu\xi_\mu = 0 \quad \Longrightarrow \quad \forall\mu: \quad R^\mu_\mu - R^0_0 = \frac{G_{\mu\mu}}{g_{\mu\mu}} - \frac{G_{00}}{g_{00}} \geq 0$$

Talk by K. Rannu

$$\textit{Model I} \quad D=5, \quad \phi=\phi(z), \quad F_{(1)} = q_{(1)}\,dx^2\wedge dx^3, \quad F_{(3)} = q_{(3)}\,dx^1\wedge dx^2$$

$$ds^2 = \frac{L^2}{z^2}\left(-g(z)dt^2 + dx_1^2 + \left(\frac{z}{L}\right)^{2-\frac{2}{\nu}} dx_2^2 + e^{c_B z^2}\left(\frac{z}{L}\right)^{2-\frac{2}{\nu}} dx_3^2 + \frac{dz^2}{g(z)}\right)$$

I. Y. Aref'eva, A. Hajilou, K. Rannu, and P. Slepov, [arXiv:2305.06345 [hep-th]]

Lifshitz + Gauss anisotropies

Null Energy Condition

$$T_{\mu\nu}\,\xi^\mu\xi^\nu \geq 0, \quad \forall\,\xi^\mu\xi_\mu = 0 \quad \Longrightarrow \quad \forall\mu: \quad R^\mu_\mu - R^0_0 = \frac{G_{\mu\mu}}{g_{\mu\mu}} - \frac{G_{00}}{g_{00}} \geq 0$$

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I. Y. Aref'eva, A. Hajilou, K. Rannu, and P. Slepov, [arXiv:2305.06345 [hep-th]]

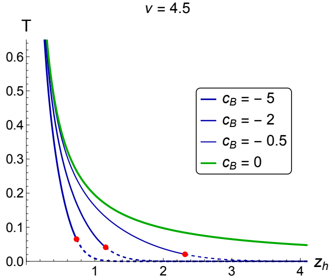
Lifshitz + Gauss anisotropies

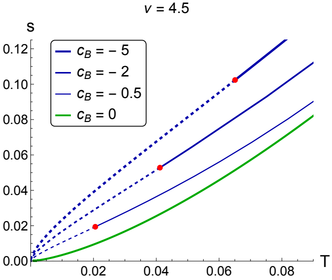
$$\text{NEC:} \qquad \begin{cases} \phi'^2 \geq 0 \\ f_{(1)}(\phi) \geq 0 \\ f_{(3)}(\phi) \geq 0 \end{cases} \iff \begin{cases} c_B \leq 0 \\ \nu \geq 1 \\ z_h \leq z_h^{max}(\nu, c_B) \text{ for } c_B < 0 \end{cases}$$

Null Energy Condition & Third Law of Thermodynamics

$$z_h \leq z_h^{max}(\nu, c_B) \text{ for } c_B < 0 \quad \Longrightarrow \quad T \geq T_{min} = T(z_h^{max})$$

$T(z_h)$ and $s(T)$ for various c_B, $[T] = [z_h]^{-1} = [c_B]^{1/2} = \mathrm{GeV}$
Red points — T_{min}. Dashed lines — regimes, forbidden by NEC





Third Law in Planck form $\quad S \to 0 \quad$ as $\quad T \to 0$

Construction of Solutions

$$\textit{Model II} \quad D=5, \quad \phi=\phi(z), \quad F_{(2)}=q_{(2)}\,dx^1\wedge dx^3, \quad F_{(3)}=q_{(3)}\,dx^1\wedge dx^2$$

$$ds^2=\frac{L^2}{z^2}\,\mathfrak{b}(z)\left(-\,g(z)dt^2+dx_1^2+\left(\frac{z}{L}\right)^{2-\frac{2}{\nu_1}}dx_2^2+\left(\frac{z}{L}\right)^{2-\frac{2}{\nu_2}}dx_3^2+\frac{dz^2}{g(z)}\right)$$

Lifshitz $+$ Lifshitz anisotropies

For arbitrary $\mathfrak{b}(z)$:

$$g(z)=C\,z^{4-\frac{2}{\nu_1}-\frac{2}{\nu_2}}\int_z^{z_h}\mathfrak{b}(\xi)^{-\frac{3}{2}}\,\xi^{-3+\frac{3}{\nu_1}+\frac{3}{\nu_2}}\,d\xi$$

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For arbitrary $\mathfrak{b}(z)$:

$$g(z)=C\,z^{4-\frac{2}{\nu_1}-\frac{2}{\nu_2}}\int_z^{z_h}\mathfrak{b}(\xi)^{-\frac{3}{2}}\xi^{-3+\frac{3}{\nu_1}+\frac{3}{\nu_2}}\,d\xi$$

For $\mathfrak{b}(z)=e^{cz^n}, \;\; n>0, \;\; c\in\mathbb{R}$:

$$g(z)=\begin{cases}1-\dfrac{I(z)}{I(z_h)}, \text{ where } I(z)=\displaystyle\int_0^z\xi^3\exp\left(-\frac{3}{2}c\xi^n\right)d\xi, & \dfrac{1}{\nu_1}+\dfrac{1}{\nu_2}=2\\ 8z^8\displaystyle\int_z^{z_h}\xi^{-9}\exp\left(-\frac{3}{2}c\xi^n\right)d\xi, & \dfrac{1}{\nu_1}+\dfrac{1}{\nu_2}=-2\end{cases}$$

Remark: $\displaystyle\int_z^{z_h}\xi^{-9}\exp\left(-\frac{3}{2}c\xi^n\right)d\xi\to+\infty$ as $z\to+0$

Conlusions

- Class of models considered leads to a pattern in $T_{\mu\mu}/g_{\mu\mu}$
- Obtained formula for diagonal metric $g_{\mu\nu}(z)$
- Procedure developed to solve for $g(z)$ in quadratures
  - demonstrated using two examples, including four-Maxwell
  - **extended to arbitrary D and $F_{(0)}$, $F_{(\mathcal{M})}$, $H_{(\mathcal{N})}$**
- Observations made can be applied to **ease NEC consideration**
- Procedure helps **systematically derive all solutions** while symbolic solvers may fail

Thank you for your attention!