Inhomogeneous phases in QCD phase diagram

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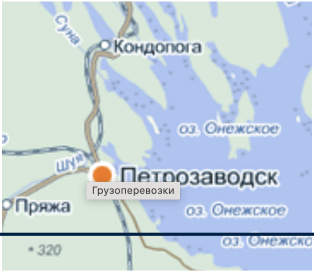
The 23d International Conferences on High-Energy Physics

Kondopoga – the place I was born

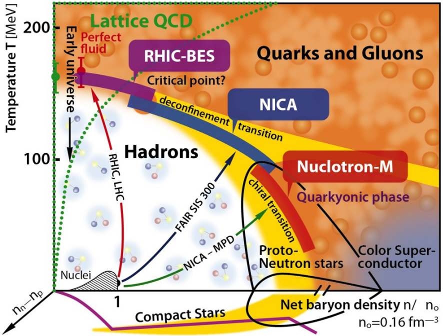


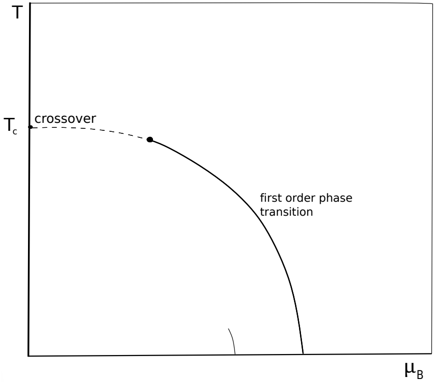
Group and Acknowlegements

K.G. Klimenko, IHEP

A. Kozhakin

QCD Phase Diagram





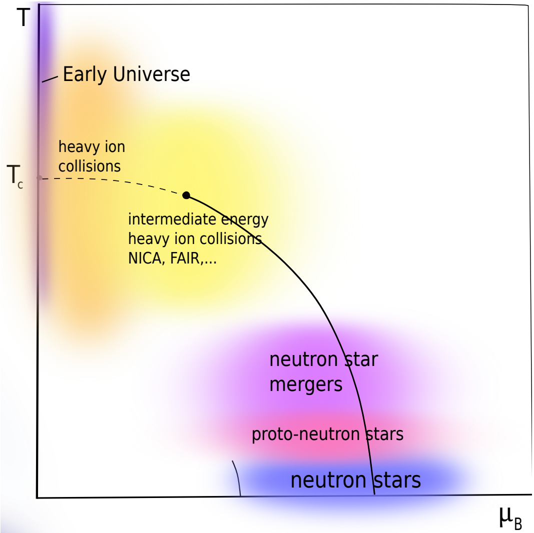
Two main phase transitions

- confinement-deconfinement
- chiral symmetry breaking phase—chriral symmetric phase

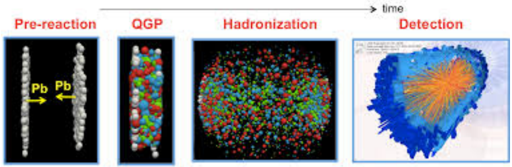
QCD Dhase Diagram

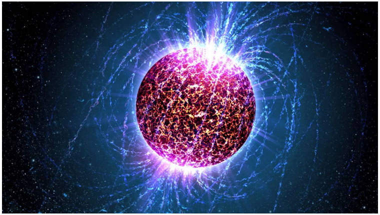
QCD at T and μ
(QCD at extreme conditions)

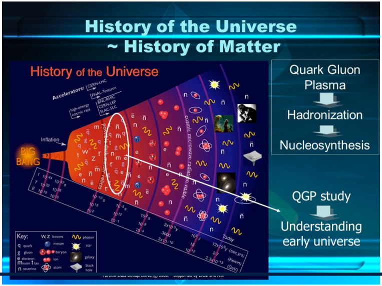
- Early Universe
- heavy ion collisions
- neutron stars
- proto- neutron stars
- neutron star mergers



QCD at extreme conditions







History of the Universe
~ History of Matter

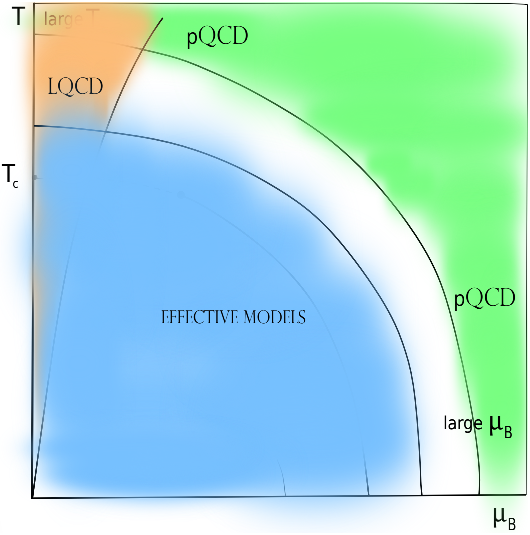
Quark Gluon Plasma
Hadronization
Nucleosynthesis

QGP study
Understanding early universe

QCD Dhase Diagram and Methods

Methods of dealing with QCD

- Perturbative QCD
- First principle calculation – lattice QCD
- Effective models
- DSE, FRG
- .....



Nambu–Jona-Lasinio (NJL) type model

Lagrangian of GN model

$$\mathcal{L} = i\bar{\psi}\partial\!\!\!/\psi + \frac{G}{2N_c}(\bar{\psi}\psi)^2 \qquad \psi = (\psi_1, \ldots, \psi_{N_c})$$

Discrete chiral symmetry: $\psi \to \gamma_5\psi, \quad \bar{\psi}\psi \to -\bar{\psi}\psi$

Auxiliary Lagrangian

$$\widetilde{\mathcal{L}} = \bar{\psi}\Big[\gamma^\rho \mathrm{i}\partial_\rho - \sigma - \mathrm{i}\gamma^5\pi\Big]\psi - \frac{N_c}{4G}\sigma^2$$

Generating functional

$$Z = \int \mathcal{D}\bar{\psi}\mathcal{D}\psi\mathcal{D}\sigma\, e^{\int dx \widetilde{\mathcal{L}}(\bar{\psi},\psi,\sigma)}$$

Chiral symmetry breaking in NJL model

$$\sigma(x) = \bar{\sigma} + \delta\sigma(x), \qquad\qquad \sigma(x) = \langle\sigma\rangle + \delta\sigma(x)$$

In the leading order of $1/N_c$ expansion or in mean field one can omit the fluctuations $\delta\sigma(x)$

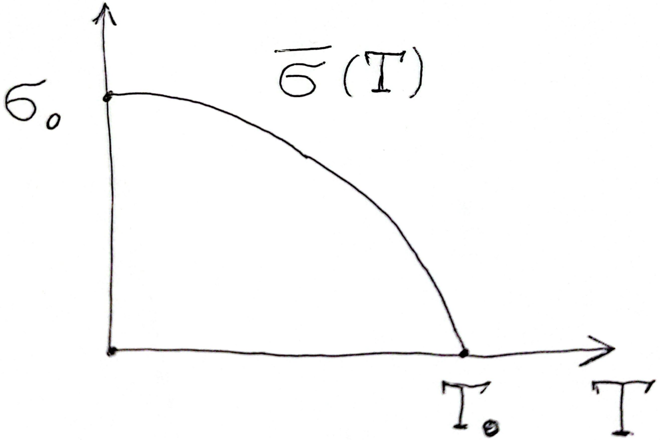
$$Z = \int \mathcal{D}\bar{\psi}\mathcal{D}\psi \, exp\Big\{\int dx \widetilde{\mathcal{L}}(\bar{\psi}, \psi, \bar{\sigma})\Big\}$$

Chiral symmetry breaking

$$\langle\bar{\psi}\psi\rangle \neq 0, \qquad\qquad \langle\sigma\rangle \sim \langle\bar{\psi}\psi\rangle \neq 0$$

$$\langle\sigma\rangle \neq 0 \quad \longrightarrow \quad \widetilde{\mathcal{L}} = \bar{\psi}\Big[\gamma^\rho \mathrm{i}\partial_\rho - \langle\sigma\rangle\Big]\psi$$

Order parameter

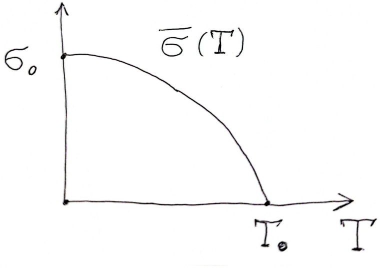


Order parameter $\bar{\sigma} = \bar{\sigma}(T, \mu)$

Inhomogeneous phase

Order parameter $\bar{\sigma}$

$$\bar{\sigma} = \bar{\sigma}(T, \mu)$$



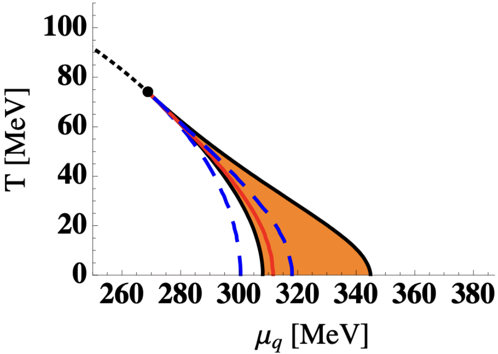
$$\bar{\sigma} = \bar{\sigma}(\vec{x})$$

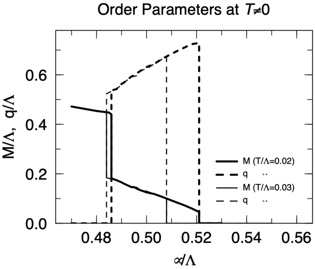
$$\bar{\sigma} = \bar{\sigma}(\vec{x}\,|\,T, \mu)$$

Dual chiral density wave in quark matter

Chiral Density Wave (CDW) or **Chiral spiral ansatz**

$$\langle \sigma \rangle = M \cos(bx) \qquad \langle \pi \rangle = M \sin(bx)$$





E. Nakano, T. Tatsumi, Phys.Rev.D 71 (2005) 114006 arXiv:hep-ph/0411350

D. Nickel, Phys.Rev.D 80 (2009) 074025 arXiv:0906.5295 [hep-ph]

Two dimensional inhomogeneous phases

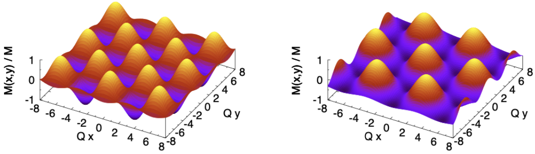


FIG. 1. Mass functions $M(x, y)$ with two-dimensional modulations in coordinate space. Left: "egg-carton" modulation on a square lattice, Eq. (12). Right: hexagonal modulation, Eq. (13).

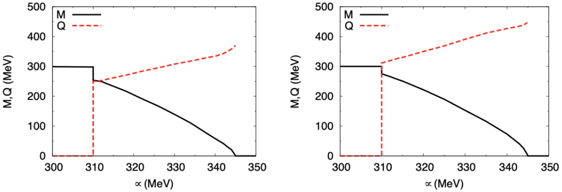
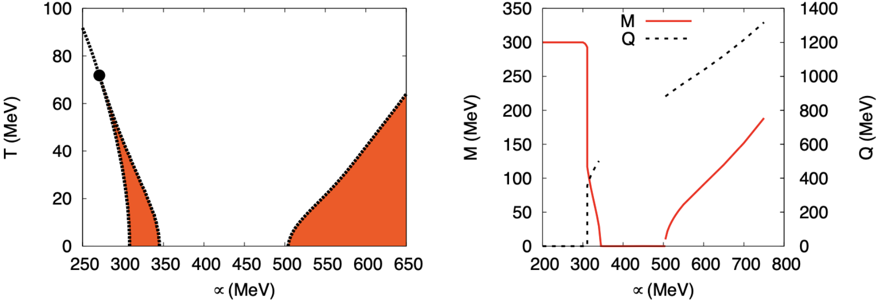


FIG. 2. Amplitude M and wave number Q at $T = 0$ as functions of the chemical potential μ after minimization of the thermodynamic potential for given shapes of the mass function. Left: "egg-carton" modulation on a square lattice, Eq. (12). Right: hexagonal ansatz, Eq. (13).

Phys.Rev.D 86 (2012) 074018

Inhomogeneous islands and continents



S. Carignano, M. Buballa, Acta Phys.Polon.Supp. 5 (2012) 641-658, arXiv:1111.4400 [hep-ph]

Inhomogeneous islands and continents in the Nambu–Jona-Lasinio model

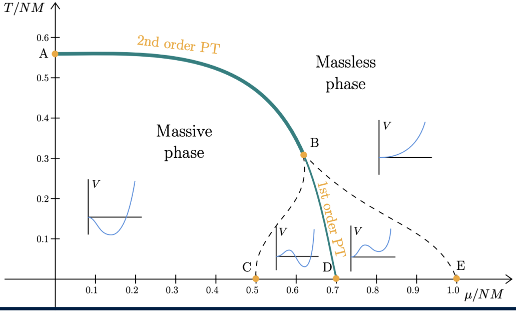
(1+1)-dim GN model

Lagrangian of (1+1)-dim GN model

$$\mathcal{L} = \bar{\psi}\left(\gamma^{\nu}\mathrm{i}\partial_{\nu} + \mu\gamma^{0}\right)\psi + \frac{G}{N_c}(\bar{\psi}\psi)^2$$

Homogeneous condensate

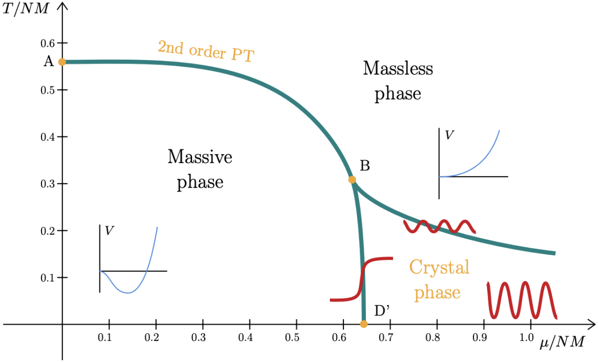
$$\langle\sigma\rangle = M = \mathrm{const}$$



(1+1)-dim GN model

For example cosine ansatz

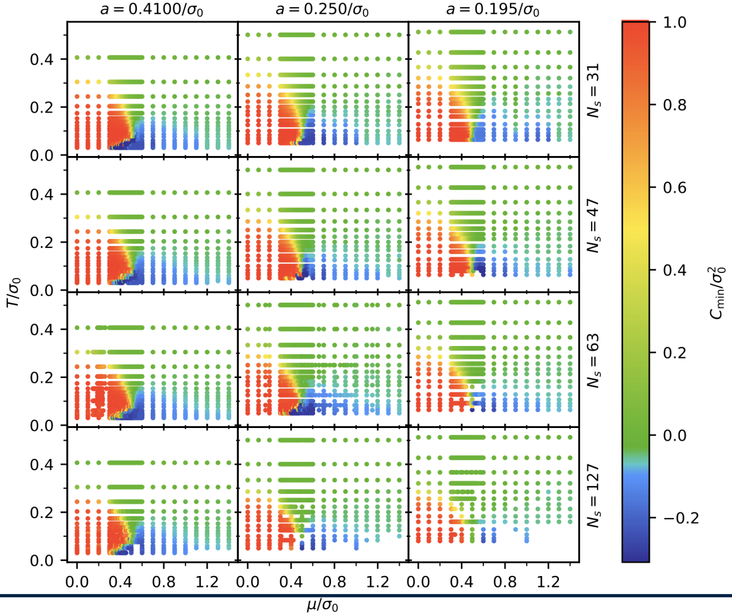
$$\langle\sigma\rangle = M\cos(bx) \qquad \sigma(x) = M\,\mathrm{sn}(Mx|\nu)$$



(μ, T)-phase diagram of GN model with possibility of inhomogeneos case

O. Schnetz, M. Thies, K. Urlichs 2004, Annals Phys. 314 (2004) 425-447

Phase diagram for $N_f = 8$: C_{min}



Large N limit and finite N

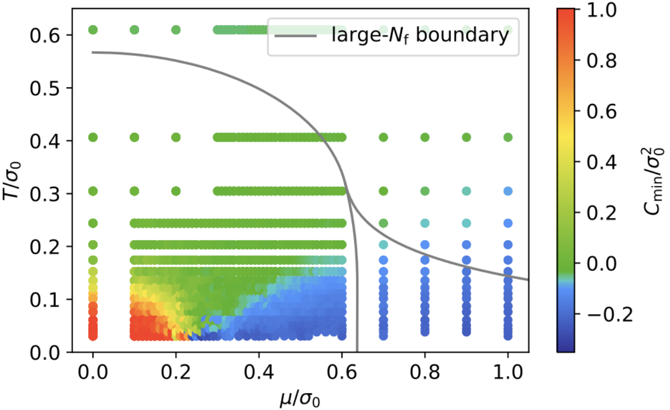
As a rule one employs **large N limit** to suppress bosonic fluctuations

they can easily **destroy** the condensation and **inhomogeneous phases**

It is extremely hard to get **full picture** with bosonic fluctuations (at finite N without mean field approximation)

It is **possible** and was done on **lattice** in (1+1)-dim GN model

Phase diagram for $N_f = 2$



the phase diagram from C_{min} for $N_f = 2$

Unexpected feature is that the temperature range with negative C_{min} grows with increasing μ, which is qualitatively different from the situation at large N_f

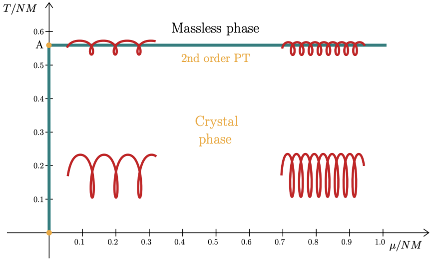
(1+1)-dim NJL model (chiral GN model)

Lagrangian of (1+1)-dim NJL model

$$\mathcal{L} = \bar{\psi}\left(\gamma^{\nu}\mathrm{i}\partial_{\nu} + \mu\gamma^{0}\right)\psi + \frac{G}{N_c}\left[(\bar{\psi}\psi)^2 + (\bar{\psi}\mathrm{i}\gamma^5\psi)^2\right]$$

Chiral spiral ansatz

$$\langle\sigma\rangle = M\cos(bx) \qquad \langle\pi\rangle = M\sin(bx)$$



Inhomogeneous phases in Chiral Gross-Neveu Model

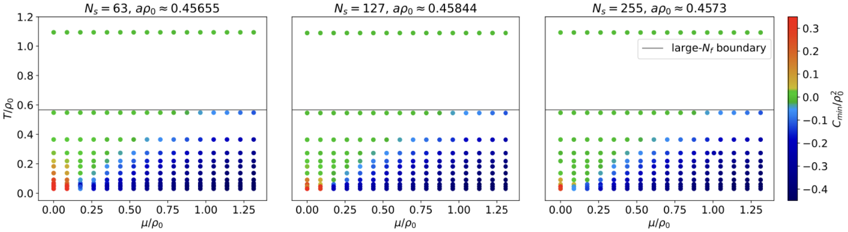
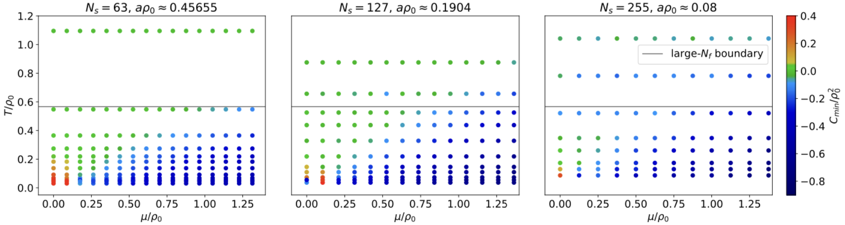


FIG. 9: Infinite-volume extrapolation: phase diagrams for fixed lattice spacing and $N_s = 63, 127$ and 255.



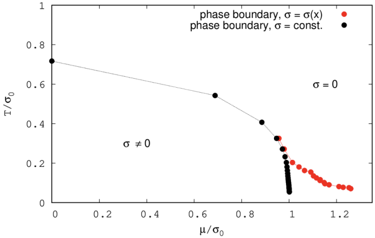
$N_f = 2$ **result**

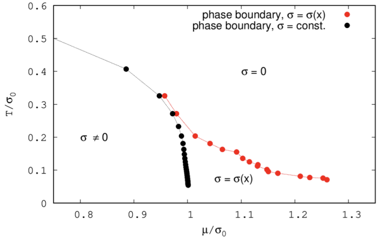
Inhomogeneous phases in (2+1)-dimensions

No papers on inhomogeneous phases in (2+1)-dim models such as GN model

A plethora of studies of inhomogeneous phases in (1+1) and (3+1) dimensional models

First results in (2+1)-dim GN Model

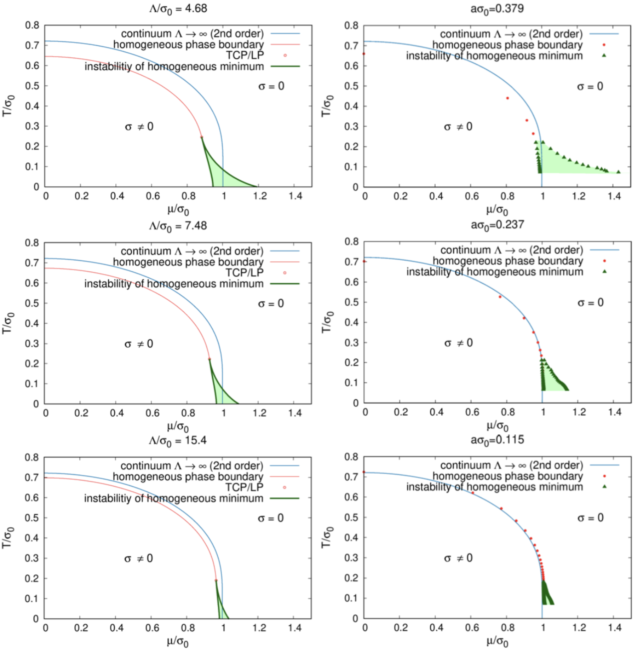




M. Winstel, J. Stoll, M. Wagner, J.Phys.Conf.Ser. 1667 (2020) 1, 012044, arXiv:1909.00064 [hep-lat]

Lattice investigation of an inhomogeneous phase of the 2+1-dimensional

Gross-Neveu model in the limit of infinitely many flavors


Λ/σ$_0$ = 4.68
aσ$_0$=0.379
Λ/σ$_0$ = 7.48
aσ$_0$=0.237
Λ/σ$_0$ = 15.4
aσ$_0$=0.115
continuum Λ → ∞ (2nd order)
homogeneous phase boundary
TCP/LP
instabilitiy of homogeneous minimum
instability of homogeneous minimum
σ = 0
σ ≠ 0
T/σ$_0$
μ/σ$_0$
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.2
0.4
0.6
0.8
1
1.2
1.4

Inhomogeneous phases in (2+1)-dim GN Model

- Call for a critical revision of the role of the regularization.

- For instance, inhomogeneous phases have also been found in the (3+1)-dim NJL model. Unlike in (2+1) dimensions, (3+1)-dim NJL model is non-renormalizable and therefore the studied with finite regulators.

- Given that inhomogeneous phases exist in the renormalized (1+1)-dim GN and NJL models, while in the (2+1)-dim GN model they are only present at finite regulator values, one might suspect that the observed inhomogeneous phases at (3+1) dimensions could be regularization artifacts.

NJL model with chiral imbalance in (2+1) dimensions

$$\psi_n \to \gamma_4 \psi_n, \quad \bar{\psi}_n \to -\bar{\psi}_n \gamma_4, \tag{11}$$

$$\psi_n \to \gamma_5 \psi_n, \quad \bar{\psi}_n \to -\bar{\psi}_n \gamma_5 \tag{12}$$

with

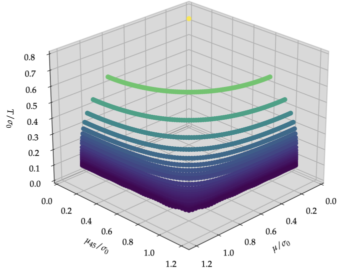
$$\gamma_{45} = \mathrm{i}\gamma_4\gamma_5 = \tau_3 \otimes \mathbb{1}_2 = \begin{pmatrix} +\mathbb{1}_2 & 0 \\ 0 & -\mathbb{1}_2 \end{pmatrix}$$

$$\gamma_4 = \tau_1 \otimes \mathbb{1}_2 = \begin{pmatrix} 0 & +\mathbb{1}_2 \\ +\mathbb{1}_2 & 0 \end{pmatrix}, \quad \gamma_5 = -\tau_2 \otimes \mathbb{1}_2 = \begin{pmatrix} 0 & +\mathrm{i}\mathbb{1}_2 \\ -\mathrm{i}\mathbb{1}_2 & 0 \end{pmatrix}. \tag{13}$$

Both γ_4 and γ_5 anticommute with γ_0, γ_1 and γ_2, thus fulfilling the necessary properties

$$Q[\mu,\sigma] = Q^{(4)}[\mu,\sigma] \to Q[\mu,\mu_{45},\sigma] = Q^{(4)}[\mu,\mu_{45},\sigma] = \gamma_\nu \partial_\nu + \gamma_0 \mu + \gamma_{45}\gamma_0\mu_{45} + \sigma$$

$$= \begin{pmatrix} Q^{(2)}[\mu+\mu_{45},\sigma] & 0 \\ 0 & \tilde{Q}^{(2)}[\mu-\mu_{45},\sigma] \end{pmatrix}.$$



L. Pannullo, M. Wagner, and M. Winstel, Symmetry 14, 265 (2022), arXiv:2112.11183 [hep-lat]

Inhomogeneous Phases in the Chirally Imbalanced (2+1)-Dimensional Gross- Neveu Model and Their Absence in the Continuum Limit

NJL model with chiral imbalance in (2+1) dimensions

- For (2+1)-dimensional GN model chiral imbalance μ_{45}, it was shown that an **isospin chemical potential** μ_I is **equivalent to the chiral chemical potential** μ_{45}

- Thus, all results can be interpreted in the context of chiral imbalance or of isospin imbalance. In particular the μ-μ_I-T **phase diagram** was **identical to the** μ-μ_{45}-T **phase diagram**

- μ_{45} seems to **disfavor inhomogeneous modulations**

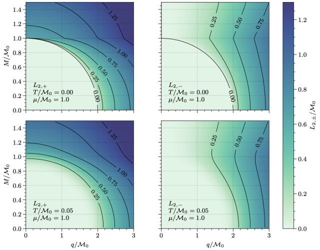
- The inhomogeneous phase **shrank** for decreasing lattice spacing and was expected to **disappear in the continuum limit**

Symmetry 14, 265 (2022)

NJL model with various terms in (2+1) dimensions

$$\mathcal{S}_{\mathrm{FF}}[\bar{\psi},\psi] = \int_0^{\beta} \mathrm{d}\tau \int \mathrm{d}^2 x \left\{ \bar{\psi}\left(\partial\!\!\!/ + \gamma_3 \mu\right)\psi - \left[\sum_{j=1}^{16} \frac{\lambda_j}{2N}\left(\bar{\psi}\, c_j\, \psi\right)^2\right]\right\},$$

$$C = (c_j)_{j=1,\ldots,16} = (1, \mathrm{i}\gamma_4, \mathrm{i}\gamma_5, \gamma_{45}, \vec{\tau}, \mathrm{i}\vec{\tau}\gamma_4, \mathrm{i}\vec{\tau}\gamma_5, \vec{\tau}\gamma_{45})$$



- **no instability towards an inhom phase.**
- **no so-called moat regime**

L. Pannullo, M. Winstel, Phys.Rev.D 108 (2023) 3, 036011, arXiv:2305.09444 [hep-ph]

Absence of inhomogeneous chiral phases in (2+1)-dimensional four-fermion and Yukawa models

Ansatz Method for inhomogeneous phases

Homogeneous condensate and effective action

Effective action satisfies

$$\left.\frac{d\Gamma[\phi]}{d\phi(x)}\right|_{\phi(x)=\phi_c} = 0 \qquad \phi_c(x) = \langle\phi\rangle = \bar{\phi}, \quad \Gamma[\bar{\phi}]$$

Effective potential

$\Gamma[\phi_c] = -\int dx\, V_{\mathrm{eff}}[\phi_c],\; V_{\mathrm{eff}}[\bar{\phi}\,] = -\frac{1}{TV}\Gamma[\bar{\phi}\,]$

find **the minimum of effective potential or TDP** with respect to **condensate** $\bar{\phi}$

For quark-antiquark condensate

$$\langle\bar{q}q\rangle \sim \langle\sigma\rangle = \bar{\sigma} = M \qquad V_{\mathrm{eff}}[\bar{\sigma}\,] = -\frac{1}{TV}\Gamma[\bar{\sigma}\,]$$

Ansatz method

$$\langle \phi \rangle = \bar{\phi}(x) = F(x, a_i)$$

Chiral spiral ansatz

$$\langle \sigma \rangle = M \cos(bx) \qquad \langle \pi \rangle = M \sin(bx)$$

where M and b parameters

Effective potential

$$V_{\text{eff}} = V_{\text{eff}}(M, b)$$

So one should find the minimum with respect to parameters M and b

Even if one gets exact V_{eff},
for example, in $1/N_c$ approximation or
mean field

Minimum of effective potential

$$V_{\text{eff}}(M, b)$$

does not necessarily gives
true ground state

Stability analysis for inhomogeneous phases

Stability analysis of inhomogeneous phase

$$\sigma(\mathbf{x}) = \bar{\sigma} + \delta\sigma(\mathbf{x}),$$

where $\bar{\sigma}$ **is ground state assuming only homogeneous condensates**

Expanding effective action with respect to $\delta\sigma$

$$\begin{aligned}
S^{(0)}_{\text{eff}} &= N_f \left(\frac{\beta V}{2\lambda} \bar{\sigma}^2 - \operatorname{Tr}(\ln(\bar{Q})) \right) \\
S^{(1)}_{\text{eff}} &= N_f \left(\frac{\beta}{\lambda} \bar{\sigma} \int d^2x \, \delta\sigma(\mathbf{x}) - \operatorname{Tr}\left(\bar{Q}^{-1} \delta\sigma \right) \right) \\
S^{(2)}_{\text{eff}} &= N_f \left(\frac{\beta}{2\lambda} \int d^2x \, \left(\delta\sigma(\mathbf{x}) \right)^2 + \frac{1}{2} \operatorname{Tr}\left(\bar{Q}^{-1} \delta\sigma \bar{Q}^{-1} \delta\sigma \right) \right),
\end{aligned}$$

$$S^{(2)}_{\text{eff}} = \frac{1}{2} \beta \int \frac{d^2q}{(2\pi)^2} |\delta\tilde{\sigma}(\mathbf{q})|^2 \Gamma^{-1}(\mathbf{q}^2),$$

Stability analysis of inhomogeneous phase

If there is instability

$$\text{If} \quad \Gamma_{min} = \min_q \Gamma^{(2)}(q) < 0$$

then **homogeneous ground state with condensate $\bar{\sigma}$** is **unstable** with respect to fluctuations

But $\bar{\sigma}$ is **true ground state, i. e. minimun,** among homogeneous configurations

So true ground state **should be** only **inhomogeneous**

Stability analysis and its limitations

- If $\quad \Gamma_{min} = \min_q \Gamma^{(2)}(q) < 0$

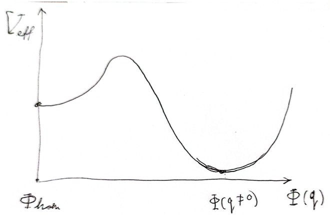
There is instability hence the true ground state **is inhomogeneous phase**

The homogeneous state is a saddle point, not a minimum. The system will definitely roll away from it. Since the instability is at $q \neq 0$, the true ground state must involve spatial modulation. This is a sufficient condition.

- But if $\quad \Gamma_{min} = \min_q \Gamma^{(2)}(q) > 0$

The homogeneous state could still be not the true ground state

The effective potential could look like this in field space:



Stability analysis of inhomogeneous phase

In general we know nothing about true ground state, inhomogeneous phase and its configuration

Only know that true ground state is inhomogeneous phase, that is all

Stability analysis and Moat Regime

Stability analysis and moat regime

Assume that $\Gamma^{(2)}(q)$ depends on q^2. Then the expansion of $\Gamma^{(2)}(q)$ around $q = 0$ is given by

$$\Gamma^{(2)}(q) = \Gamma^{(2)}(0) + \frac{q^2}{2} \frac{d^2\Gamma^{(2)}(q)}{dq^2}\bigg|_{q=0}$$

Instability to inhomogeneous phase if

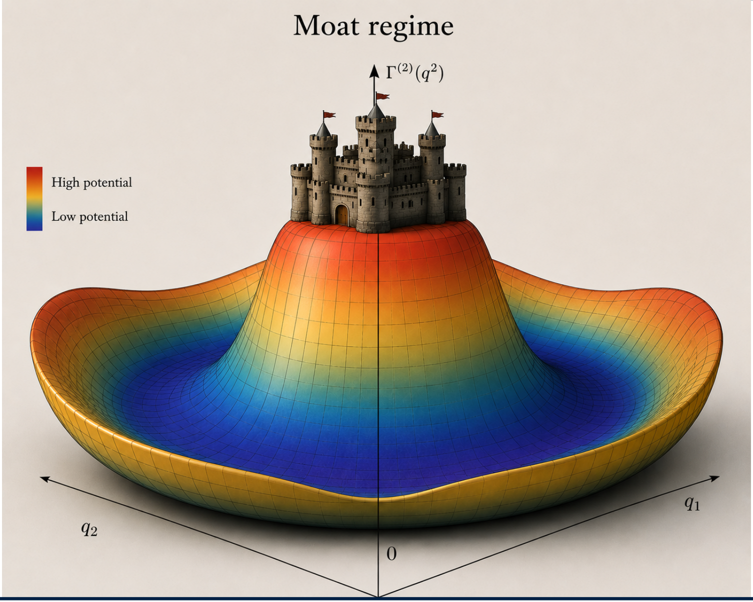
$$\Gamma_{min} = \min_q \Gamma^{(2)}(q) < 0$$

One as a rule define

$$Z = \frac{d^2\Gamma^{(2)}(q)}{dq^2}\bigg|_{q=0}$$

And if $Z < 0$ then this regime is called **moat regime**

Moat Regime in QCD Phase Diagram: $\Gamma^{(2)}(q)$

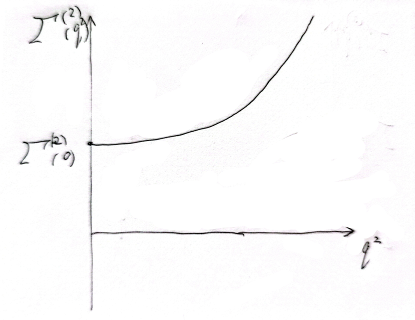


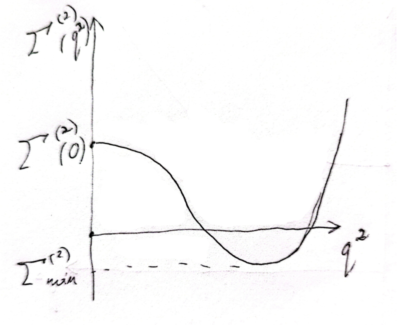
Moat Regime is considered as an

Indication or a Precursor

of Inhomogeneous phase

Stability analysis and moat regime

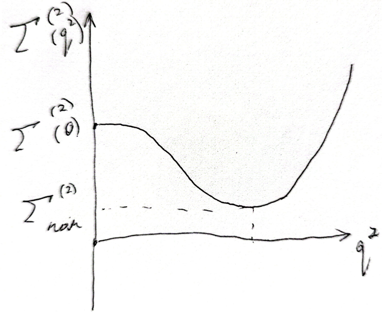


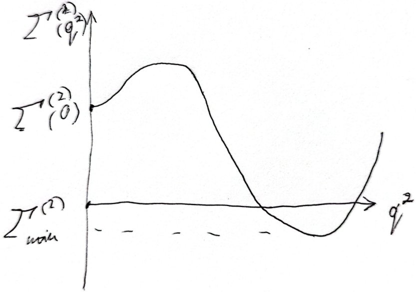


$$Z = \left.\frac{d^2\Gamma^{(2)}(q)}{dq^2}\right|_{q=0} > 0, \quad \Gamma_{min} > 0 \qquad\qquad Z < 0, \quad \Gamma_{min} < 0$$

- **No inhomogeneous phase**
  - **no moat regime**
- **Inhomogeneous phase**
  - **moat regime**

Stability analysis and moat regime



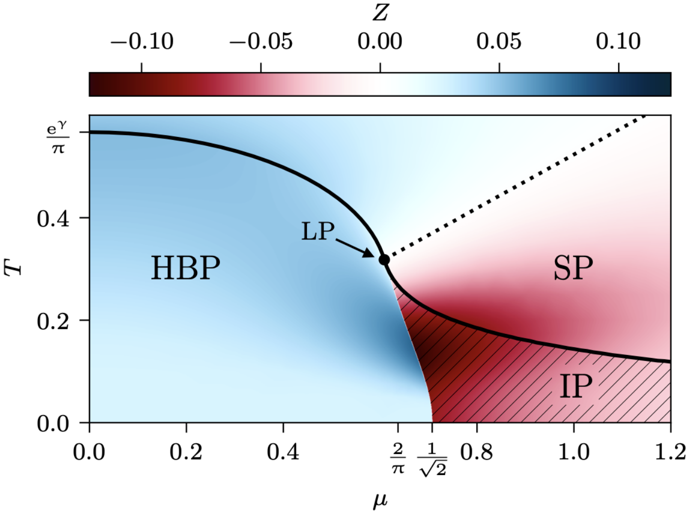


$$Z = \left.\frac{d^2\Gamma^{(2)}(q)}{dq^2}\right|_{q=0} < 0, \quad \Gamma_{min} > 0$$

$$Z > 0, \quad \Gamma_{min} < 0$$

- **No inhomogeneous phase**
  - **moat regime**
- **Inhomogeneous phase**
  - **no moat regime**

Testing Stability Analysis in (1+1)- dim GN model

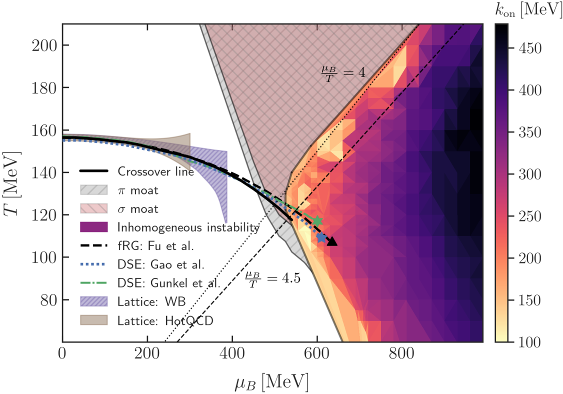


A. Koenigstein, L. Pannullo, S. Rechenberger, M. Steil, M. Winstel, J.Phys.A 55 (2022) 37, 375402, arXiv:2112.07024 [hep-ph]

Detecting inhomogeneous chiral condensation from the bosonic two-point function in the (1+1)-dimensional Gross-Neveu model in the mean-field approximation

Inhomogeneous phase in FRG

- **Standard crossover** $\mu_B/T < 4$
- **moat regime** appears at $\mu_B/T \approx 4$
- Instability at finite momentum for $\mu_B/T > 4.5$ **inhomogeneous phase**
- onset chemical potential μ_B of the instability region on the crossover line shows a mild regulator dependence: $\mu_B \approx 540 - 620$ MeV, consistent with systematic error estimate of 10%



Moat regimes: pions - hatched grey, sigma - hatched red. The region with signatures of inhomogeneous condensation is shown with a heatmap whose colour indicate value of k_{on} which is the lowest value of the RG-scale that can be reached before the instability terminates the flow.

Regularization and fits

(3+1)-dim NJL model is effective model

It is **non-renormalizable**

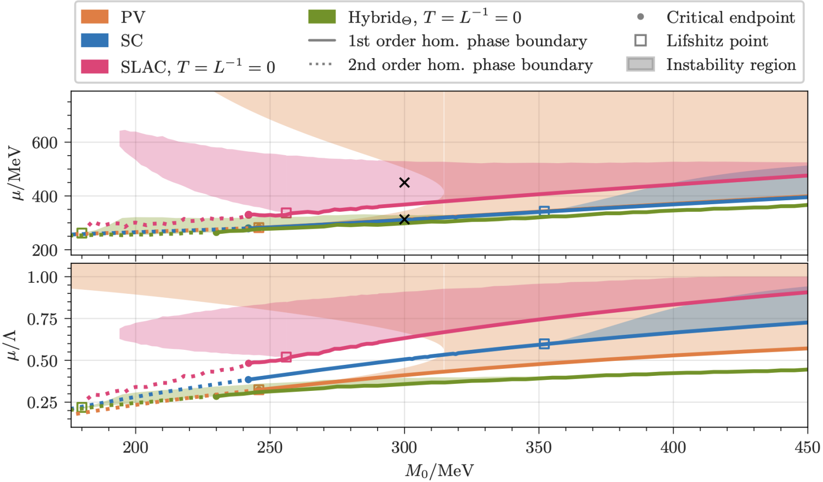
it is valid up to some energy scale

And one cannot reduce the regulator,
the regulator and the fit is the part of the model

(G, Λ) from experimental data such as m_π, f_π,...

One should relate (f_π, M_0) to (G, Λ)

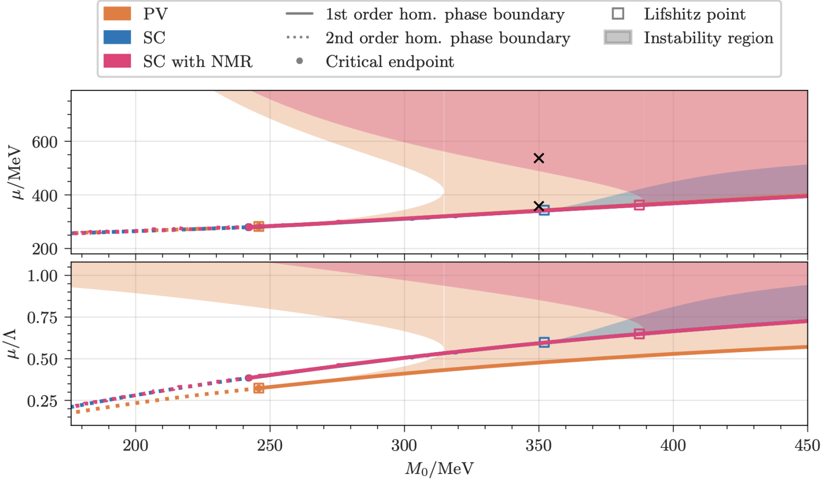
Inhomogeneous phase in (3+1)-dim NJL model



L. Pannullo, M. Wagner, M. Winstel, Phys.Rev.D 110 (2024) 7, 076006, arXiv:2406.11312 [hep-ph]

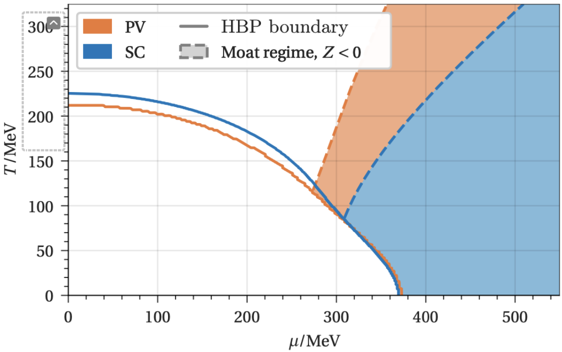
Regularization effects in the Nambu-Jona-Lasinio model: Strong scheme dependence of inhomogeneous phases and persistence of the moat regime

Inhomogeneous phase: No medium regularization



Phys.Rev.D 110 (2024) 7, 076006

Moat Regime in (3+1)-dim NJL model



Homogeneous phase boundaries and the moat regime for both the PV and the SC schemes.

Phys.Rev.D 110 (2024) 7, 076006

- (1+1)-dim — inhomogeneous phases
- (2+1)-dim — no inhom phases if no regulator
- (3+1)-dim — inhom phases

(d+1)-dimensional NJL type model

GN model in $1 < d < 3$ dimensions

$$\bar{U}_{\text{eff}}(\bar{\sigma}, \mu, d) = \frac{\bar{\sigma}^2}{2\lambda} - \frac{1}{\beta V} \ln \mathrm{Det}\left(\not{\partial} + \gamma_0 \mu + \bar{\sigma}\right) =$$

$$= \frac{\bar{\sigma}^2}{2\lambda} - \frac{N_\gamma}{2} \int \frac{\mathrm{d}^d p}{(2\pi)^d} \left[E - \Theta(\mu^2 - E^2)(E - |\mu|)\right]$$

GN model in d dimensions

$$\bar{U}_{\text{eff}}(\bar{\sigma}, \mu, d) = \frac{N_\gamma}{2^d \pi^{\frac{d}{2}}} \left[\frac{(d+1)\Gamma\left(-\frac{d+1}{2}\right)}{8\sqrt{\pi}} \left(-\frac{\bar{\sigma}_0^{d-1} \bar{\sigma}^2}{2} + \frac{|\bar{\sigma}|^{d+1}}{d+1} \right) + \right.$$

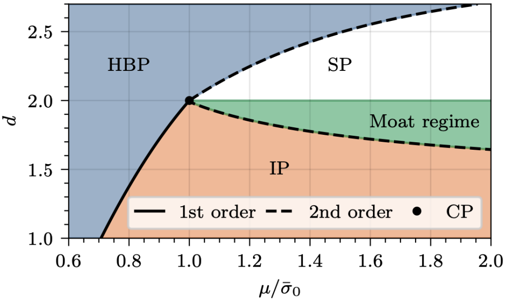
$$\left. + \frac{\Theta\left(\bar{\mu}^2\right)}{d\Gamma\left(\frac{d}{2}\right)} |\bar{\sigma}|^{d+1} \left|\frac{\bar{\mu}}{\bar{\sigma}}\right|^d \left({}_2F_1\left(-\tfrac{1}{2}, \tfrac{d}{2}; \tfrac{d+2}{2}; -\tfrac{\bar{\mu}^2}{\bar{\sigma}^2}\right) - \left|\frac{\mu}{\bar{\sigma}}\right| \right) \right]$$

L. Pannullo, Phys.Rev.D 108 (2023) 3, 036022

arXiv:2306.16290 [hep-ph]

Inhomogeneous condensation in the Gross-Neveu model in noninteger spatial dimensions $1 < d < 3$

GN model in $1 < d < 3$ dimensions



$(\mu,\, d)$-phase diagram of GN model with possibility of inhomogeneous case

L. Pannullo, Phys.Rev.D 108 (2023) 3, 036022
arXiv:2306.16290 [hep-ph]
A. Koenigstein, L. Pannullo, Phys.Rev.D 109 (2024) 5, 056015

arXiv:2312.04904 [hep-ph]

GN model in $1 < d < 3$ dimensions

$$\mathcal{S}[\bar{\psi},\psi]=\int_0^\beta \mathrm{d}\tau\int \mathrm{d}^d x\left[\bar{\psi}(\not{\partial}+\gamma_0\mu)\psi-\frac{\lambda}{2N}\left(\bar{\psi}\psi\right)^2\right]$$

GN model in d dimensions

$$S[\bar{\psi},\psi]=\int_0^\beta d\tau\int d^d x\left[\bar{\psi}(\gamma\partial+\gamma_0\mu)\psi-\frac{\lambda}{2N}\left\{(\bar{\psi}\psi)^2+(i\bar{\psi}\gamma_5\psi)^2\right\}\right]$$

NJL model (or chiral GN model) in d dimensions

But ambiguities with γ_5, $\{\gamma_\mu,\gamma_5\}$ make stability analysis of NJL model in d dimensions less clear

Phys.Rev.D 108 (2023) 3, 036022 arXiv:2306.16290 [hep-ph]

(2+1)-dimensional NJL model

(2+1)-dimensional NJL model with Lagrangian

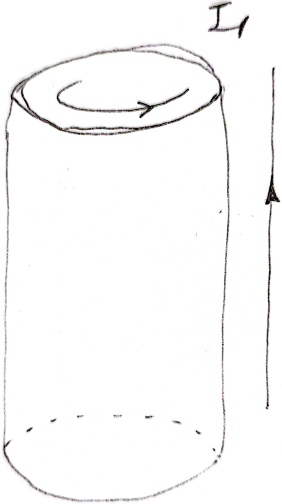
$$L = \bar{q}\Big[\gamma^\rho \mathrm{i}\partial_\rho + \mu\gamma^0\Big]q + \frac{G}{N}\Big[(\bar{q}q)^2 + (\bar{q}\mathrm{i}\gamma^5 q)^2\Big],$$

Chiral density wave (CDW) or chiral spiral

$$\langle\sigma(t,\vec{r})\rangle = M\cos(2bx_1), \quad \langle\pi(t,\vec{r})\rangle = M\sin(2bx_1)$$

where M and b are time- and $\vec{r}$- independent quantities.

(2+1)-dim NJL model with compactified dimension



(2+1)-dimensional NJL model

$$L = \bar{q}\Big[\gamma^{\rho}\mathrm{i}\partial_{\rho} + \mu\gamma^{0}\Big]q + \frac{G}{N}\Big[(\bar{q}q)^2 + (\bar{q}\mathrm{i}\gamma^5 q)^2\Big]$$

with one compactified spacial dimension

of circumference L

- at small L closer to (1+1) dimensions
- at $L \to \infty$ it is (2+1) dimensional model

TDP with compactified spacial dimensions

(2+1)-dimensional NJL model with Lagrangian

$$L = \bar{q}\Big[\gamma^{\rho} \mathrm{i}\partial_{\rho} + \mu\gamma^{0}\Big]q + \frac{G}{N}\Big[(\bar{q}q)^{2} + (\bar{q}\mathrm{i}\gamma^{5}q)^{2}\Big]$$

Chiral density wave (CDW) or chiral spiral

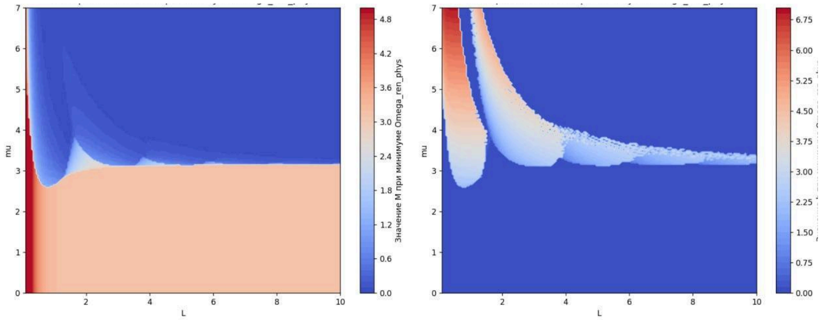
$$\langle\sigma(t,\vec{r})\rangle = M\cos(2bx_1), \quad \langle\pi(t,\vec{r})\rangle = M\sin(2bx_1)$$

M and b are time and $\vec{r}$- independent quantities.

Final renormalized expression for the TDP

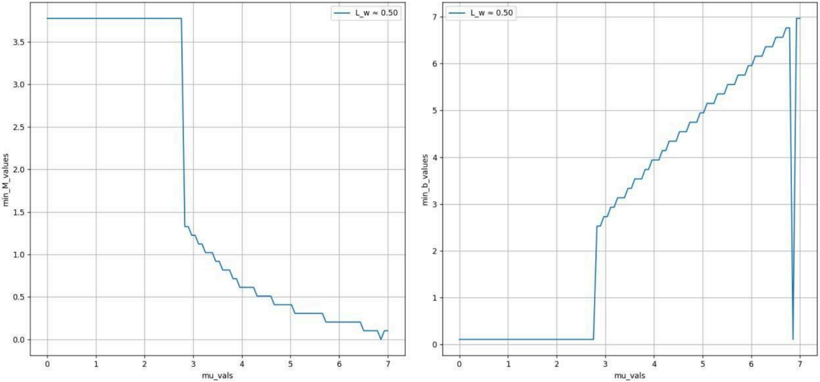
$$\Omega^{ren}_{phys}(M,b) \equiv \frac{M^{2}}{2g} + \Delta V_{phys}(M,b) + \widetilde{V}_{L}(M,b) + W_{\mu L}(M,b)$$

Inhomogeneous phase: $g < 0$



There are several **inhomogeneous phases** in phase diagram

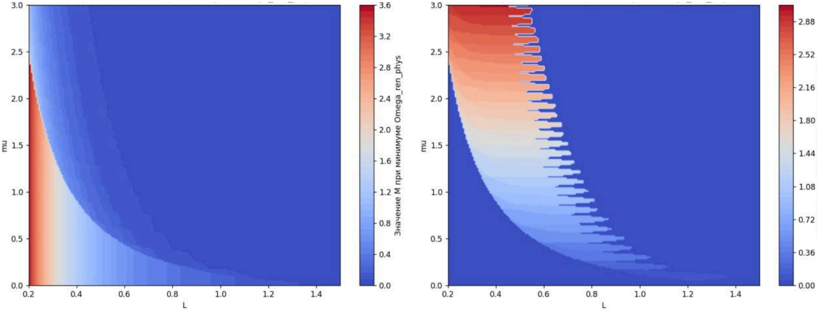
Inhomogeneous phase: $g < 0$ at fixed L



M and b as a function of μ at $L = 0.5$

There are **inhomogeneous phases** in wide interval of μ

Inhomogeneous phase: $g > 0$



- At small enough L there is chiral symmetry breaking
- But first inhomogeneous chiral symmetry breaking
- Inhomogeneous phase even at rather small μ

Thanks for the Attention

Thanks for Your Attention