Hadrons of $\mathcal{N} = 2$ QCD from non-Abelian String on 2D $\mathcal{N} = 2$ Black Hole

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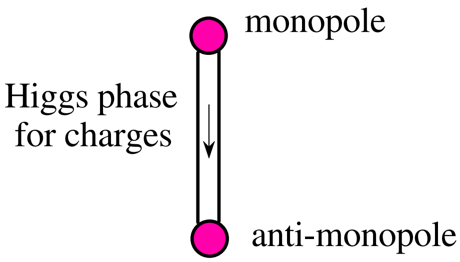
Introduction

Nambu, Mandelstam, 't Hooft and Polyakov 1970's:
Confinement is a dual Meissner effect upon condensation of monopoles.

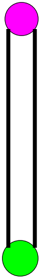
Electric charges condense $\rightarrow$ magnetic Abrikosov-Nielsen-Olesen flux tubes (strings) are formed $\rightarrow$ monopoles are confined



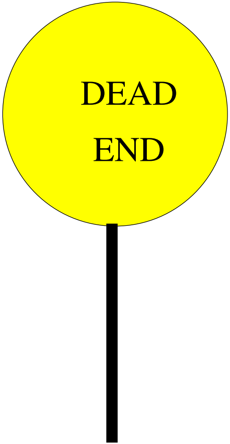
Nambu, Mandelstam, 't Hooft and Polyakov:

Dual Meissner effect:

Monopoles condense $\rightarrow$ electric Abrikosov-Nielsen-Olesen flux tubes are formed $\rightarrow$ electric charges are confined



No progress for many years...



QCD:

- No monopoles
- No confining strings
- Strong coupling

Breakthrough discovery come from supersymmetry.

Seiberg and Witten 1994 : Exact solution of $\mathcal{N} = 2$ supersymmetric QCD

Supersymmetric gauge theories can be considered as a ''theoretical laboratory" to develop insights in the dynamics of non-Abelian gauge theories.

Supersymmetric theories are "simplier" then real-world QCD
Many aspects are determined by exact solutions.

Seiberg and Witten 1994 : Confinement in the monopole vacuum of $\mathcal{N} = 2$ QCD

Cascade gauge symmetry breaking:

- SU(N)$\rightarrow$ U(1)$^{N-1}$ condensate of adjoint scalars
  Example: SU(2)$\rightarrow$ U(1)
- U(1)$^{N-1} \rightarrow 0$ condensate of monopoles

At the last stage Abelian Abrikosov-Nielsen-Olesen flux tubes are formed.

Abelian confinement

In the search for a non-Abelian confinement
Non-Abelian vortex strings
were found in $\mathcal{N}=2$ U(N) QCD
Hanany, Tong 2003
Auzzi, Bolognesi, Evslin, Konishi, Yung 2003
Shifman Yung 2004
Hanany Tong 2004

Non-Abelian string : Orientational zero modes
Rotation of color flux inside SU(N).
Non-Abelian vortex string is BPS and preserves
$\mathcal{N}=(2,2)$ supersymmetry on its world sheet.

$\mathcal{N} = 2$ SQCD:

In monopole vacua Abelian strings confine quarks

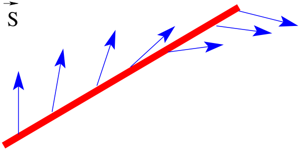
In quark vacua non-Abelian strings confine monopoles

In both cases an "observer" will see colorless hadrons

= stringy states

Next problem:

How to quantize confining solitonic string outside critical dimension???



Shifman and Yung, 2015: Non-Abelian vortex in $\mathcal{N} = 2$ supersymmetric QCD can behave as a critical superstring

Idea:

Non-Abelian string has more moduli then Abrikosov-Nielsen-Olesen string.

It has translational + orientaional moduli

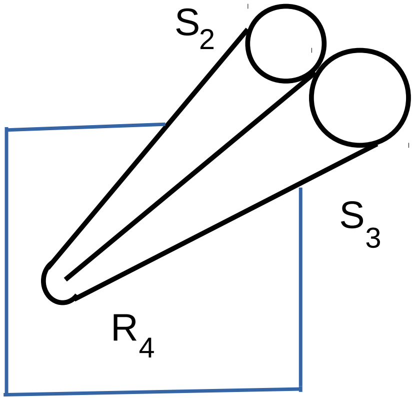
We can fulfill the criticality condition: In $\mathcal{N} = 2$ QCD with $U(N = 2)$ gauge group and $N_f = 4$ quark flavors.

- The solitonic non-Abelian vortex have six orientational and size moduli, which, together with four translational moduli, form a ten-dimensional space.
- For $N_f = 2N$ 2D world sheet theory is conformal.

For $N = 2$ and $N_f = 4$ the target space of the 2D sigma model on the string world sheet is

$$R^4 \times Y_6,$$

where Y_6 is a non-compact Calabi-Yau manifold studied by Candelas, Witten and Vafa, namely **conifold.**



We study states of closed type IIA string propagating on $R^4 \times Y_6$ and interpreted them as hadrons in 4D $\mathcal{N} = 2$ QCD.

Conifold $==$ non-compact CY.

Looking for states with normalizable wave function over Y_6
String states localized near the conifold singularity. They are 4D SQCD states

Massless baryon in 4D SQCD $=$ complex structure modulus b of the conifold promoted to the 4D scalar field

To find the massive spectrum of string states we used Little String Theory approach
Ghoshal, Vafa, 1995; Giveon Kutasov 1999 proposed that
Critical string on a conifold at strong coupling is equivalent to non-critical string on

$$\mathcal{R}^4 \times \mathcal{R}_\phi \times S^1,$$

with linear in the Liouville field ϕ dilaton
$\mathcal{N} = 2$ supersymmetric Liouville theory

Recently it was proven in a direct way

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Coulomb branches of world sheet weighted $CP(N-1)$ ($\mathbb{WCP}(N, N)$) models on non-compact CY manifolds are described by $\mathcal{N} = 2$ Liouville theory with background charge depending on N.

Now using the $\mathcal{N}=2$ Liouville theory approach we make a step towards broadening the class of 4D $\mathcal{N}=2$ SQCDs where hadrons can be described as string states of the non-Abelian vortex.

We introduce quark masses in $\mathcal{N}=2$ SQCD and changing values of mass parameters interpolate between SQCDs with different gauge groups and numbers of quark flavors.

$$U(2), N_f = 4 \;\; SQCD \quad \to \quad U(N), N_f = 2N \;\; SQCD \;\; N = even$$

$\mathbb{WCP}(N, N)$ models

World sheet sigma models on non-Abelian strings in $\mathcal{N} = 2$ SQCD with $N_f = 2N$ are $\mathbb{WCP}(N, N)$ models. Can be understood as Higgs branches of U(1) gauge theory, $e_0 \to \infty$ (*Witten, 1993*). Conformal in the massless limit.

$$S = \int d^2x \left\{ \left|\nabla_\alpha n^i\right|^2 + \left|\tilde{\nabla}_\alpha \rho^j\right|^2 - \frac{1}{4e_0^2} F_{\alpha\beta}^2 + \frac{1}{e_0^2} \left|\partial_\alpha \sigma\right|^2 \right.$$
$$+ \frac{1}{2e_0^2} D^2 - \left|\sqrt{2}\sigma + m_i\right|^2 \left|n^i\right|^2 + \left|\sqrt{2}\sigma + \tilde{m}_j\right|^2 \left|\rho^j\right|^2$$
$$\left. + D \left(\left|n^i\right|^2 - \left|\rho^j\right|^2 - \mathrm{Re}\,\beta \right) - \frac{\vartheta}{2\pi} F_{01} \right\},$$

where $i = 1, \ldots, N$, $j = 1, \ldots, N$ and the complex scalar fields n^i and ρ^j have charges $\mathtt{Q} = +1$ and $\mathtt{Q} = -1$

$$\nabla_\alpha = \partial_\alpha - iA_\alpha\,, \qquad \tilde{\nabla}_\alpha = \partial_\alpha + iA_\alpha\,,$$

$$-\frac{\beta}{2}\int d^2\tilde{\theta}\sqrt{2}\,\Sigma = -\frac{\beta}{2}\,(D - iF_{01}), \qquad \beta = \mathrm{Re}\,\beta + i\,\frac{\vartheta}{2\pi}$$

$$\Sigma = \sigma + \sqrt{2}\theta_R\bar{\lambda}_L - \sqrt{2}\bar{\theta}_L\lambda_R + \sqrt{2}\theta_R\bar{\theta}_L(D - iF_{01})$$

Twisted masses m_i and $\tilde{m}_j$ coincide with quark masses of $2N$ flavors in 4D SQCD.

Dimension of the Higgs branch in the $m_i = \tilde{m}_j = 0$ limit

$$\mathrm{dim}_R\mathcal{H} = 4N - 1 - 1 = 2\,(2N - 1)$$

The model is conformal and $\mathcal{N} = (2,2)$ supersymmetric $\Rightarrow$ target space is Ricci-flat and Kähler $\Rightarrow$ Calabi-Yau

For $N = 2$ $\mathrm{dim}_R\mathcal{H} = 6$ – conifold 6+4=10 – critical non-Abelian string

Interpolation procedure

Classical vacuum structure (at $\mathrm{Re}\,\beta > 0$)

$$\sqrt{2}\sigma = -m_{i_0}, \qquad |n^{i_0}|^2 = \mathrm{Re}\,\beta, \qquad i_0 = 1, ..., N.$$

Fields n^i , $i \neq i_0$ and fields ρ^j have masses $|m_i - m_{i_0}|$ and $|m_j - m_{i_0}|$ respectively.
Take $N = 2K$, $\tilde{m}_i = m_i$ and $\mathbb{Z}_K$-symmetric quark masses

$$\{m_i, \tilde{m}_j\}_{i,j=1}^{N} = \{\underbrace{m_2, m_2, \ldots, m_{2K}, m_{2K}}_{N=2K}; \underbrace{\tilde{m}_2, \tilde{m}_2, \ldots, \tilde{m}_{2K}, \tilde{m}_{2K}}_{N=2K}\},$$

$$m_{2k} = m \exp\left(\frac{2\pi i k}{K}\right), \quad m_{2k-1} = m_{2k} \quad k = 1, \ldots K$$

Starting point: $M \to \infty$

$K-1$ pairs of n and ρ fields decouple. We have
$\mathbb{WCP}(2,2)$ model

Final point: $M \to 0$

We have $\mathbb{WCP}(N,N)$ model $N = 2K$.

In 4D SQCD:

Starting point: $M \to \infty$
K non-interacting copies of $\mathcal{N} = 2$ U(2) SQCD with $N_f = 4$

Final point: $M \to 0$ $\quad \mathcal{N} = 2$ U(N) SQCD with $N_f = 2N$

The starting point is the
critical non-Abelian string on the conifold

$\mathcal{N} = 2$ Liouville theory from $\mathbb{WCP}(N, N)$ model

Massless theory

Take $\mathbb{WCP}(N, N)$ at $\beta = 0$. Fields n and ρ become "massive" at $\sigma \neq 0$ and can be integrated out.
Witten, 1979 for $\mathrm{CP}(N-1)$ model.

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Similar calculation for $\mathbb{WCP}(N, N)$ model gives

$$S^{\sigma}_{\mathrm{eff}} = \frac{1}{4\pi} \int d^2x \sqrt{h} \, \left(\frac{1}{2} \, h^{\alpha\beta} (\partial_\alpha \phi \partial_\beta \phi + \partial_\alpha Y \partial_\beta Y) - \frac{Q}{2} \phi \, R^{(2)} \right) ,$$

where $h^{\alpha\beta}$ is the world sheet metric and $R^{(2)}$ is the world sheet Ricci scalar.

$$\sigma = \gamma \, e^{-\frac{\phi + iY}{Q}}$$

Here Y is a compact variable, $Y + 2\pi Q \sim Y$.

$$Q = \sqrt{2(N-1)}, \qquad \text{for} \quad N = 2 \quad Q = \sqrt{2}$$

This is the bosonic part of the $\mathcal{N} = 2$ Liouville action with linear dilaton

$$\Phi(\phi) = -\frac{Q}{2}\phi$$

The $\mathcal{N} = 2$ Liouville interaction superpotential comes from the 2D FI term in the $\mathbb{WCP}(N, N)$ model. For $N = 2$

$$S_{\rm FI} = -\frac{\beta}{\sqrt{2}} \int d^2x d^2\tilde{\theta}\, \Sigma + c.c. = b \int d^2x d^2\tilde{\theta}\, e^{-\frac{\phi + iY}{Q}} + c.c,$$

where b is the complex structure modulus of the conifold.

Mass deformation

Now consider $\mathbb{WCP}(N, N)$ model with nonzero twisted masses and integrate out n^i and ρ^J fields. We get

$$S_{\text{eff}} = \frac{1}{4\pi} \int d^2x \, g_{cl}(\phi, Y) \left(\frac{1}{2} (\partial_\alpha \phi)^2 + \frac{1}{2} (\partial_\alpha Y)^2 \right)$$

where we make a substitution

$$\sigma + \frac{m_{i_0}}{\sqrt{2}} = \gamma \, e^{-\frac{\phi + iY}{Q}}$$

. and

$$g_{cl}(\phi) \approx 1 + \exp\left(-\frac{2\phi}{Q}\right) \sum_{i \neq i_0} \frac{|b|^2}{|M_i - M_{i_0}|^2}, \qquad M_i = -\frac{b \, m_i}{\sqrt{2}\gamma}$$

Gravity equations

The bosonic part of the action of the type-II supergravity in the string frame is given by

$$S = \frac{1}{2\kappa^2} \int d^D x \sqrt{-G}\, e^{-2\Phi} \left\{ R + 4G^{MN}\partial_M \Phi \partial_N \Phi_N + \cdots \right\}$$

Einstein's equations:

$$R_{MN} + 2D_M D_N \Phi = 0$$

Dilaton equation:

$$R = 4G^{MN}\partial_M \Phi \partial_N \Phi - 4G^{MN} D_M D_N \Phi + p,$$

where $p = \frac{D-10}{2}$.
Minkowski 4D $\times$ deformed Liouville theory. $D = 6,\ p = -2$
Ansatz for the internal metric:

$$ds^2_{\rm int} = g(\phi, Y) \left\{ d^2\phi + d^2 Y \right\}$$

Solutions to gravity equations

Solution for for the dilaton:

$$\Phi(\phi) = -\frac{Q}{2}\,\phi + \frac{1}{2}\,\ln g$$

and for the metric warp factor:

$$g(\phi) = \frac{1}{1 - e^{-Q(\phi-\phi_0)}},$$

These solutions satisfy initial conditions $g_{cl}(\phi) \approx 1 + \exp\left(-\frac{2\phi}{Q}\right) \sum_{i \neq i_0} \frac{|b|^2}{|M_i - M_{i_0}|^2}$ induced by the mass deformation with

$$\phi_0 = \frac{1}{Q} \ln \left(\sum_{i \neq i_0} \frac{|b|^2}{|M_i - M_{i_0}|^2} \right) = \frac{1}{Q} \log \frac{(K^2 - 1)|b|^2}{12|M|^2}$$

only if $Q = \sqrt{2}$. $M = \frac{b\,m}{\sqrt{2}\gamma}$

Trumpet geometry

Mass-deformed Liouville theory

$$S_{\rm ws} = \frac{1}{4\pi}\int d^2x\sqrt{h}\ \left\{\frac{1}{2}\,g(\phi)\left[(\partial_\alpha\phi)^2 + (\partial_\alpha Y)^2\right] +\ \Phi(\phi)\,R^{(2)}\right\}$$

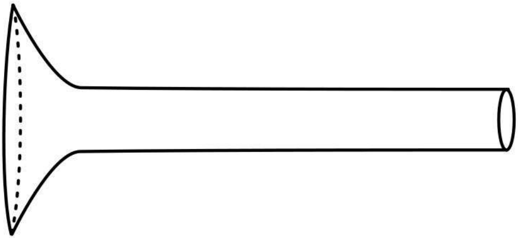
Change variables $e^{\frac{Q}{2}(\phi-\phi_0)} = \cosh\rho$. We get

$$S_{\rm ws} = \frac{k}{4\pi}\int d^2x\sqrt{h}\ \left((\partial_\alpha\rho)^2 + \coth^2\rho\,(\partial_\alpha\vartheta)^2\right)$$
$$+\frac{1}{4\pi}\int d^2x\sqrt{h}\,\Phi(\rho)\,R^{(2)}, \qquad k = \frac{2}{Q^2}.$$

with $\vartheta = \frac{Q}{2}\,Y$, $\quad \vartheta \sim \vartheta + 2\pi/k$.
The dilaton takes the form

$$\Phi(\rho) = -\frac{Q}{2}\phi_0 - \log\left(\sinh\rho\right)$$

The target manifold looks like a "trumpet"



Trumpet geometry. Asymptotically, at $\rho \to \infty$, it turns into a cylinder of radius Q.

T-duality

Trumpet is T-dual to the 2D $\mathcal{N} = 2$ supersymmetric black hole, which is the $\mathrm{SL}(2, \mathbb{R})/\mathrm{U}(1)$ coset WZNW theory with the level k of supersymmetric Kač-Moody algebra.

$$S_{\mathrm{BH}} = \frac{k}{4\pi} \int d^2x \sqrt{h} \left\{ (\partial_\alpha \rho)^2 + \tanh^2 \rho \, (\partial_\alpha \theta)^2 \right\} + \frac{1}{4\pi} \int d^2x \sqrt{h} \, \Phi(\rho) R^{(2)}$$

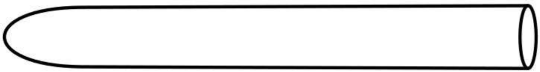
with the dilaton

$$\Phi(\rho) = \Phi_0 - \log \cosh \rho$$

$R \to 1/R$ transformation:

$$\sqrt{2k} \tanh \rho \to \alpha' \frac{\coth \rho}{\sqrt{2k}}, \qquad \alpha' = 2.$$

Cigar geometry of 2D black hole (Eucledian version)



$R(\rho \to \infty) = \sqrt{2k}$. In our theory $Q = \sqrt{2} \Rightarrow k = 1$.
Witten 1991 : Black hole mass

$$M_{BH} = \frac{Q}{2} e^{-2\Phi_0} = \frac{1}{\sqrt{2k}} e^{-2\Phi_0}$$

For the mass deformation

$$\Phi_0^{(M)} = -\frac{Q}{2}\phi_0 = -\frac{1}{2}\ln\frac{(K^2-1)|b|^2}{12|M|^2}$$

and

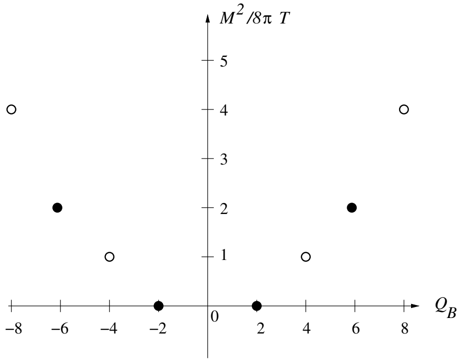
$$M_{BH}^{(M)} = \frac{Q}{2}\frac{(K^2-1)|b|^2}{12 g_0 |M|^2}.$$

The spectrum

The spectrum of $\mathcal{N} = 2\ \mathrm{SL}(2, R)/\mathrm{U}(1)$ coset with a cigar geometry is known.

$$T_{j,m,p_{mu}}(\rho \to \infty) \sim e^{ip_\mu x^\mu}\, e^{2(j\rho + im\theta)}, \qquad \Delta_{j,m} = \frac{1}{k}\left\{m^2 - j(j+1)\right\}$$

with $j = -\frac{1}{2}, -1.$ $\quad m = \pm\{j, j-1, j-2, ...\},$ $\quad B = 4m$



We see that as we reduce the mass parameter M the mass spectrum of the string states does not change.

Strange!

In fact in $\mathcal{N}=2$ SQCD it is natural to assume

$$m_H^2 = m^2_{\text{non-BPS}}(\alpha') + |Z_{BPS}(m_i, \tilde{m}_j)|^2$$

Turns out that

$$Z_{BPS} = i\sum_i \left(q_i^{(n)} m_i - q_i^{(\rho)} \widetilde{m}_i\right) = 0 \quad \text{for} \quad m_i = \widetilde{m}_i$$

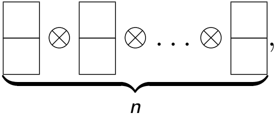
However we expect that the number of states on each level should increase as we reduce M.

Number of states from SQCD side

The global symmetry group of SQCD at $M \to 0$ is

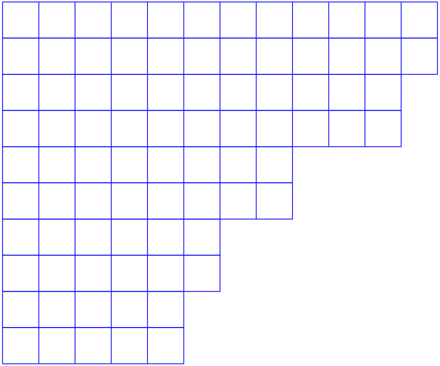
$$SU(N)_n \times SU(N)_\rho \times U(1)_B, \qquad \vec{q}^{(n)} = \vec{q}^{(\rho)}$$

Allowed irreducible representations are from the product



where $B = 2n$.

Allowed representations are of the form of "double-step staircase"



Universal leading behavior in the limit $B \gg N \gg 1$

$$\log \dim \lambda_B \sim \frac{1}{2} N^2 \log n \sim \frac{1}{2} N^2 \log B \sim \frac{1}{2} N^2 \log E$$

Number of string states on the black hole

To find the number of states we continue to a Eucledian space and consider the compact dimension θ as a temperature circle, $R(\rho \to \infty) = \sqrt{2k}$. $T = \frac{1}{2\pi\sqrt{2k}}$ And calculate the entropy.
Problem: Hagedorn behavior

$$Z = \int_0^\infty dE\,\omega(E)\,e^{-\frac{E}{T}}, \qquad \omega(E) \sim E^\alpha \exp\left(\frac{E}{T_H}\right)$$

Suskind 1993, Horovitz and Polchinski 1997:
Hagedorn behavior $\longleftrightarrow$ black hole / excited strings phase transition
Giveon, Kutasov, Rabinovichi, Sever 2005 : $k_c = 1$ - our value!

Thermal scalar

Atick and Witten 1988:
In the Euclidean formulation, the Hagedorn behavior is described by the so-called thermal scalar, which is a winding string mode around the thermal circle.
It becomes massless and non-normalizable along the cigar as $T \to T_H$.
Action for the thermal scalar (lowest winding string mode)

$$S_b^E = \frac{1}{2\kappa^2} \int d\rho d\vartheta \sqrt{G}\, e^{-2\Phi} \left\{ g^{nn'} \partial_n \bar{T}_b \partial_{n'} T_b - |T_b|^2 \right\}$$

Entropy

$$s = \left(1 - R \frac{\partial}{\partial R} \right) \log Z \approx \left(R \frac{\partial}{\partial R} - 1 \right) S_b^E$$

The divergence at $k \to 1$ comes from the infrared region $\rho \to \infty$

$$\int d\rho \sinh(2\rho)|T_b|^2 = \int d\rho\, |\Psi_b|^2, \qquad \Psi_b^{(t)}(\rho)|_{\rho\to\infty} \approx e^{-(k-1)\rho}$$

Finally we get

$$s = \frac{\sqrt{2k}}{k-1} M_{BH}^{\rm total}$$

At small M we have

$$s \approx \frac{1}{k-1} \frac{(K^2-1)|b|^2}{12 g_0 |M|^2}$$

As we reduce M the entropy becomes "more divergent".

The number of states at each high energy level

$$\omega(E) \approx \exp\left\{ \frac{E}{T_H} + \frac{(K^2-1)|b|^2}{24\, g_0^2 |M|^2} \log E + \cdots \right\} \approx$$

$$\approx \exp\left(\frac{E}{T_H} + \frac{N^2}{96} \frac{|b|^2}{\lambda^2} \log E + \cdots \right)$$

at large $N = 2K$, where we assume

$$g_0 \to \infty, \qquad M \to 0, \qquad \lambda = g_0 |M| = \text{fixed}$$

Conclusions

The phase diagram for 4D SQCD. The fundamental domain of 4D coupling τ_{SW},

$$\tau_{SW} = i\frac{8\pi}{g^2} + \frac{\theta_{4D}}{\pi}$$

is shown in the the horizontal plane. SQCD is in the Higgs phase on this plane. The stringy phase where $\langle b \rangle \neq 0$ is schematically shown by the cone.