Optimization of factorization scale for QED e^+e^- annihilation process

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Outline

1. Motivation
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3. Conclusions
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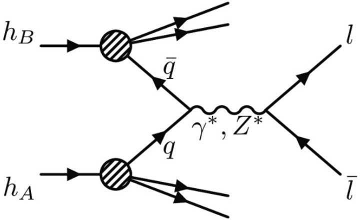
Motivation

- Results of perturbative calculations often depend on factorization and renormalization scales and scheme choices as in QCD as well as in QED (and in EW)
- Existing analytic results in QED provide a unique possibility to check analytically and numerically those dependencies
- QED Drell-Yan-like process $e^+e^- \to \gamma^*, Z^* \to \mu^+\mu^-$ is the best case for the tests
- Different prescriptions for the factorization scale choice can be compared
- Several other issues can also be addressed: dependence on renormalization scale, scheme choices, exponentiation, effect of scale variation by factor 2, etc.
- Optimization of the scale and scheme choices is required both in QCD and QED/EW

Drell-Yan process in QCD

Schematically

$$d\sigma(h_A h_B \to \gamma^*, Z^* \to \mu^+\mu^-) = \sum_{a,b=q,\bar{q},g} D_{ah_A} \otimes D_{bh_B} \otimes d\hat{\sigma}(ab \to \gamma^*, Z^* \to \mu^+\mu^-)$$



[1] V.A. Matveev, R.M. Muradyan, A.N. Tavkhelidze, Production of Muon Pairs in Strong Interactions. . . , JINR Preprint P2-4578 '1969

[2] S.D. Drell, T.-M. Yan, Massive Lepton-Pair Production in Hadron-Hadron Collisions at High Energies, PRL '1970

Drell-Yan-like process in QED: e^+e^- annihilation

The factorization structure is exactly the same as in QCD:

$$\frac{d\sigma_{e^+e^-\to\gamma^*,Z^*\to\mu^+\mu^-}}{ds'} = \frac{1}{s}\sigma^{(0)}(s') \sum_{a,b=e^\pm,\gamma} D_{ae^-} \otimes D_{be^+} \otimes \tilde{\sigma}_{ab\to\gamma^*,Z^*\to\mu^+\mu^-}$$

Complete analytic results as function of the final state invariant mass are known in $O(\alpha)$, $O(\alpha^2)$, and in the leading and next-to-leading logarithmic approximations in several higher orders (terms $\sim m_e^2/s$ are neglected).
Note, that only initial state radiation is discussed here.
See [J. Ablinger, J. Blümlein, A. De Freitas and K. Schönwald, "Subleading Logarithmic QED Initial State Corrections to $e^+e^- \to \gamma^* Z_0$ to $O(\alpha^6 L^5)$," NPB 955 (2020) 115045] and [A.Arbuzov, U.V., PRD 2024]

Drell-Yan-like process in QED: ingredients

$$\frac{d\sigma_{e^+e^-\to\gamma^*,Z^*\to\mu^+\mu^-}}{ds'} = \frac{1}{s}\sigma^{(0)}(s') \sum_{a,b=e^\pm,\gamma} D_{ae^-} \otimes D_{be^+} \otimes \tilde{\sigma}_{ab\to\gamma^*,Z^*\to\mu^+\mu^-}$$

- $\sigma^{(0)}(s')$ is the Born cross-section, $\tilde{\sigma}$is the partonic one (normalized to Born)
$$\tilde{\sigma} = \delta(1-z) + \alpha\bar{\sigma}^{(1)}(z) + \alpha^2\bar{\sigma}^{(2)}(z) + \ldots$$
- D_{ae} are electron PDFs, they are solutions of DGLAP evolution eqs.
- $\alpha \equiv \alpha_{QED}(\mu_R)$
- $s = (p_{e^+} + p_{e^-})^2, \quad s' = (p_{\mu^+} + p_{\mu^-})^2 \equiv zs$

QED DGLAP evolution equations

$$D_{ba}\left(x, \frac{\mu_R^2}{\mu_F^2}\right) = \delta_{ab}\delta(1-x) + \frac{\alpha}{2\pi} d_{ba}^{(1)}(x, \mu_R^2/m_e^2) + \ldots$$
$$+ \sum_{c=e^{\pm},\gamma} \int\limits_{\mu_R^2}^{\mu_F^2} \frac{dt}{t} \int\limits_{x}^{1} \frac{dy}{y} P_{bc}(y,t) D_{ca}\left(\frac{x}{y}, \frac{\mu_R^2}{t}\right)$$

a, b, c are massless partons ($\sim e^{\pm}, \gamma$)
μ_F is a factorization (energy) scale
μ_R is a renormalization (energy) scale
D_{ba} is a parton distribution function (PDF)
$d_{ba}^{(m)}$ are initial conditions
P_{bc} is a splitting function or kernel of the DGLAP equation
Analytic QED PDFs in the leading and next-to-leading log approximations are known [A.Arbuzov, U.V., JPG '2023], see also [S.Frixione et al., JHEP'2020]

QED splitting functions

The perturbative splitting functions are

$$P_{ba}(x, \bar{\alpha}(t)) = \frac{\bar{\alpha}(t)}{2\pi} P_{ba}^{(0)}(x) + \left(\frac{\bar{\alpha}(t)}{2\pi} \right)^2 P_{ba}^{(1)}(x) + \mathcal{O}(\alpha^3)$$

$$P_{ee}^{(0)}(x) = \left[\frac{1 + x^2}{1 - x} \right]_+$$

They come from loop calculations, e.g., $P_{ba}^{(1)}(x)$ comes from 2-loops

The splitting functions can be obtained by reduction of the ones known in QCD to the abelian case of QED

$\bar{\alpha}(t)$ is the QED running coupling constant in the $\overline{\mathrm{MS}}$ scheme

Drell-Yan-like process in QED: perturbative expansion

$$\frac{d\sigma_{e^+e^-\to\gamma^*,Z^*\to\mu^+\mu^-}}{ds'} = \frac{1}{s}\sigma^{(0)}(s') \sum_{a,b=e^\pm,\gamma} D_{ae^-} \otimes D_{be^+} \otimes \tilde{\sigma}_{ab\to\gamma^*,Z^*\to\mu^+\mu^-}$$

Large logs $L \equiv \ln(\mu_F^2/\mu_R^2)$ come from running α and anomalous dimensions

$$\frac{d\sigma_{\bar{e}e}^{\mathrm{NNLL}}(s')}{ds'} = \frac{\sigma_{\bar{e}e}^{(0)}(s')}{s}\left\{\delta(1-z) + \sum_{\substack{k=1\\ k\geq l\geq k-2}}^{\infty} \left(\frac{\alpha}{2\pi}\right)^k c_{kl}(z) L^l\right\} + \mathcal{O}\left(\frac{m_e^2}{s}\right)$$

where c_{kl} are functions of $z = s'/s$ and they contain

$$\delta(1-z),\ \zeta(m),\ N_f,\ \left[\frac{\ln^m(1-z)}{1-z}\right]_+,\ \frac{\ln^m(z)}{z},\ \mathrm{HPLs}\ldots$$

Short notation $h_{kl} = \left(\frac{\alpha}{2\pi}\right)^k L^l c_{kl}(z) \cdot 100\%$

Drell-Yan-like process in QED: the general structure

$$\frac{d\sigma_{e^+e^-\to\gamma^*,Z^*\to\mu^+\mu^-}}{ds'} = \frac{1}{s}\sigma^{(0)}(s')\sum_{a,b=e^\pm,\gamma} D_{ae^-}\otimes D_{be^+}\otimes\tilde{\sigma}_{ab\to\gamma^*,Z^*\to\mu^+\mu^-}$$

Schematically

$$\begin{array}{llll}
c_{00}\alpha^0 \ \text{(Born)} & & & \\
c_{11}\alpha^1 L^1 & c_{10}\alpha^1 L^0 & & \\
c_{22}\alpha^2 L^2 & c_{21}\alpha^2 L^1 & c_{20}\alpha^2 L^0 & \\
c_{33}\alpha^3 L^3 & c_{32}\alpha^3 L^2 & c_{31}\alpha^3 L^1 & c_{30}\alpha^3 L^0 \\
\quad\ldots & & &
\end{array}$$

where $\alpha = \alpha(\mu_R)$ and $L = \ln(\mu_F^2/\mu_R^2)$ and coefficients c_{kl} in different orders are functions of kinematical variables and N_f

Factorization scale choices

The final result of calculation in all orders in α and L would not depend on μ_F But a fixed-order result for an observable does depend on μ_F (and μ_R) Many different methods for choosing μ_F were proposed:

- CSS — Conventional Scale Setting (μ_F= hard momentum transfer)
- FAC — Fastest Apparent Convergence [G. Grunberg; N. Krasnikov]
- PMS — Principle of Minimal Sensitivity [P.M. Stevenson]
- BLM — Brodsky-Lepage-Mackenzie (absorb β_0 -dependent terms)
- PMC — Principle of Maximal Conformality [S.Brodsky et al.], see also [A.Kataev, S.Mikhailov, PRD'2015]
- ...

Factorization scale choice in $e^+e^- \to \gamma^*, Z^* \to \mu^+\mu^-$

We advocate the FAC-like prescription, i.e., hide the bulk of one-loop corrections into the scale choice

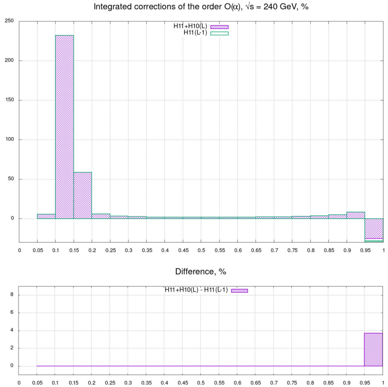
$$\frac{d\sigma^{(1)}_{\bar{e}e}(s')}{ds'} = \frac{\alpha}{\pi}\left\{\left[\frac{1+z^2}{1-z}\right]_+\left(\ln\frac{s}{m_e^2}-1\right)+\delta(1-z)\left(2\zeta(2)-\frac{1}{2}\right)\right\} \Rightarrow \mu_F^2 = s \text{ or } \frac{s}{e}$$

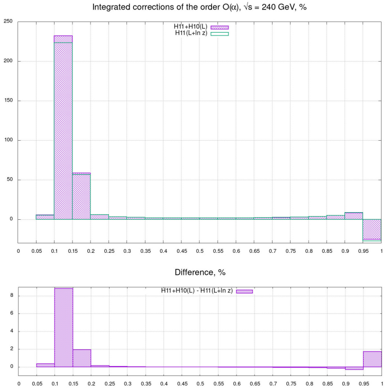
$$\frac{d\sigma^{(1)}_{\bar{e}e}(s')}{ds'} = \frac{\alpha}{\pi}\left(P^0_{ee}(x)\ln\frac{s}{m_e^2}+d^{(1)}_{ee}(x)+\frac{1}{2}\tilde{\sigma}^{(1)}_{ee}(x)\right)$$

Remind QCD Drell-Yan processes where we usually take $\mu_F^2 = s' \equiv zs \sim M_Z$, i.e., the energy scale of the hard subprocess (CSS choice)
F.Berends at al. and J. Blümlein et al. used $\mu_F^2 = zs$

Numerical results for differential distribution: $\mathcal{O}(\alpha^1)$

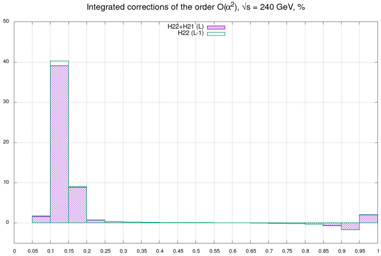
Try to reproduce NLO contributions by adjusting μ_F in LO
[U.V., A. Arbuzov, 2511.00437]

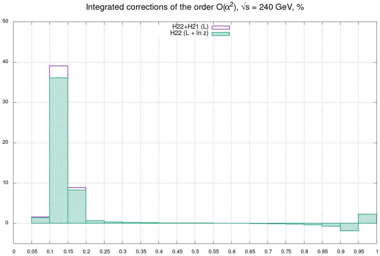


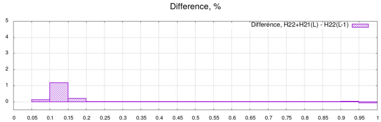


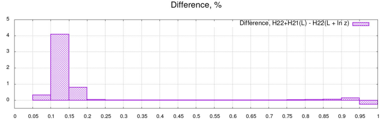
Numerical results for differential distribution: $\mathcal{O}(\alpha^2)$

Try to reproduce NLO contributions by adjusting μ_F in LO



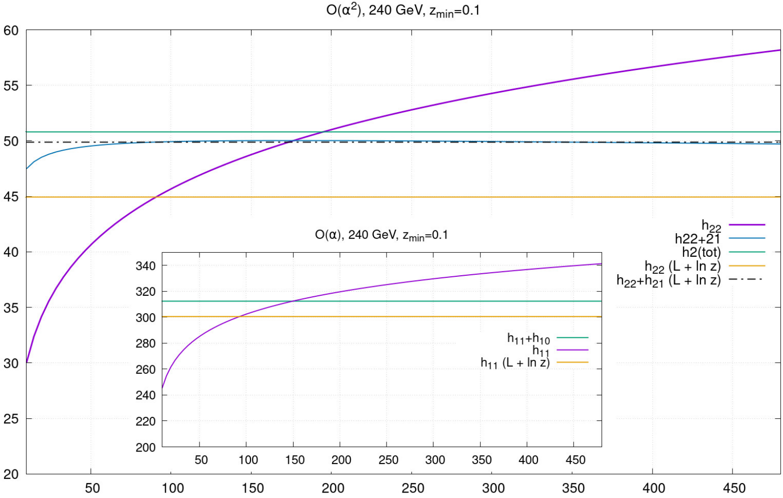






Numerical results for total cross section at 240 GeV

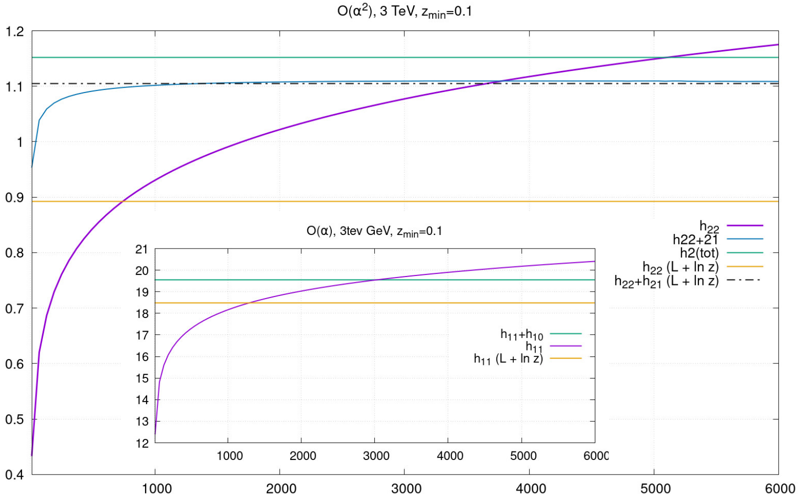
Compare LO, NLO, and NNLO results as functions of μ_F



Corrections in % vs. factorization scale μ_F for $\sqrt{s} = 240$ GeV, $z_{min} = 0.1$

Numerical results for total cross section at $\sqrt{s} = 3$ TeV

Compare LO, NLO, and NNLO results as functions of μ_F



Corrections in % vs. factorization scale μ_F for $\sqrt{s} = 3$ TeV, $z_{min} = 0.1$

Factorization scale choice: preliminary conclusions

Sensitivity to factorization scale choice is relevant numerically even in QED
Higher-order calculations are required to reduce the uncertainty
The comparison of several concrete schemes shows:

- CSS — Conventional Scale Setting (μ_F= hard momentum transfer) - fails
- FAC — Fastest Apparent Convergence - good
- PMS — Principle of Minimal Sensitivity - reasonable
- BLM — Brodsky-Lepage-Mackenzie - not applicable
- PMC — Principle of Maximal Conformality - not applicable
- ...

Factorization scale variation $\mu_F \to \mu_F/2,\ \mu_F \times 2$

Calculated (Δ) and estimated (δ) by variation of factorization scale by factor of 2 corrections of the orders $\mathcal{O}(\alpha^2)$ and $\mathcal{O}(\alpha^3)$

Calculated and estimated uncertainties in %

	$\mathcal{O}(\alpha^2)$				$\mathcal{O}(\alpha^3)$	
	LL		NLL		LL	
	Δ_2^{LL}	δ_2^{LL}	Δ_2^{NLL}	δ_2^{NLL}	Δ_3^{LL}	δ_3^{LL}
$\sqrt{s} = M_z$, $z_{min} = 0.1$	0.436	0.524	0.0064	0.0250	0.0250	0.0499
$\sqrt{s} = M_z$, $z_{min} = 0.5$	0.436	0.5246	0.0063	0.0250	0.0249	0.0499
$\sqrt{s} = M_z$, $z_{min} = 0.9$	0.440	0.529	0.0063	0.0252	0.0249	0.0497
$\sqrt{s} = 240$ GeV, $z_{min} = 0.1$	2.468	5.569	0.5178	0.1479	0.5833	0.1015
$\sqrt{s} = 240$ GeV, $z_{min} = 0.5$	0.1142	0.1057	0.0088	0.0061	0.0014	0.0002
$\sqrt{s} = 240$ GeV, $z_{min} = 0.9$	0.073	0.0403	0.0019	0.0039	0.0216	0.0294

$$\Delta_2^{\mathrm{LO}} = h_{21}, \quad \delta_2^{\mathrm{LO}} = \frac{|h_{22} - h_{22}(1/2)| + |h_{22} - h_{22}(2)|}{2}$$

$$\Delta_2^{\mathrm{NLO}} = h_{20}, \quad \delta_2^{\mathrm{NLO}} = \frac{|h_{22} + h_{21} - (h_{22} + h_{21})(1/2)|}{2} + \frac{|h_{22} + h_{21} - (h_{22} + h_{21})(2)|}{2}$$

$$\Delta_3^{\mathrm{LO}} = h_{32}, \quad \delta_3^{\mathrm{LO}} = \frac{|h_{33} - h_{33}(1/2)| + |h_{33} - h_{33}(2)|}{2}$$

Conclusions

- Current and future high-precision HEP experiments challenge theory. New calculations of two-loop and higher-order corrections within QED and full SM are required
- There is progress in NLO and NNLO QED PDFs and fragmentation functions
- QED provides explicit results and serves for various tests
- Optimization of factorization scale and scheme choices is important as in QCD as well as in QED (and in EW)
- There is no perfect choice, compromises are inevitable

Next steps:

- NLO exponentiation in the proposed DIS-like scheme (in progress)
- Implementation into ZFITTER (in progress)
- Muon and (light) hadron contributions (in progress)
- BFKL effects in QED are large (should be interesting) Monte Carlo applications (within SANC)

Thank you for your attention!

Contributions of different orders

$$\frac{d\sigma_{\bar{e}e}}{ds'} = \sigma^{(0)}\Big[D_{\bar{e}\bar{e}} \otimes D_{ee} \otimes \sigma_{\bar{e}e} + D_{\gamma e} \otimes D_{ee} \otimes \sigma_{e\gamma} + D_{ee} \otimes D_{e\bar{e}} \otimes \sigma_{ee} +$$
$$+ D_{\gamma e} \otimes D_{\bar{e}\bar{e}} \otimes \sigma_{\bar{e}\gamma} + D_{\gamma e} \otimes D_{\gamma\bar{e}} \otimes \sigma_{\gamma\gamma} + D_{\gamma e} \otimes D_{e\bar{e}} \otimes \sigma_{e\gamma}$$
$$+ D_{\bar{e}e} \otimes D_{\bar{e}\bar{e}} \otimes \sigma_{\bar{e}\bar{e}} + D_{\bar{e}e} \otimes D_{\gamma\bar{e}} \otimes \sigma_{\bar{e}\gamma} + D_{\bar{e}e} \otimes D_{e\bar{e}} \otimes \sigma_{\bar{e}e}\Big]$$

Таблица: Contributions of different orders

i \ j	$\bar{e}$	γ	e
e	$D_{ee}D_{\bar{e}\bar{e}}\sigma_{e\bar{e}}$ LO (1)	$D_{ee}D_{\gamma\bar{e}}\sigma_{e\gamma}$ NLO ($\alpha^2 L$)	$D_{ee}D_{e\bar{e}}\sigma_{ee}$ NNLO ($\alpha^4 L^2$)
γ	$D_{\gamma e}D_{\bar{e}\bar{e}}\sigma_{\gamma\bar{e}}$ NLO ($\alpha^2 L$)	$D_{\gamma e}D_{\gamma\bar{e}}\sigma_{\gamma\gamma}$ NNLO ($\alpha^4 L^2$)	$D_{\gamma e}D_{e\bar{e}}\sigma_{\gamma e}$ NLO ($\alpha^4 L^3$)
$\bar{e}$	$D_{\bar{e}e}D_{\bar{e}\bar{e}}\sigma_{\bar{e}\bar{e}}$ NNLO ($\alpha^4 L^2$)	$D_{\bar{e}e}D_{\gamma\bar{e}}\sigma_{\bar{e}\gamma}$ NLO ($\alpha^4 L^3$)	$D_{\bar{e}e}D_{e\bar{e}}\sigma_{\bar{e}e}$ LO ($\alpha^4 L^4$)

Factorization in NLO

In *Berends(1987)*, *Blumlein(2011)* factorization scale was chosen

$$\mu_F^2 = sz.$$

Then the large log

$$L = \ln(s/m_e^2) + \ln z$$

In the expression for one-loop cross section in *Blumlein(2011)*, the variable $y = z/x$ was exchanged into x:

$$\left[\bar{\delta}^{(1)}_{\bar{e}e}(sx)\right]^* = \frac{\alpha}{\pi}\left\{\left[\frac{1+y^2}{1-y}\right]_+ \ln y + 2(1+y^2)\left[\frac{\ln(1-y)}{1-y}\right]_+ + \right.$$
$$\left. +\delta(1-y)\left(2\zeta_2 - \frac{1}{2}\right)\right\}$$

.

In the approachof *Berends(1987)*, *Blumlein(2011)* this logarithm is integrated. In our calculations $\ln z$ is nt integrated in convolutions (the variable of integration is y):

$$\bar{\delta}^{(1)}_{\bar{e}e}(sx) = \frac{\alpha}{\pi}\left\{\left[\frac{1+y^2}{1-y}\right]_+ (\ln z - \ln y) + 2(1+y^2)\left[\frac{\ln(1-y)}{1-y}\right]_+ + \right.$$
$$\left. +\delta(1-y)\left(2\zeta_2 - \frac{1}{2}\right)\right\}$$

e^+e^--annihilation

$$\frac{d\sigma^{\mathrm{NLO}}_{\bar{e}e}(s')}{ds'} = \sigma^{(0)}_{\bar{e}e}\left\{1 + \sum_{i=1}^{\infty}\left(\frac{\alpha}{2\pi}\right)^i \sum_{j=i-1}^{i} c_{ij} L^j + \mathcal{O}(\alpha^i L^{i-2})\right\}$$

$$h_{ij} = \left(\frac{\alpha}{2\pi}\right)^i L^j c_{ij}$$

$$\sigma_{e^+e^-} = \sum_{i,j}\left(h^{\Delta}_{ij}\sigma^{(0)}(1) + \int_{z_{min}}^{1-\Delta} dz\sigma^{(0)}(z) h^{\theta}_{ij}(z)\right)$$

$$h^{num}_{ij} = h^{\Delta}_{ij}\sigma^{(0)}(1) + \int_{z_{min}}^{1-\Delta} dz\sigma^{(0)}(z) h^{\theta}_{ij}(z)$$

$\Delta = 10^{-7},\ 10^{-8}$

Numerical estimations, %, for $\mu_F^2 = s$

z_{min} is defined by experimental conditions

$\sqrt{s} = 160$ GeV, $z_{min} = 0.5$

h_{11}	h_{10}	h_{22}	h_{21}	h_{20}	h_{33}	h_{32}	h_{44}	h_{43}	h_{55}
γ									
9.81624	0.26017	-1.28618	0.20722	0.01521	-0.00845	-0.00714	0.00530	-0.00153	-0.00020
Pairs									
0	0	0.13182	-0.05573	-0.02155	-0.03105	0.01278	-0.00171	0.00046	-0.00100
Full									
9.81624	0.26017	-1.15435	0.15149	-0.00631	-0.03949	0.00563	0.00359	-0.00107	-0.00120

$\sqrt{s} = 240$ GeV, $z_{min} = 0.5$

h_{11}	h_{10}	h_{22}	h_{21}	h_{20}	h_{33}	h_{32}	h_{44}	h_{43}	h_{55}
γ									
3.45872	0.51562	-1.05495	0.14324	-0.00510	0.02506	-0.01176	0.00273	-0.00067	-0.00018
Pairs									
0	0	0.05956	-0.02900	-0.00366	-0.02628	0.01132	-0.00057	-0.00017	-0.00080
Full									
3.45872	0.51562	-0.99539	0.11424	-0.00876	-0.00122	-0.00044	0.00216	-0.00084	-0.00098

Numerical estimations, %, for $\mu_F^2 = s$

$\sqrt{s} = 160$ GeV, $z_{min} = 0.1$

h_{11}	h_{10}	h_{22}	h_{21}	h_{20}	h_{33}	h_{32}	h_{44}	h_{43}	h_{55}
γ									
335.7996	-12.6201	17.1479	0.6467	-0.11553	-1.8223	0.5781	-0.0348	-0.0102	0.0056
Pairs									
0	0	8.3539	-1.8438	-0.13215	0.0043	-0.0647	-0.0606	0.0357	-0.0922
Full									
335.7996	-12.6201	25.5019	-1.1971	- 0.24768	-1.8182	0.5135	-0.0953	0.0255	-0.0870

$\sqrt{s} = 240$ GeV, $z_{min} = 0.1$

h_{11}	h_{10}	h_{22}	h_{21}	h_{20}	h_{33}	h_{32}	h_{44}	h_{43}	h_{55}
γ									
324.1463	-11.7621	27.1298	-0.9909	-0.0809	-0.6242	0.6140	-0.0664	0.0149	0.0012
Pairs									
0	0	25.3337	-1.4772	-0.43752	-0.1401	-0.0207	-0.0269	0.0460	-0.1792
Full									
324.1463	-11.7621	52.4635	-2.4680	-0.51778	-0.7643	0.5934	-0.0932	0.0609	-0.1780

Factorization scale choice

Redistribution of the corrections:

$c_{00}\alpha^0$ (Born)

$c_{11}\alpha^1 L^1$	$c_{10}\alpha^1 L^0$		
$c_{22}\alpha^2 L^2$	$c_{21}\alpha^2 L^1$	$c_{20}\alpha^2 L^0$	
$c_{33}\alpha^3 L^3$	$c_{32}\alpha^3 L^2$	$c_{31}\alpha^3 L^1$	$c_{30}\alpha^3 L^0$

$\dots$

Matching equality for $\mathcal{O}(\alpha^2)$:

$$\left(\frac{\alpha}{2\pi}\right)^2 L^2 c_{22} + \left(\frac{\alpha}{2\pi}\right)^2 L c_{21} + \left(\frac{\alpha}{2\pi}\right)^2 c_{20}$$
$$= \left(\frac{\alpha}{2\pi}\right)^2 (L+\Delta L)^2 \hat{c}_{22} + \left(\frac{\alpha}{2\pi}\right)^2 (L+\Delta L)\hat{c}_{21} + \left(\frac{\alpha}{2\pi}\right)^2 \hat{c}_{20}$$

c_{ij} - coefficients for the initial μ_F choice, $\hat{c}_{ij}$ - for the new choice $\hat{\mu}_F$, $\Delta L = \ln(\hat{\mu}_F^2/\mu_F^2)$ is the shift of the large logarithm value.

$$\hat{c}_{22} = c_{22},$$
$$\hat{c}_{21} = c_{21} - 2\Delta L c_{22},$$
$$\hat{c}_{20} = c_{20} - \Delta L c_{21} + (\Delta L)^2 c_{22}$$