Cosmological dynamics in the scalar-tensor theory of gravity with a generalized non-minimal coupling

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Scalar-tensor theories

$$S = \int d^4x\sqrt{-g}\left[F(\phi)R - Z(\phi)g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - 2U(\phi)\right] + S_m\left[\psi_m, g_{\mu\nu}\right]$$

- generalizations of the Brans-Dicke theories
- the scalar field is
  - minimally coupled with ordinary matter (physical or Jordan frame)
  - non-minimally coupled with the scalar curvature by the term $F(\phi)R$

Notice: Non-minimal coupling of the scalar field with the scalar curvature is provided by the terms $F(\phi)R$

Horndeski theory

In 1974, *Gregory Walter Horndeski* derived the action of the most general scalar-tensor theories with second-order equations of motion
[G.Horndeski, *Second-Order Scalar-Tensor Field Equations in a Four-Dimensional Space*, IJTP **10**, 363 (1974)]

Horndeski Lagrangian:

$$L_{\rm H} = \sqrt{-g}\,(\mathcal{L}_2 + \mathcal{L}_3 + \mathcal{L}_4 + \mathcal{L}_5)$$

$$\mathcal{L}_2 = G_2(\phi, X)\ ,$$
$$\mathcal{L}_3 = G_3(\phi, X)\,\Box\phi\ ,$$
$$\mathcal{L}_4 = G_4(\phi, X)R - 2G_{4,X}(\phi, X)(\Box\phi^2 - \phi^{\mu\nu}\phi_{\mu\nu})\ ,$$
$$\mathcal{L}_5 = G_5(\phi, X)G_{\mu\nu}\phi^{\mu\nu} + \tfrac{1}{3}G_{5,X}(\phi, X)(\Box\phi^3 - 3\,\Box\phi\,\phi_{\mu\nu}\phi^{\mu\nu} + 2\,\phi_{\mu\nu}\phi^{\mu\sigma}\phi^{\nu}{}_{\sigma})\,,$$

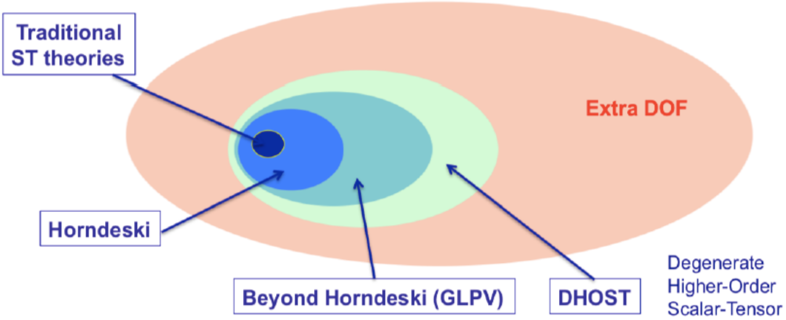
where $\phi_\mu = \nabla_\mu\phi$, $\phi_{\mu\nu} = \nabla_\mu\nabla_\nu\phi$, $X = -\frac{1}{2}(\nabla\phi)^2$,
and $G_a(\phi, X)$ are four arbitrary functions,

Subclasses of the Horndeski theory

$$\mathcal{L}_H = \mathcal{L}\{G_2, G_3, G_4, G_5\}$$

- Hilbert-Einstein action (GR):
  $G_4(\phi, X) = \frac{1}{2}M_{Pl}^2 \quad \rightarrow \quad \mathcal{L}_H \sim \frac{1}{2}M_{Pl}^2 R$
- Nonminimal coupling: $G_4(\phi, X) = f(\phi) \quad \rightarrow \quad \mathcal{L}_H \sim f(\phi)R$
- GR with a scalar field: $G_2(\phi, X) = \epsilon X - V(\phi)$
- k-essence: $G_2 = K(\phi, X)$
- Kinetic gravity braiding (KGB):
  $G_3 = B(\phi, X) \quad \rightarrow \quad \mathcal{L}_H \sim B(\phi, X)\Box\phi$
- Nonminimal kinetic coupling:
  $G_5(\phi, X) = \eta\phi \quad \rightarrow \quad \mathcal{L}_H \sim \eta G^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi$
- Fab Four, Gallileons, etc.

Extended scalar-tensor theories



Landscape of scalar-tensor theories
D. Langlois, Dark energy and modified gravity in degenerate higher-order scalar-tensor (DHOST) theories: A review Int. J. Mod. Phys. D 28 (2019), no. 05 1942006

DHOST theories

$$S = \int d^4x \sqrt{-g}\Big[F_{(2)}(\phi, X)R + P(\phi, X) + Q(\phi, X)\Box\phi$$

$$+F_{(3)}(\phi, X)\, G_{\mu\nu}\phi^{\mu\nu} + \sum_{a=1}^{5} A_a(\phi, X)L_a^{(2)} + \sum_{a=1}^{10} B_a(\phi, X)L_a^{(3)}\Big]$$

$$L_1^{(2)} = \phi_{\mu\nu}\phi^{\mu\nu}\,, \qquad L_2^{(2)} = (\Box\phi)^2\,, \qquad L_3^{(2)} = (\Box\phi)\phi^\mu\phi_{\mu\nu}\phi^\nu\,,$$
$$L_4^{(2)} = \phi^\mu\phi_{\mu\rho}\phi^{\rho\nu}\phi_\nu\,, \qquad L_5^{(2)} = (\phi^\mu\phi_{\mu\nu}\phi^\nu)^2\,.$$

$$L_1^{(3)} = (\Box\phi)^3\,, \quad L_2^{(3)} = (\Box\phi)\,\phi_{\mu\nu}\phi^{\mu\nu}\,, \quad L_3^{(3)} = \phi_{\mu\nu}\phi^{\nu\rho}\phi_\rho^\mu\,,$$
$$L_4^{(3)} = (\Box\phi)^2\,\phi_\mu\phi^{\mu\nu}\phi_\nu\,, \quad L_5^{(3)} = \Box\phi\,\phi_\mu\phi^{\mu\nu}\phi_{\nu\rho}\phi^\rho\,, \quad L_6^{(3)} = \phi_{\mu\nu}\phi^{\mu\nu}\phi_\rho\phi^{\rho\sigma}\phi_\sigma\,,$$
$$L_7^{(3)} = \phi_\mu\phi^{\mu\nu}\phi_{\nu\rho}\phi^{\rho\sigma}\phi_\sigma\,, \quad L_8^{(3)} = \phi_\mu\phi^{\mu\nu}\phi_{\nu\rho}\phi^\rho\,\phi_\sigma\phi^{\sigma\lambda}\phi_\lambda\,,$$
$$L_9^{(3)} = \Box\phi\,(\phi_\mu\phi^{\mu\nu}\phi_\nu)^2\,, \quad L_{10}^{(3)} = (\phi_\mu\phi^{\mu\nu}\phi_\nu)^3\,.$$

Non-minimal coupling of the scalar field with curvature

Notice: There are only two qualitatively different terms describing non-minimal coupling of the scalar field with curvature: $M(\phi, X)R$ and $N(\phi, X)G^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi$.

- $M(\phi, X)R$ — Brans-Dicke-like theories
- $N(\phi, X)G^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi$ — theories with non-minimal derivative coupling

Theory with a generalized nonminimal coupling

Action:

$$S = \frac{1}{2}\int d^4x\sqrt{-g}\left[(1-\xi\phi^2)R - (g^{\mu\nu} + \eta G^{\mu\nu})\nabla_\mu\phi\nabla_\nu\phi - 2V(\phi)\right]$$

Units: $M_{Pl}^2 = 1$

Dynamic systems in the literature:

- $\eta = 0$, $\xi \neq 0$
  Szydlowski et al, 2007-2014; Sami et al, 2012, 2014; and many others
- $\eta \neq 0$, $\xi = 0$
  Skugoreva, S.S., Toporensky, 2013; Matsumoto, S.S., 2015, 2018

Cosmological model

Ansatz:

$$ds^2 = -dt^2 + a^2(t)d\mathbf{x}^2,$$
$$\phi = \phi(t)$$

$a(t)$ *cosmological factor*, $H = \dot{a}/a$ *Hubble parameter*

Field equations:

$$3H^2 F = \frac{1}{2}\dot{\phi}^2 + V(\phi) - \frac{9}{2}\eta H^2\dot{\phi}^2 + 6\xi H\phi\dot{\phi} \tag{5}$$

$$(2\dot{H} + 3H^2)F = -\frac{1}{2}\dot{\phi}^2 + V(\phi) - \frac{1}{2}\eta\left((2\dot{H} + 3H^2)\dot{\phi}^2 + 4H\dot{\phi}\ddot{\phi}\right) + \xi\left(4H\phi\dot{\phi} + 2\phi\ddot{\phi} + 2\dot{\phi}^2\right) \tag{6}$$

$$\ddot{\phi} + 3H\dot{\phi} - 3\eta\left(H^2\ddot{\phi} + 2H\dot{H}\dot{\phi} + 3H^3\dot{\phi}\right) + \xi\phi(\dot{H} + 2H^2) = -V_\phi \tag{7}$$

where $F = 1 - \xi\phi^2$

Dynamic system

Dynamic variables

$$x = \frac{\dot\phi^2}{6H^2 F}, \quad y = \frac{3\eta\dot\phi^2}{2F}, \quad g = \frac{2\xi\phi\dot\phi}{HF}, \quad z = \frac{V(\phi)}{3H^2 F}, \quad v = \frac{\dot\phi}{H\phi}$$

Note that now the modified Friedmann equation reads $x + y + g + z = 1$

Dynamic system

$$\frac{dy}{dN} = y(2\delta + g), \quad \frac{dg}{dN} = g(v + \delta + \varepsilon + g), \quad \frac{dv}{dN} = v(\delta + \varepsilon - v),$$

$$x = \frac{gv}{12\xi}, \quad z = 1 - x - y - g,$$

where $N = \ln a$ is the number of e-folds,

and $\varepsilon = -\frac{\dot H}{H^2}$, $\delta = \frac{\ddot\phi}{H\dot\phi}$ are the slow-roll parameters (functions of y, g, v)

Stationary points $(y' = 0, g' = 0, v' = 0)$

The case $V(\phi) = 0$ $(z = 0)$ (no potential)

1. $y = 1,\ g = 0,\ v = 0, \quad \varepsilon = \frac{3}{2};$
2. $y = 0,\ g = 1,\ v = 0, \quad \varepsilon = 2;$
3. $y = 1,\ g = 0,\ v = \frac{3}{2}, \quad \varepsilon = \frac{3}{2};$
4. $y = \frac{r^2}{24\xi} - r + 1,\ g = r,\ v = -\frac{r}{2}$, where

$$\frac{r^2}{24\xi} - r + 1 = 0, \quad \varepsilon = 3 - r$$

$$\frac{r^3}{12\xi} + \left(1 - \frac{1}{2\xi}\right) r^2 - r + 6 = 0, \quad \varepsilon = 0.$$

NB: $\varepsilon = 0$ $(H = \text{const})$ $\forall \xi$

The stationary point classification for $V(\phi) = 0$

No	Type	Condition	w_{eff}
1	Repeller	$\eta < 0$	0
2	Attractor	$\xi > 0$	$\frac{1}{3}$
3	Saddle	$\eta < 0$	0
$4a_1$	Attractor	$\xi < 0$	$1 - 8\xi + 8\sqrt{\xi(\xi - \frac{1}{6})} > 1$
$4a_2$	Saddle	$\xi < 0 \cup \frac{1}{6} < \xi < \frac{3}{16}$	$1 - 8\xi - 8\sqrt{\xi(\xi - \frac{1}{6})} \in (-1, 1)$
		$\xi > \frac{1}{6}$	$1 - 8\xi + 8\sqrt{\xi(\xi - \frac{1}{6})} \in (-\frac{1}{3}, \frac{1}{3})$
$4a_3$	Repeller	$\xi > \frac{3}{16}$	$1 - 8\xi - 8\sqrt{\xi(\xi - \frac{1}{6})} < -1$
$4b_1$	Attractor	$\eta > 0,\ \xi > 0$	-1
$4b_2$	Saddle	$\eta > 0,\ 0 < \xi < \frac{1}{6}$	-1
		$\eta < 0,\ \frac{1}{6} < \xi < \xi_* \approx 0.19$	-1
$4b_3$	Repeller	$\eta > 0,\ \xi < \frac{3}{16}$	-1
		$\eta < 0,\ \frac{3}{16} < \xi < \xi_* \approx 0.19$	-1

Phase portraits ($\eta > 0,\ V = 0$)

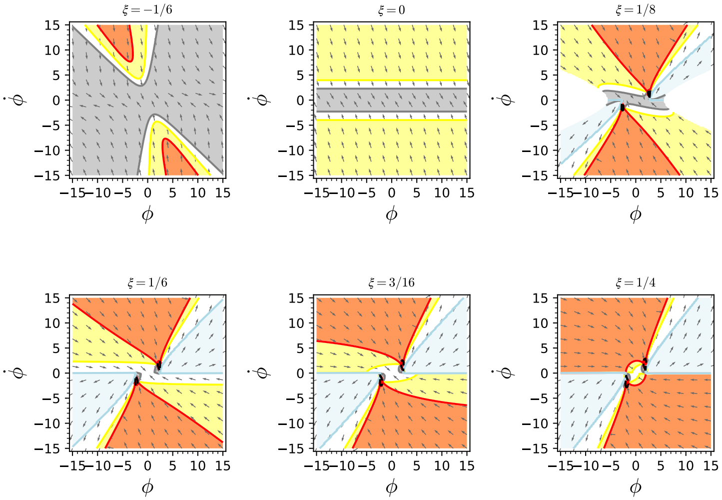
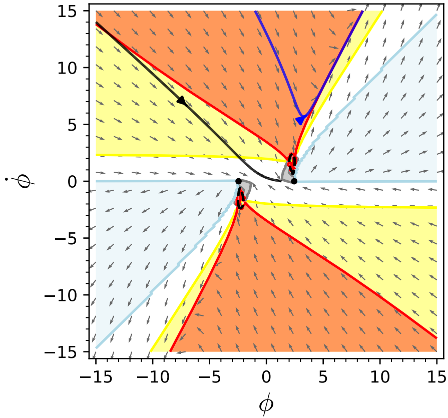


Figure: White and gray zones – expansion with deceleration, $w \leq 1/3$ and $w > 1/3$. Yellow and red zones – expansion with acceleration, $w < -1/3$ and $w < -1$, $\dot{H} > 0$. Here $\eta = 1/9$.

Examples of phase trajectories and corresponding cosmological evolution in the case $V(\phi) \equiv 0$



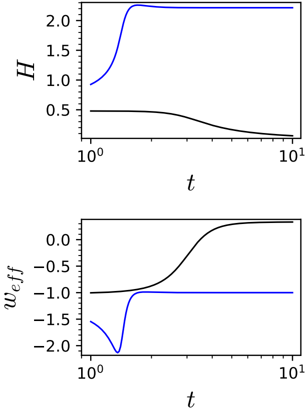


Figure: The case $V(\phi) \equiv 0$ with $\eta = 1/9$, $\xi = 1/6$. Initial conditions: (i) $\phi_i = -15, \dot{\phi}_i = 14$ (black curve), (ii) $\phi_i = -1, \dot{\phi}_i = 15$ (blue curve).

The case $V(\phi) = \frac{1}{2}m^2\phi^2$

Generally, one has 9 stationary points, and among them one special:

$$y = r_1, \quad g = r_2, \quad v = -\frac{r_2}{2}, \quad \varepsilon = 0$$

where $r_1 = -\frac{3}{2} \cdot \frac{r_2^2(1-\frac{1}{4\xi})+r_2}{r_2+3}$, $r_1\left(\frac{r_2^2}{24\xi} - r_2 - r_1 + 1\right) = -\frac{\eta V_0}{8\xi^2}\, r_2^2$

Since $\varepsilon = 0$, one has $H = \mathrm{const}$ for *all* ξ, i.e. a quasi-de Sitter asymptotic.

Supposing $|\eta m^2| \ll 1$, one has:

$$\begin{cases} H^2 \approx -\frac{(4\xi-1)^2 m^2}{2\xi(6\xi-1)(16\xi-3)}, & \\ H^2 \approx \frac{1}{3\eta}\left(1 + \frac{1}{3\xi}\right), & |\xi| \gg 1, \\ H^2 \approx \frac{1}{9\eta}\left(1 - \frac{10}{3}\xi\right) + \frac{1}{9}m^2, & |\xi| \ll 1. \end{cases}$$

Examples of phase trajectories and corresponding cosmological evolution in the case $V(\phi) = \frac{1}{2}m^2\phi^2$

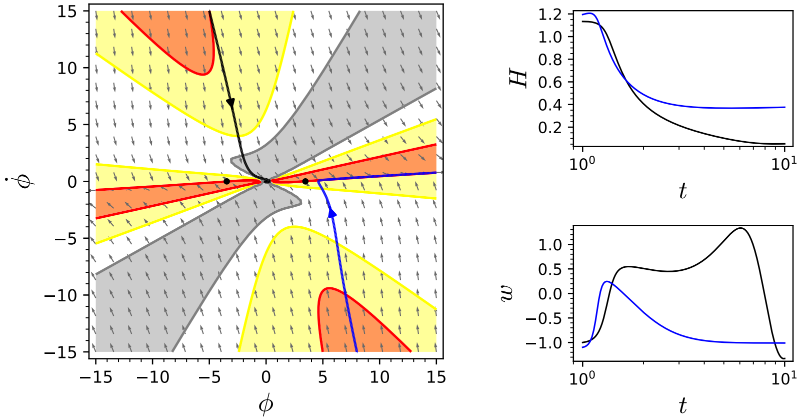


Figure: The case $V(\phi) = \frac{1}{2}m^2\phi^2$ with $\eta = 1/9$, $\xi = -1/12$, $m = 1/3$. Initial conditions: (i) $\phi_i = -5, \dot\phi_i = 15$ (black curve), (ii) $\phi_i = 8, \dot\phi_i = 15$ (blue curve).

Conclusions

- At early stage of the Universe evolution, $t \to -\infty$, there exists the general quasi-de Sitter stage (primary kinetic inflation) in case $\eta > 0$ and *any* ξ for potentials $V(\phi) \propto \phi^n$ with $n \leq 2$, including $V(\phi) \equiv 0$. More steep potentials with $n > 2$ destroy the kinetic inflation.
- A late time evolution depends on initial conditions. In the case $\eta > 0$ there exist scenarios with a stable quasi-de Sitter stage (secondary inflation) at $t \to +\infty$, such that.

$$\begin{cases} H^2 \approx \frac{1}{3\eta}\left(1 + \frac{1}{3\xi}\right), & |\xi| \gg 1, \\ H^2 \approx \frac{1}{9\eta}\left(1 - \frac{10}{3}\xi\right) + \frac{1}{9}m^2, & |\xi| \ll 1. \end{cases}$$

THANKS FOR YOUR ATTENTION!