Extending Anisotropic Einstein-Dilaton-Maxwell Gravity for Holographic QCD

K.A. Rannu

Peoples Friendship University of Russia (RUDN)
Steklov Mathematical Institute (MI RAS)

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with I.Ya. Aref'eva, V. Zlobin A. Golubtsova, V. Nerovnova
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Black hole thermodynamics

0-th law	thermal equilibrium, T	mechanical equilibrium, κ
1-st law	$dE = TdS - pdv + \mu dN$	$dM = \frac{\kappa}{8\pi} dA + \Omega dJ + \Phi dQ$
2-nd law	$\delta S \geq 0$	$\delta A \geq 0$
3-rd law (in Planck form)	$S \to 0$ as $T \to 0$	$S = \frac{A}{4} \to 0$ as $T \to 0$???

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Schwarzschild BH — no 3-rd law
SchwAdS (global coordinates) — no 3-rd law
SchwPAdS (Poincare patch) — the 3-rd law

$\Downarrow$

This is good for AdS/CFT applications

$\Downarrow$

What for other models, that are interesting for holography?

Application to holography

$$T = \frac{|g'|}{4\pi}\bigg|_{z=z_h} \quad \longrightarrow \quad \text{QGP temperature (Maldacena)}$$

$$s = \frac{1}{4}\sqrt{\prod_{i=1}^{3} g_{ii}}\,\bigg|_{z=z_h} \quad \longrightarrow \quad \text{muliplicity of process (Landau)}$$

$$F = \int_{z_h}^{\infty} s\, dT \quad \longrightarrow \quad \text{background BH-BH phase transition}$$

$$A_t = \mu - \rho z^2 + \ldots \quad \longrightarrow \quad \text{chemical potential, density}$$

Overview

1. Maximal models
2. Particular models
   - Model I: 5D, Gauss anisotropy + Lifshitz anisoptopy
   - Model II: 5D, Lifshitz anisotropy + Lifshitz anisoptopy
   - Model III: 6D, Lifshitz anisotropy + Lifshitz anisoptopy
3. D-dimensional models with electric and magnetic ansatz
4. Conclusions

Holographic model of anisotropic plasma in magnetic field at nonzero chemical potential

I.Aref'eva, KR'18; IA, KR, P.Slepov'21

$$S = \int d^5x \sqrt{-g}\left[R - \frac{f_0(\phi)}{4} F_{(0)}^2 - \frac{f_1(\phi)}{4} F_{(1)}^2 - \frac{f_B(\phi)}{4} F_{(3)}^2 - \frac{1}{2}\partial_M\phi\partial^M\phi - V(\phi)\right]$$

$$ds^2 = \frac{L^2}{z^2}\,\mathfrak{b}(z)\left[-g(z)\,dt^2 + dx^2 + \left(\frac{z}{L}\right)^{2-\frac{2}{\nu}} dy_1^2 + e^{c_B z^2}\left(\frac{z}{L}\right)^{2-\frac{2}{\nu}} dy_2^2 + \frac{dz^2}{g(z)}\right]$$

$$A_{(1)\mu} = A_t(z)\delta_\mu^0 \qquad A_t(0) = \mu \qquad F_{(1)} = q_1\, dy^1 \wedge dy^2 \qquad F_{(B)} = q_3\, dx \wedge dy^1$$

$\mathfrak{b}(z) = e^{2\mathcal{A}(z)}$ $\Leftrightarrow$ quarks mass

"Bottom-up approach"

Heavy quarks (c, b)

$\mathcal{A}(z) = -\,cz^2/4$ — *Andreev, Zakharov'06*

$\mathcal{A}(z) = -\,cz^2/4 + p(c_B)z^4$ — *IA, Hajilou, Rannu, Slepov' 23*

Light quarks (u, d, s)

$\mathcal{A}(z) = -\,a\ln(bz^2+1)$ — *Li, Yang, Yuan'17*

$\mathcal{A}(z) = -\,a\ln\big((bz^2+1)(dz^4+1)\big)$ — *Zhu, Chen, Zhou, Zhang, Huang'25*

Model extension

$$S = \int d^D x \sqrt{-g_D}\left[R - \frac{f_{(0)}(\phi)}{2\cdot 2!}F_{(0)}^2 - \sum_{\mathcal{M}}\frac{f_{(\mathcal{M})}(\phi)}{2\cdot 2!}F_{(\mathcal{M})}^2 - \sum_{\mathcal{M}}\frac{\mathfrak{f}_{(\mathcal{M})}(\phi)}{2\cdot 2!}\mathfrak{F}_{(\mathcal{M})}^2 - \right.$$
$$\left. - \sum_{\mathcal{N}}\frac{h_{(\mathcal{N})}(\phi)}{2\cdot 3!}H_{(\mathcal{N})}^2 - \frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi - V(\phi)\right]$$

$$ds^2 = \frac{L^2\,\mathfrak{b}(z)}{z^2}\left(-g(z)dt^2 + \sum_{i=1}^{D-2}\mathfrak{g}_i(z)dx_i^2 + \frac{dz^2}{g(z)}\right)$$

Scalar field	$\phi = \phi(z),\ V(\phi) = V\big(\phi(z)\big)$
Electric ansatz	$F_{\mu\nu}^{(0)} = \partial_\mu A_\nu^{(0)} - \partial_\nu A_\mu^{(0)},\ A_\mu^{(0)} = A_t(z)\delta_\mu^0,\ \mu = \overline{0, D-1}$
Magnetic ansatz	$F_{(\mathcal{M})} = q_{(\mathcal{M})}\epsilon_{\mathcal{M}pk}\,dx^p \wedge dx^k,\ \mathcal{M}, p, k = \overline{1,3}$
Magnetic ansatz	$\mathfrak{F}_{(\mathcal{M})} = \mathfrak{q}_{(\mathcal{M})}\,dx^{\mathcal{M}} \wedge dx^j,\ \mathcal{M} = \overline{1,3},\ j = \overline{4, D-2}$
Magnetic ansatz	$H_{(\mathcal{N})} = Q_{(\mathcal{N})}\epsilon_{\mathcal{N}pkl}\,dx^p \wedge dx^k \wedge dx^l,\ \mathcal{N}, p, k, l = \overline{1, D-2}$

Model extension

$$S = \int d^D x \sqrt{-g_D} \Bigg[R - \frac{f_{(0)}(\phi)}{2 \cdot 2!} F_{(0)}^2 - \sum_{\mathcal{M}} \frac{f_{(\mathcal{M})}(\phi)}{2 \cdot 2!} F_{(\mathcal{M})}^2 - \sum_{\mathcal{M}} \frac{\mathfrak{f}_{(\mathcal{M})}(\phi)}{2 \cdot 2!} \mathfrak{F}_{(\mathcal{M})}^2 -$$

$D = 6$

$$- \sum_{\mathcal{N}} \frac{h_{(\mathcal{N})}(\phi)}{2 \cdot 3!} H_{(\mathcal{N})}^2 - \frac{1}{2} \partial_\mu \phi \, \partial^\mu \phi - V(\phi) \Bigg]$$

$$ds^2 = \frac{L^2 \, \mathfrak{b}(z)}{z^2} \left(- g(z) dt^2 + \sum_{i=1}^{D-2} \mathfrak{g}_i(z) dx_i^2 + \frac{dz^2}{g(z)} \right)$$

Scalar field	$\phi = \phi(z), \; V(\phi) = V\big(\phi(z)\big)$
Electric ansatz	$F_{\mu\nu}^{(0)} = \partial_\mu A_\nu^{(0)} - \partial_\nu A_\mu^{(0)}, \; A_\mu^{(0)} = A_t(z) \delta_\mu^0, \; \mu = \overline{0,5}$
Magnetic ansatz	$F_{(\mathcal{M})} = q_{(\mathcal{M})} \epsilon_{\mathcal{M}pk} \, dx^p \wedge dx^k, \; \mathcal{M}, p, k = \overline{1,3}$
Magnetic ansatz	$\mathfrak{F}_{(\mathcal{M})} = \mathfrak{q}_{(\mathcal{M})} \, dx^{\mathcal{M}} \wedge dx^4, \; \mathcal{M} = \overline{1,3}$
Magnetic ansatz	$H_{(\mathcal{N})} = Q_{(\mathcal{N})} \epsilon_{\mathcal{N}pkl} \, dx^p \wedge dx^k \wedge dx^l, \; \mathcal{N}, p, k, l = \overline{1,4}$

6D with 10 magnetic ansatz: metric and dilaton equation

$$ds^2 = -g_{00}(z)dt^2 + g_{11}(z)dx_1^2 + g_{22}(z)dx_2^2 + g_{33}(z)dx_3^2 + g_{44}(z)dx_4^2 + g_{55}(z)dx_5^2$$

$$\phi'' + \frac{\phi'}{2}\sum_{\mu}(-1)^{\delta_{\mu 5}}\frac{g'_{\mu\mu}}{g_{\mu\mu}} - \\ - g_{55}\left(\sum_{\mathcal{K}}\frac{q_{\mathcal{K}}^2}{2}\frac{\partial f_{\mathcal{K}}}{\partial\phi}\prod_{i\neq\mathcal{K}}\frac{1}{g_{ii}} + \sum_{\mathcal{M}}\frac{\mathfrak{q}_{\mathcal{M}}^2}{2g_{\mathcal{M}\mathcal{M}}g_{44}}\frac{\partial\mathfrak{f}_{\mathcal{M}}}{\partial\phi} + \sum_{\mathcal{N}}\frac{Q_{\mathcal{N}}^2}{2}\frac{\partial H_{\mathcal{N}}}{\partial\phi}\prod_{j\neq\mathcal{N}}\frac{1}{g_{jj}} + \frac{\partial V(\phi)}{\partial\phi}\right) = 0$$

$$\mathcal{K},\mathcal{M} = \overline{1,3}, \quad \mathcal{N} = \overline{1,4}, \quad \mu,\nu = \overline{0,5}, \quad i = \overline{1,3}, \quad j = \overline{1,4}, \quad k,l = \overline{1,5}$$

6D with 10 magnetic ansatz: Eisntein equations

$$\sum_j \left(\frac{2g''_{jj}}{g_{jj}} - \frac{g'^2_{jj}}{g^2_{jj}} \right) + \sum_{k<l} (-1)^{\delta_{5k}} (-1)^{\delta_{5l}} \frac{g'_{kk}}{g_{kk}} \frac{g'_{ll}}{g_{ll}} +$$

$$+ g_{55} \left(\sum_{\mathcal{K}} f_{\mathcal{K}} q^2_{\mathcal{K}} \prod_{i \neq \mathcal{K}} \frac{1}{g_{ii}} + \sum_{\mathcal{M}} \frac{\mathfrak{f}_{\mathcal{M}} \mathfrak{q}^2_{\mathcal{M}}}{g_{\mathcal{M}\mathcal{M}} g_{44}} + \sum_{\mathcal{N}} H_{\mathcal{N}} Q^2_{\mathcal{N}} \prod_{j \neq \mathcal{N}} \frac{1}{g_{jj}} + \frac{\phi'^2}{g_{55}} + 2V(\phi) \right) = 0$$

$$\sum_{\mu \neq j} (1 - \delta_{\mu 5}) \left(\frac{2g''_{\mu\mu}}{g_{\mu\mu}} - \frac{g'^2_{\mu\mu}}{g^2_{\mu\mu}} \right) + \sum_{\mu < \nu \neq j} (-1)^{\delta_{5\mu}} (-1)^{\delta_{5\nu}} \frac{g'_{\mu\mu}}{g_{\mu\mu}} \frac{g'_{\nu\nu}}{g_{\nu\nu}} +$$

$$+ g_{55} \left(\sum_{\mathcal{K}} (-1)^{\delta_{\mathcal{K}j}} (-1)^{1+\delta_{4j}} f_{\mathcal{K}} q^2_{\mathcal{K}} \prod_{i \neq \mathcal{K}} \frac{1}{g_{ii}} + \sum_{\mathcal{M}} (-1)^{\delta_{\mathcal{M}j}} (-1)^{\delta_{4j}} \frac{\mathfrak{f}_{\mathcal{M}} \mathfrak{q}^2_{\mathcal{M}}}{g_{\mathcal{M}\mathcal{M}} g_{44}} + \right.$$

$$\left. + \sum_{\mathcal{N}} (-1)^{1+\delta_{\mathcal{N}j}} H_{\mathcal{N}} Q^2_{\mathcal{N}} \prod_{i \neq \mathcal{N}} \frac{1}{g_{ii}} + \frac{\phi'^2}{g_{55}} + 2V(\phi) \right) = 0, \qquad j = \overline{1,4}$$

$$\frac{g'_{00}}{g_{00}} \sum_j \frac{g'_{jj}}{g_{jj}} + \prod_{k<l} (1 - \delta_{l5}) \frac{g'_{kk}}{g_{kk}} \frac{g'_{ll}}{g_{ll}} +$$

$$+ g_{55} \left(\sum_{\mathcal{K}} f_{\mathcal{K}} q^2_{\mathcal{K}} \prod_{i \neq \mathcal{K}} \frac{1}{g_{ii}} + \sum_{\mathcal{M}} \frac{\mathfrak{f}_{\mathcal{M}} \mathfrak{q}^2_{\mathcal{M}}}{g_{\mathcal{M}\mathcal{M}} g_{44}} + \sum_{\mathcal{N}} H_{\mathcal{N}} Q^2_{\mathcal{N}} \prod_{j \neq \mathcal{N}} \frac{1}{g_{jj}} - \frac{\phi'^2}{g_{55}} + 2V(\phi) \right) = 0$$

6D solution

$$ds^2 = \mathfrak{B}^2(z)\left(-g(z)\,dt^2 + \mathfrak{g}_1(z)\,dx_1^2 + \mathfrak{g}_2(z)\,dx_2^2 + \mathfrak{g}_3(z)\,dx_3^2 + \mathfrak{g}_4(z)\,dx_4^2 + \frac{dz^2}{g(z)}\right)$$

$$\phi(z) = \pm \int \sqrt{2g_{zz}\left(\frac{G_{zz}}{g_{zz}} + \frac{G_{tt}}{g_{tt}}\right)}\,dz$$

$$A_t(z) = \mu\left(1 - \frac{I(z)}{I(z_h)}\right), \quad \mathfrak{s}(z) = \prod_{\gamma\neq z,t}\sqrt{g_{\gamma\gamma}(z)}, \; I(z) = \int_0^z \frac{\mathfrak{B}^2(\xi)}{\mathfrak{s}(\xi)\,f_0(\phi(\xi))}\,d\xi$$

$$g(z) = \mathfrak{G}_i(z)\Bigg[C\int_{z_h}^z \frac{d\xi}{\mathfrak{s}(\xi)\,\mathfrak{G}_i(\xi)} + + 2\,q_i^2\int_{z_h}^z \frac{d\xi}{\mathfrak{s}(\xi)\,\mathfrak{G}_i(\xi)}\int_{z_h}^{\xi}\frac{\mathfrak{s}(\eta)}{\mathfrak{B}^2(\eta)}\,f_i(\phi(\eta))\prod_{j=1,\,j\neq i}^{3}\frac{1}{\mathfrak{g}_j(\eta)}\,d\eta + \left(\frac{\mu}{I(z_h)}\right)^2\int_{z_h}^z \frac{d\xi}{\mathfrak{s}(\xi)\,\mathfrak{G}_i(\xi)}\int_{z_h}^{\xi}\frac{\mathfrak{B}^2(\eta)}{\mathfrak{s}(\eta)\,f_0(\phi(\eta))}\,d\eta\Bigg], \; i = 1,2,3$$

Talk by V. Zlobin $\quad \mathfrak{G}_i = \frac{1}{\mathfrak{g}_i}\prod_{j=1,j\neq i}^{3}\mathfrak{g}_j \qquad \lim_{z\to 0+} g(z) = 1$

Particular models

Model I $\quad D=5 \qquad F_3 = q_3\, dx^1 \wedge dx^2, \quad F_1 = q_1\, dx^2 \wedge dx^3$

$$ds^2 = \frac{L^2}{z^2}\left(-\,g(z)dt^2 + dx_1^2 + \left(\frac{z}{L}\right)^{2-\frac{2}{\nu}} dx_2^2 + e^{c_B z^2}\left(\frac{z}{L}\right)^{2-\frac{2}{\nu}} dx_3^2 + \frac{dz^2}{g(z)}\right)$$

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Model II $\quad D=5$

i $\quad F_2 = q_2\, dx^1 \wedge dx^3, \quad F_3 = q_3\, dx^1 \wedge dx^2$

ii $\quad F_1 = q_1\, dx^2 \wedge dx^3, \quad F_3 = q_3\, dx^1 \wedge dx^2$

$\mathfrak{b}(z) = e^{cz^n}$ **iii** $\quad F_1 = q_1\, dx^2 \wedge dx^3, \quad F_2 = q_2\, dx^1 \wedge dx^3$

$$ds^2 = \frac{L^2}{z^2}\,\mathfrak{b}(z)\left(-g(z)dt^2 + dx_1^2 + \left(\frac{z}{L}\right)^{2-\frac{2}{\nu_1}} dx_2^2 + \left(\frac{z}{L}\right)^{2-\frac{2}{\nu_2}} dx_3^2 + \frac{dz^2}{g(z)}\right)$$

Particular models

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Model III $\quad D=6 \quad \mathfrak{b}(z)=1,\ \mathfrak{F}_3=\mathfrak{q}_3\, dx^3\wedge dx^4,\ H_1=Q_1\, dx^2\wedge dx^3\wedge dx^4$

$$ds^2=\frac{L^2}{z^2}\,\mathfrak{b}(z)\left(-g(z)dt^2+dx_1^2+\left(\frac{z}{L}\right)^{2-\frac{2}{\nu_1}}dx_2^2+\left(\frac{z}{L}\right)^{2-\frac{2}{\nu_2}}dx_3^2+\left(\frac{z}{L}\right)^{2-\frac{2}{\nu_2}}dx_4^2+\frac{dz^2}{g(z)}\right)$$

Model I solution

$$g(z) = e^{c_B z^2} \frac{\int_z^{z_h} e^{-\frac{3c_B}{2}\xi^2} \xi^{1+\frac{2}{\nu}} \, d\xi}{\int_0^{z_h} e^{-\frac{3c_B}{2}\xi^2} \xi^{1+\frac{2}{\nu}} \, d\xi}, \quad 1 + \frac{1}{\nu} > 0$$

$$s(z_h) = \frac{1}{4} \left(\frac{L}{z_h}\right)^{1+\frac{2}{\nu}} e^{\frac{c_B z_h^2}{2}}, \quad T(z_h) = \begin{cases} \frac{1}{4\pi} \frac{e^{-\frac{c_B}{2} z_h^2} z_h^{1+\frac{2}{\nu}}}{\int_0^{z_h} e^{-\frac{3c_B}{2}\xi^2} \xi^{1+\frac{2}{\nu}} \, d\xi} \\ \frac{1}{2\pi z_h}\left(1 + \frac{1}{\nu}\right), \; c_B = 0 \end{cases}$$

$$f_1(\phi(z)) = -2\left(\frac{L}{z}\right)^{4/\nu} \frac{e^{c_B z^2}}{L^2 q_1^2} \left(1 - \frac{1}{\nu}\right) \left[g'(z)\, z - \left(2 + \frac{2}{\nu} - c_B z^2\right) g(z)\right]$$

$$f_3(\phi(z)) = 2\left(\frac{L}{z}\right)^{2/\nu} \frac{c_B}{q_3^2} \left[g'(z)\, z - \left(\frac{2}{\nu} - c_B z^2\right) g(z)\right]$$

$$\phi'(z)^2 = -\frac{2c_B^2}{z^2}\left(z^4 + \frac{3\nu - 2}{c_B \nu} z^2 + \frac{2 - 2\nu}{c_B^2 \nu^2}\right)$$

Model I results

3-rd law

$$c_B = 0: \quad s(T) = \frac{1}{4}\left(\frac{2\pi L T}{1+\frac{1}{\nu}}\right)^{1+\frac{2}{\nu}} \to 0 \text{ as } T \to 0,\ 1 + \frac{2}{\nu} > 0,\ 1 + \frac{1}{\nu} > 0$$

$$c_B < 0: \quad s(T) \to 0 \text{ as } T \to 0,\ 1 + \frac{1}{\nu} > 0$$

$$c_B > 0: \quad s(z_h) \to +\infty \text{ as } T(z_h) \to 0,\ z_h \to +\infty$$

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NEC

$$f_1(\phi) \geq 0, \quad f_3(\phi) \geq 0, \quad \phi'^2 \geq 0 \implies \begin{cases} c_B = 0,\ \nu \geq 1 \\ c_B < 0,\ \nu \geq 1,\ z_h \leq z_h^{max} \end{cases}$$

$$z_h^{max} = \sqrt{\frac{2 - 3\nu - \sqrt{9\nu^2 - 4\nu - 4}}{2 c_B \nu}}$$

Model I results

3-rd law

$$c_B = 0: \quad s(T) = \frac{1}{4}\left(\frac{2\pi LT}{1+\frac{1}{\nu}}\right)^{1+\frac{2}{\nu}} \to 0 \text{ as } T \to 0,\ 1+\frac{2}{\nu} > 0,\ 1+\frac{1}{\nu} > 0$$

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$$z_h^{max} = \sqrt{\frac{2 - 3\nu - \sqrt{9\nu^2 - 4\nu - 4}}{2c_B\nu}}$$

$$\Downarrow$$

$$c_B < 0: \quad z_h \leq z_h^{max} \quad \Rightarrow \quad T \geq T_{min} = T(z_h^{max})$$

Model I results

3-rd law

$$c_B = 0: \quad s(T) = \frac{1}{4}\left(\frac{2\pi LT}{1+\frac{1}{\nu}}\right)^{1+\frac{2}{\nu}} \to 0 \text{ as } T \to 0,\ 1+\frac{2}{\nu} > 0,\ 1+\frac{1}{\nu} > 0$$

$$c_B < 0: \quad s(T) \to 0 \text{ as } T \to 0,\ 1+\frac{1}{\nu} > 0$$

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$$f_1(\phi) \geq 0, \quad f_3(\phi) \geq 0, \quad \phi'^2 \geq 0 \implies \begin{cases} c_B = 0,\ \nu \geq 1 \\ c_B < 0,\ \nu \geq 1,\ z_h \leq z_h^{max} \end{cases}$$

$$z_h^{max} = \sqrt{\frac{2 - 3\nu - \sqrt{9\nu^2 - 4\nu - 4}}{2c_B\nu}}$$

$$\Downarrow$$

$$c_B < 0: \quad z_h \leq z_h^{max} \quad \Rightarrow \quad T \geq T_{min} = T(z_h^{max})$$

$$c_B = 0: \quad \nu \geq 1$$

Model II solution

$$\mathfrak{b}(z) = e^{cz^n}, \quad g_a(z) = C_a\, z^{\alpha_a} \int_z^{z_h} \mathfrak{b}(\xi)^{-\frac{3}{2}} \xi^{1+\frac{1}{\nu_1}+\frac{1}{\nu_2}-\alpha_a} d\xi, \quad a = \mathbf{i},\ \mathbf{ii},\ \mathbf{iii},$$

$$\alpha_{\mathbf{i}} = 4 - \frac{2}{\nu_1} - \frac{2}{\nu_2}, \quad \alpha_{\mathbf{ii}} = \frac{2}{\nu_1} - \frac{2}{\nu_2}, \quad \alpha_{\mathbf{iii}} = \frac{2}{\nu_2} - \frac{2}{\nu_1}$$

$$s_{div,\,\mathbf{i}}(T) = \frac{1}{2\pi\, L} \cdot \frac{1}{T}, \quad s_{div,\,\mathbf{ii}}(T) = \frac{1 + 1/\nu_1}{4\pi\, L} \cdot \frac{1}{T}$$

$$c = 0, \quad s_{conv,\,\mathbf{i}}(T) = \frac{1}{4}(\pi L\, T)^3, \quad s_{conv,\,\mathbf{ii}}(T) = \frac{1}{4}\left(\frac{2\pi L\, T}{1 + \frac{1}{\nu_1}}\right)^{1+\frac{2}{\nu_1}}$$

$$c < 0, \quad 0 < n < 1, \quad X = \frac{3n|c|}{8\pi\, T}$$

$$s_{conv,\,\mathbf{i}}(T) = \frac{L^3\, e^{\frac{4-n}{n-1}}}{4} X^{-\frac{3}{1-n}} \exp\left(-\frac{3|c|}{2} X^{n/(1-n)}\right)\left[1 + O\left(T^{\,n/(1-n)}\right)\right]$$

$$s_{conv,\,\mathbf{ii}}(T) = \frac{L^{1+\frac{2}{\nu_1}}\, e^{\frac{2+2/\nu_1-n}{n-1}}}{4} X^{-\frac{1+2/\nu_1}{1-n}} \exp\left(-\frac{3|c|}{2} X^{n/(1-n)}\right)\left[1 + O\left(T^{\,n/(1-n)}\right)\right]$$

$$c > 0, \quad s_{conv,\,\mathbf{i}}(T) = \frac{L^3}{16\pi} \frac{n\,(3c/2)^{\frac{4}{n}}}{\Gamma\left(\frac{4}{n}\right)} \cdot \frac{1}{T} + O\left(\left(\ln\frac{1}{T}\right)^{(1-n)/n}\right)$$

$$s_{conv,\,\mathbf{ii}}(T) = \frac{L^{1+\frac{2}{\nu_1}}}{16\pi} \frac{n\,(3c/2)^{\frac{2+2/\nu_1}{n}}}{\Gamma\left(\frac{2+2/\nu_1}{n}\right)} \cdot \frac{1}{T} + O\left(\left(\ln\frac{1}{T}\right)^{(1-n)/n}\right)$$

Model II results

3-rd law

$$s_{div,\,\mathbf{i}}(T),\ s_{div,\,\mathbf{ii}}(T) \sim \frac{1}{T} \to \infty \quad \text{as } T \to 0$$

$$c = 0: \quad s_{conv,\,\mathbf{i}}(T) \to 0, \quad s_{conv,\,\mathbf{ii}}(T) \to 0,\ 1 + \frac{2}{\nu_1} > 0$$

$$c < 0: \quad s_{conv}(T) \to 0,\ 0 < n < 1, \quad \mathfrak{b}(z) = e^{cz^n}$$

$$c > 0: \quad s_{conv,\,\mathbf{i/ii}}(z_h) \to +\infty, \quad T_{conv,\,\mathbf{i/ii}}(z_h) \to 0 \quad \text{as } z_h \to +\infty$$

Model II results

3-rd law

$$s_{div,\,\mathbf{i}}(T),\ s_{div,\,\mathbf{ii}}(T) \sim \frac{1}{T} \to \infty \quad \text{as } T \to 0$$

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NEC

$$c = 0: \begin{cases} \mathbf{i} & \nu_1 = \nu_2 = 1 \\ \mathbf{ii,\ iii} & \nu_1 = \nu_2 \geq 1 \end{cases} \qquad c < 0: \begin{cases} \mathbf{i} & \nu_1 = \nu_2 = 1 \\ \mathbf{ii,\ iii} & \nu_1 = \nu_2 \geq 1 \end{cases}$$

$$c > 0: \ \nu_1 = \nu_2 \geq 1,\ z_h \leq \mathbf{min}(z_1,\, z_2) = \left(\frac{3(1+n) \pm \sqrt{3}\sqrt{\frac{8 - 8\nu_1 + 3(1+n)^2\nu_1^2}{\nu_1^2}}}{3\,c\,n} \right)^{1/n} > 0$$

Model II results

3-rd law

$$s_{div,\,\mathbf{i}}(T),\ s_{div,\,\mathbf{ii}}(T) \sim \frac{1}{T} \to \infty \quad \text{as } T \to 0$$

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$$\Downarrow$$

$$c = 0: \begin{cases} \mathbf{i} & \nu_1 = \nu_2 = 1 \\ \mathbf{ii,\ iii} & \nu_1 = \nu_2 \geq 1 \end{cases} \qquad c < 0: \begin{cases} \mathbf{i} & \nu_1 = \nu_2 = 1, \quad 0 < n < 1 \\ \mathbf{ii,\ iii} & \nu_1 = \nu_2 \geq 1, \quad 0 < n < 1 \end{cases}$$

Model III solution

$$g(z) = C_1 \int_z^{z_h} \frac{\xi^{1+\frac{1}{\nu_1}+\frac{2}{\nu_2}}}{\mathfrak{b}^2(\xi)}\, d\xi$$

$$\mathfrak{b}(z) = 1 : \quad g(z) = 1 - \left(\frac{z}{z_h}\right)^{2+\frac{1}{\nu_1}+\frac{2}{\nu_2}}, \quad 2 + \frac{1}{\nu_1} + \frac{2}{\nu_2} > 0$$

$$\phi(z) = \phi_0 \pm \log\left(\frac{z}{L}\right) \sqrt{\frac{2}{\nu_1^2}(\nu_1 - 1) + \frac{4}{\nu_2^2}(\nu_2 - 1)}$$

$$h_1(\phi) = \frac{2}{L^2 Q_1^2}\left(1 - \frac{1}{\nu_1}\right)\left(2 + \frac{1}{\nu_1} + \frac{2}{\nu_2}\right) \exp\left[\left(\frac{2}{\nu_1} + \frac{4}{\nu_2}\right) \frac{\mp(\phi - \phi_0)}{\sqrt{\frac{2}{\nu_1^2}(\nu_1 - 1) + \frac{4}{\nu_2^2}(\nu_2 - 1)}}\right]$$

$$\mathfrak{f}_3(\phi) = \frac{2}{L^2 \mathfrak{q}_3^2}\left(\frac{1}{\nu_1} - \frac{1}{\nu_2}\right)\left(2 + \frac{1}{\nu_1} + \frac{2}{\nu_2}\right) \exp\left[\frac{4}{\nu_2} \frac{\mp(\phi - \phi_0)}{\sqrt{\frac{2}{\nu_1^2}(\nu_1 - 1) + \frac{4}{\nu_2^2}(\nu_2 - 1)}}\right]$$

$$V(\phi) \equiv -\frac{1}{L^2}\left(4 + \frac{4}{\nu_1} + \frac{6}{\nu_2} + \frac{1}{\nu_1^2} + \frac{2}{\nu_2^2} + \frac{3}{\nu_1 \nu_2}\right)$$

$$s(T) = 4^{\frac{1}{\nu_1}+\frac{2}{\nu_2}} \left(\frac{L\,\pi\,T}{2 + \frac{1}{\nu_1} + \frac{2}{\nu_2}}\right)^{1+\frac{1}{\nu_1}+\frac{2}{\nu_2}}$$

Model III results

3-rd law

$$s(T) \sim T^{1+\frac{1}{\nu_1}+\frac{2}{\nu_2}} \to 0, \; 1+\frac{1}{\nu_1}+\frac{2}{\nu_2} > 0, \; 2+\frac{1}{\nu_1}+\frac{2}{\nu_2} > 0$$

Model III results

3-rd law

$$s(T) \sim T^{\,1+\frac{1}{\nu_1}+\frac{2}{\nu_2}} \to 0,\ 1+\frac{1}{\nu_1}+\frac{2}{\nu_2}>0,\ 2+\frac{1}{\nu_1}+\frac{2}{\nu_2}>0$$

NEC

$$\phi'^2 \geq 0, \quad h_1(\phi) \geq 0, \quad \frac{h_1(\phi)\,Q_1^2}{g_{x_2x_2}} + \mathfrak{f}_3(\phi)\,\mathfrak{q}_3^2 \geq 0$$

$$\frac{1}{\nu_1}\left(1-\frac{1}{\nu_1}\right)+\frac{2}{\nu_2}\left(1-\frac{1}{\nu_2}\right) \geq 0,$$

$$\left(1-\frac{1}{\nu_1}\right)\left(2+\frac{1}{\nu_1}+\frac{2}{\nu_2}\right) \geq 0, \quad \left(1-\frac{1}{\nu_2}\right)\left(2+\frac{1}{\nu_1}+\frac{2}{\nu_2}\right) \geq 0$$

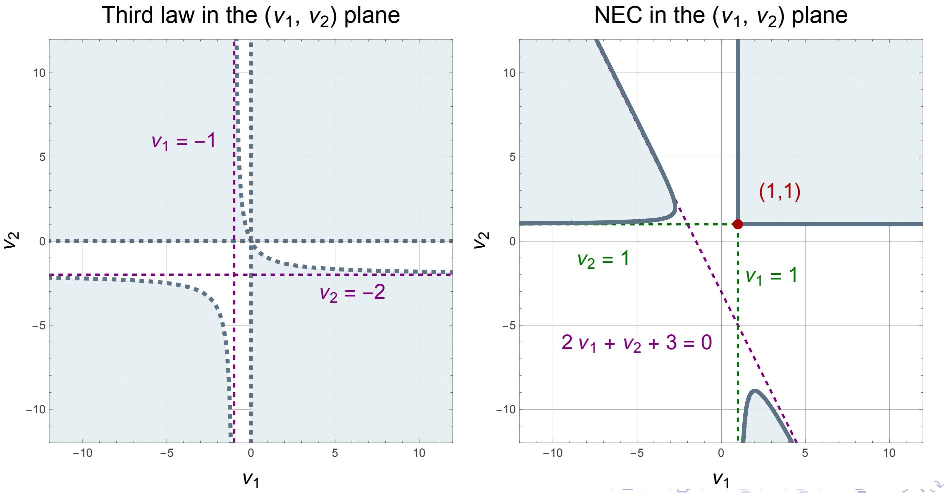
$$\nu_1, \nu_2 \geq 0 \ \Rightarrow \ \nu_1, \nu_2 \geq 1$$

$$\mathrm{sgn}\big(h_1(\phi)\big) = \mathrm{sgn}\left(1-\frac{1}{\nu_1}\right) = 0,\ +1,$$

$$\mathrm{sgn}\big(\mathfrak{f}_3(\phi)\big) = \mathrm{sgn}\left(\frac{1}{\nu_1}-\frac{1}{\nu_2}\right) = -1,\ 0,\ +1$$

Model III results

$$1 + \frac{1}{\nu_1} + \frac{2}{\nu_2} > 0 \qquad \text{NEC} \;\Rightarrow\; \text{3-rd law}$$



Results

Model I D = 5 Lifshitz + Gauss

NEC $c_B \leq 0,\ \nu \geq 1:$ $c_B < 0,\ z_h \leq z_h^{max},\ T_{min} = T(z_h^{max}) > 0$

$c_B = 0$ horizon unbounded

3-rd law + NEC $c_B = 0,\ \nu \geq 1$

Model II D = 5 2 Lifshitz

NEC **i** $\nu_1 = \nu_2 = 1$, **ii**, **iii** $\nu_1 = \nu_2,\ z_h^{max}:\ \nu_1 = \nu_2 > 1,\ c > 0$

3-rd law + NEC
$$\begin{cases} \textbf{i} & \nu_1 = \nu_2 = 1, \quad c < 0,\ 0 < n < 1 \\ \textbf{ii, iii} & \nu_1 = \nu_2 \geq 1, \quad c < 0,\ 0 < n < 1 \end{cases}$$

Model III D = 6 2 Lifshitz

$$\text{NEC} \Rightarrow \text{3-rd law}: \quad \mathfrak{b} = 1,\ 1 + \frac{1}{\nu_1} + \frac{2}{\nu_2} > 0$$

$$\mathfrak{b} \neq 1,\ ???$$

D-dimensional models: electric + magnetic ansatz

Electric and 2-form magnetic ansatz

$$S = \int d^D x \sqrt{-g_D} \left(R - \frac{f_{(0)}(\phi)}{4} F_{(0)}^2 - \frac{f_{(1)}(\phi)}{4} F_{(1)}^2 - \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - V(\phi) \right)$$

$$ds^2 = \frac{L^2 b(z)}{z^2} \left(- g(z) dt^2 + \sum_{i=1}^{d} dx_i^2 + z^{2-\frac{2}{\nu}} \sum_{j=1}^{2} dy_j^2 + \frac{dz^2}{g(z)} \right)$$

2- and 3-forms magnetic ansatz

$$S = \int d^D x \sqrt{-g_D} \left(R - \frac{1}{2} \partial_\mu \phi \, \partial^\mu \phi - \frac{1}{4} f_{(2)}(\phi) F_{(2)}^2 - \frac{1}{4} f_{(H)}(\phi) H^2 - V(\phi) \right)$$

$$ds^2 = \frac{L^2 \mathfrak{b}(z)}{z^2} \left(- g(z) dt^2 + \sum_{i=1}^{d} dx_i^2 + \frac{e^{c_B z^2}}{(z/L)^{\frac{2}{\nu}-2}} \sum_{j=1}^{3} dy_j^2 + \frac{dz^2}{g(z)} \right)$$

Scalar field $\phi = \phi(z), \; V(\phi) = V\big(\phi(z)\big)$

Electric ansatz $F_{\mu\nu}^{(0)} = \partial_\mu A_\nu^{(0)} - \partial_\nu A_\mu^{(0)}, \; A_\mu^{(0)} = A_t(z) \delta_\mu^0, \; \mu = \overline{0, D-1}$

Magnetic ansatz $F_{(1)} = q \, dy_1 \wedge dy_2, \; F_{(2)} = q \, dy_2 \wedge dy_3, \; H = q_H \, dy_1 \wedge dy_2 \wedge dy_3$

arXiv:2604.08441 [hep-th]

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Scalar field $\phi = \phi(z), \; V(\phi) = V\big(\phi(z)\big)$ Analytic solutions

Electric ansatz $F_{\mu\nu}^{(0)} = \partial_\mu A_\nu^{(0)} - \partial_\nu A_\mu^{(0)}, \; A_\mu^{(0)} = A_t(z) \delta_\mu^0, \; \mu = \overline{0, D-1}$

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3-rd law conditions

Electric and 2-form magnetic ansatz

$$\mathfrak{b}(z) = 1$$

$$F^2_{(0)} = 0: \quad s(T) = \frac{1}{4}\left(\frac{\frac{2}{\nu} + D - 3}{4\pi T}\right)^{4-D-\frac{2}{\nu}}, \ D \geqslant 5, \ \nu \geqslant 1, \ 4 - D - \frac{2}{\nu} < 0$$

$$F^2_{(0)} \neq 0: \quad s \to 0 \text{ as } T \to 0, \quad f_0 = e^{k(\phi-\phi_0)} = z^{\kappa}, \ \kappa < -1$$

$$\mathfrak{b}(z) = e^{-cz^2}, \ D = 5: \quad c = 0$$

2- and 3-forms magnetic ansatz

$$\mathfrak{b}(z) = e^{-cz^2}$$

$$s(z_h) = \frac{1}{4}\left(\frac{z_h}{L}\right)^{5-D-\frac{3}{\nu}} e^{\frac{z_h^2}{2}(c_B - c(D-2))}, \ c_B = c\,(D-2), \ \nu \geq 1, \ D \geq 5$$

$$\mathfrak{b}(z) = 1: \quad g(z) = 1 + C\,z^{1+\frac{3}{\nu}} - \left(\frac{z}{z_h}\right)^{2+\frac{4}{\nu}}\left(1 + C\,z_h^{1+\frac{3}{\nu}}\right), \ C = 0; \quad V = 0$$

arXiv:2604.08441 [hep-th]

Conclusions

- Analytic solution can be obtained in quadratures
- In 5D with Gauss 3-rd law fulfills for $\nu \geq 1$, $c_B = 0$
- Magnetic anisotropy in Gauss form $c_B > 0$ spoils the 3-rd law
- Anisotropy spoils NEC for 2 Lifshitz
- In 5D with Gauss $c_B < 0$ and for 2 Lisfitz $c < 0$, $n \geq 1$ spoil the 3-rd law for NEC fulfillment
- 6D with 2 Lishitz allows 3-rd law and NEC for $\nu < 1$, $\nu_1 \neq \nu_2$, but for $\mathfrak{b} = 1$

In 5D Gaussian warp-factor and anisotropy affect the 3-rd law

D > 5 models can solve these problems
for some warp-factors and anisotropies

Thank you
for your attention