Slow-roll approximations in Einstein-Gauss-Bonnet gravity

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• In the Einstein–Gauss–Bonnet (EGB) gravity models, the slow-roll approximation has been extended [1] by taking into account the first-order slow-roll parameter $\delta_1 = -2\,H^2\,\xi'/U_0$, which is proportional to the first derivative of the Gauss-Bonnet coupling function ξ with respect to the e-folding number. The tested models were formulated in terms of field.

• We apply the extended slow-roll approximations to the attractor model [2] formulated in terms of e-folding numbers.

• We compare the standard slow-roll approximation with the extended slow-roll approximations taking into account correction of the first order and the exact expression for potential.

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[1] E. O. Pozdeeva, M. A. Skugoreva, A. V. Toporensky and S. Y. Vernov, JCAP **09**, 050 (2024) doi:10.1088/1475-7516/2024/09/050 [arXiv:2403.06147 [gr-qc]].

[2] E. O. Pozdeeva, Eur. Phys. J. C **85** (2025) no.2, 217 doi:10.1140/epjc/s10052-025-13895-7 [arXiv:2411.16194 [gr-qc]]

The Einstein-Gauss-Bonnet gravity described by the following action:

$$S = \int d^4x\sqrt{-g}\left[U_0 R - \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi) - \frac{1}{2}\xi(\phi)\mathcal{G}\right], \qquad (1)$$

where $U_0 > 0$ is a constant, the functions $V(\phi)$ and $\xi(\phi)$ are differentiable ones, R is the Ricci scalar, $\mathcal{G} = R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 4R_{\mu\nu}R^{\mu\nu} + R^2$ is the Gauss–Bonnet term.
We apply the slow-roll parameters [3] during our consideration:

$$\varepsilon_1 = -\frac{\dot{H}}{H^2} = \frac{1}{2}\frac{d\ln(H^2)}{dN}, \qquad \varepsilon_{i+1} = -\frac{d\ln|\varepsilon_i|}{dN}, \quad i \geqslant 1, \qquad (2)$$

$$\delta_1 = \frac{2}{U_0}\xi_{,\phi}H\psi = -\frac{2}{U_0}\xi_{,\phi}H^2\chi, \qquad \delta_{i+1} = -\frac{d\ln|\delta_i|}{dN}, \quad i \geqslant 1. \qquad (3)$$

[3] Z.-K. Guo and D.J. Schwarz, Slow-roll inflation with a Gauss-Bonnet correction, Phys. Rev. D 81 (2010) 123520 [1001.1897];
C. van de Bruck and C. Longden, Higgs Inflation with a Gauss-Bonnet term in the Jordan Frame, Phys. Rev. D 93 (2016) 063519 [1512.04768]

The Friedmann equation can be presented in terms of slow-roll parameters

$$12U_0H^2\left(1-\delta_1\right)=\psi^2+2V=H^2\chi^2+2V\,,\psi=\dot{\phi} \tag{4}$$

$$4U_0\dot{H}\left(1-\delta_1\right)=\,-\psi^2+2U_0H^2\delta_1\left(\delta_2+\varepsilon_1-1\right). \tag{5}$$

From the Second Friedmann equation we got

$$\chi^2=2U_0\left[2\varepsilon_1-\delta_1+\delta_1\left(\delta_2-\varepsilon_1\right)\right]\,. \tag{6}$$

where $\chi=-H\psi$
Substituting this expression to the first Friedmann equation and neglecting multiplications of slow-roll parameters one can get

$$H^2\approx\frac{V}{U_0(6(1-\delta_1)-(2\epsilon_1-\delta_1))} \tag{7}$$

Here one can suppose that this is the best approximation extended up to first order of slow-roll parameters.

However we remember that the spectral index n_s and the tensor-to-scalar ratio r are connected with the slow-roll parameters as follows:

$$n_s = 1 - 2\varepsilon_1 - \frac{2\varepsilon_1\varepsilon_2 - \delta_1\delta_2}{2\varepsilon_1 - \delta_1} = 1 - 2\varepsilon_1 - \frac{d\ln(r)}{dN} = 1 + \frac{d}{dN}\ln\left(\frac{H^2}{U_0 r}\right), \qquad (8)$$

$$r = 8|2\varepsilon_1 - \delta_1|. \qquad (9)$$

Here we know that the r is a very small number in inflationary scenarios. So, we suppose that the difference $2\varepsilon_1 - \delta_1$ is a second order of smallest and get the supposition

$$H^2 \approx \frac{V}{6U_0(1-\delta_1)} \qquad (10)$$

Such approximation is equivalent to neglecting δ^2 in the First Friedmann equation. Using relation $\frac{1}{1-\delta_1} \approx (1+\delta_1)$ the (10) can be approximated once again such as

$$H^2 \approx \frac{V}{6U_0}(1+\delta_1) \qquad (11)$$

in the supposition $\delta_1 \ll 1$.

Test

- Inflationary models with monomial potential $V = V_0\,\phi^n$, $\xi = \frac{CU_0^2}{V+\Lambda}$, where C and Λ are positive constants, $n = 2, 4$.

- $V_1 \sim 6U_0(1+\delta_1)H^2$ (in the start), excellent inflationary parameters in the start

- $V_2 \sim \frac{6U_0H^2}{(1-\delta_1)}$ (exit from the inflation), excellent inflationary parameters in the start

- $V_{sl} = 6U_0H^2$ big values of the field,excellent inflationary parameters in the start

- numerical

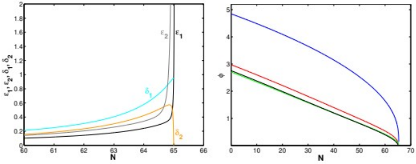

Fig. 1. The inflationary model with $V(\phi) = V_0\phi^2$. Slow-roll parameters as functions of N at the end of inflation found by numerical integration of equations of motion without any approximation are presented in the left panel. Values of the function $\phi(N)$ in units of $M_{\rm Pl}$ obtained numerically or using slow-roll approximations are presented in the right panel. The black line is the result of the numerical integration of system (2.4). The blue curve is obtained in the standard approximation using Eq. (3.5), red — in the approximation I using Eq. (4.10), green — in the approximation II by Eq. (4.21). The initial values $\phi(0) = \phi_0$ are given in Table 1.

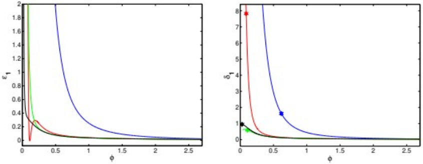

Fig. 2. The slow-roll parameters $\varepsilon_1(\phi)$ (left panel) and $\delta_1(\phi)$ (right panel) for the model with $V(\phi) = V_0\phi^2$. The black line is the result of the numerical integration of the system (2.4), blue curves are obtained in the standard approximation, red curves in the approximation I, and green curves in the approximation II. The stars denote the end of the inflation (when $\varepsilon_1 = 1$). Values of ϕ are given in units of $M_{\rm Pl}$.

Table. Numerical and approximate values of parameters, characterizing the inflationary dynamic in the model with the quartic potential.

Parameter	Numerical results	Standard approximation	Approximation I	Approximation II
$\phi_0/M_{\rm Pl}$	1.4019	4.9705	1.4898	1.3974
$10^9 A_s(\phi_0)$	2.096	117.2	2.599	2.017
$n_s(\phi_0)$	0.965	0.953	0.965	0.965
$r(\phi_0)$	0.0044	0.0120	0.0045	0.0045
$\phi_{end}/M_{\rm Pl}$	0.2017	0.8899	0.3048	0.3037
$\delta_1(\phi_{end})$	0.885	1.80	4.23	0.577
$N(\phi_{end})$	60.6	60.6	60.6	60.6

Formulation in terms of e-folding number derivatives

- The Friedmann equations in terms of e-folding number derivatives:

$$12\,H^2\,\left(U_0+2\xi'\,H^2\right)=H^2\Phi+2V, \tag{12}$$

$$2\,(H^2)'\left(U_0+2\xi'\,H^2\right)=H^2\Phi-4H^2\left(H^2\xi''+\left(\frac{(H^2)'}{2}+H^2\right)\xi'\right), \tag{13}$$

where $'=\frac{d}{dN}$, $\Phi=\chi^2$, $\chi=\frac{d\phi}{dN}$.
Field equation in the spatially flat Friedmann Universe

$$\frac{H^2}{2}\Phi'+\frac{(H^2)'}{2}\Phi-3\,H^2\,\Phi=-V'-12\,H^2\,\xi'\left(-\frac{(H^2)'}{2}+H^2\right). \tag{14}$$

In the slow-roll regime ($\ddot{\phi} \ll 3H\dot{\phi}$ or equivalently $\ddot{\phi}\phi' = \frac{H^2}{2}\Phi' + \frac{(H^2)'}{2}\Phi \ll 3\,H^2\,\Phi$) the equation (14) can be reduced to

$$3\,H^2\,\Phi = V' + 12\,(H^2)^2\,\xi'\,(1-\epsilon_1) \tag{15}$$

leading to

$$\Phi_{1,2} = \frac{V'_{1,2} + 12\,(H^2)^2\,\xi'\,(1-\epsilon)}{3H^2} \tag{16}$$

for the extended slow-roll approximations and

$$\Phi_{sl} = \frac{V'_{sl} + 12\,(H^2)^2\,\xi'}{3H^2} \tag{17}$$

for the standard slow-roll approximation ($\frac{(H^2)'}{2} \ll H^2$ or $\epsilon_1 \ll 1$).

The obtained evolution equations allows to get expression for $\Phi = \chi^2$ using (13)

$$\Phi_{exact} = \frac{2\,U_0\,(H^2)' + 6\,H^2\,(H^2)'\,\xi' + 4\,(H^2)^2\,(\xi'' + \xi')}{(H^2)} \tag{18}$$

and for the potential V substituting (18) to (12)

$$V_{exact} = -\left(3\,\xi'\,H^2 + U_0\right)\,\left(H^2\right)' + 2\,\left(5\,\xi' - \xi''\right)\left(H^2\right)^2 + 6\,U_0\,(H^2). \tag{19}$$

- $\Phi_2 = \Phi_{exact}$.

Model in terms of e-folding numbers

• We start our reconstruction using the following form of the Hubble parameter

$$H^2 = (H_0^2)\exp\left(-\frac{3}{2}\frac{C_\beta}{(N+N_0)}\right), \xi = \frac{\xi_0\,(H_0)^2}{(H^2)}. \quad (20)$$

where C_β, N_0, H_0^2, ξ_0 are model constants.
• The model was proposed [4] to reconstruct inflationary parameters of cosmological attractors models [5] and the first of all $R+R^2$ inflationary scenario [6]

[4] E. O. Pozdeeva, Eur. Phys. J. C **80** (2020) no.7, 612 doi:10.1140/epjc/s10052-020-8176-3 [arXiv:2005.10133 [gr-qc]]

[5] M. Galante, R. Kallosh, A. Linde and D. Roest, Phys. Rev. Lett. **114** (2015) no.14, 141302 doi:10.1103/PhysRevLett.114.141302 [arXiv:1412.3797 [hep-th]]

[6] A. A. Starobinsky, Phys. Lett. B **91** (1980), 99-102 doi:10.1016/0370-2693(80)90670-X

We substitute these expressions for H^2 and ξ into the formulas for the first slow-roll parameters:

$$\epsilon_1 = \frac{3}{4}\frac{C_\beta}{(N+N_0)^2}, \;\; \delta_1 \;=\; \frac{3\,\xi_0\,H_0^2}{U_0}\frac{C_\beta}{(N+N_0)^2}. \tag{21}$$

To get exit from inflation at $N = N_e$ $(\epsilon_1 = 1)$ we put $C_\beta = \frac{4\,(N_e+N_0)^2}{3}$.

The inflationary parameters: the tensor-to-scalar ratio, the spectral index of scalar perturbation and the amplitude of scalar perturbation:

$$r \approx 8|2\epsilon_1 - \delta_1| \approx \frac{16\,N_0^2}{(N+N_0)^2}\left|1 - \frac{4\,\xi_0\,H_0^2}{U_0}\right|, \quad (22)$$

$$n_s \approx 1 - 2\epsilon_1 + \frac{d\ln(r)}{dN} \approx 1 - \frac{2}{N+N_0}\left(1 + \frac{N_0}{N+N_0}\right), \quad (23)$$

$$A_s \approx \frac{H^2}{\pi^2\,U_0\,r} \approx \frac{H_0^2(N+N_0)^2\exp\left(-\frac{2\,N_0^2}{(N+N_0)}\right)}{16\pi^2\,U_0\,N_0^2\left|1 - \frac{4\,\xi_0\,H_0^2}{U_0}\right|}. \quad (24)$$

We derive the expressions for the potential of the exponential model under consideration using approximations and the exact formula:

$$V_{sl} = 6 U_0 H_0^2 \exp\left(-\frac{2{N_0}^2}{N + N_0}\right), \tag{25}$$

$$V_1 = V_{sl}\left(1 + \frac{4 H_0^2\, \xi_0\, {N_0}^2}{U_0\, \left(N + N_0\right)^2}\right)^{-1}, \tag{26}$$

$$V_2 = V_{sl}\left(1 - \frac{4\, H_0^2\, \xi_0\, {N_0}^2}{U_0\, \left(N + N_0\right)^2}\right), \tag{27}$$

$$V_{exact} = V_{sl}\left(1 - \frac{N_0^2}{3\,(N + N_0)^2} - \frac{2 H_0^2 \xi_0 {N_0}^2 (5\,(N + N_0)^2 - N_0^2 + 2\,(N + N_0))}{3\, U_0\, (N + N_0)^4}\right)$$

We substitute the potentials to corresponding expressions for $\Phi = (\phi')^2$ and get:

$$\Phi_{sl} = \frac{4\,{N_0}^2\left(U_0 - 2\,H_0^2\,\xi_0\right)}{\left(N+N_0\right)^2}, \tag{28}$$

$$\Phi_1 = \frac{4\,U_0^2\,N_0^2\left(1 + \frac{4\xi_0\,H_0^2(N+N_0)}{U_0(N+N_0)^2+4\xi_0\,H_0^2\,N_0^2}\right)}{U_0(N+N_0)^2 + 4\xi_0\,H_0^2\,N_0^2} + \frac{8\,H_0^2\xi_0\,N_0^2\left(\frac{N_0^2}{(N+N_0)^2} - 1\right)}{(N+N_0)^2}, \tag{29}$$

$$\Phi_2 = \frac{4\,N_0^2}{(N+N_0)^2}\left(U_0 - 2\,H_0^2\xi_0 + \frac{4\,H_0^2\,\xi_0}{(N+N_0)} - \frac{2\,H_0^2\xi_0\,N_0^2}{(N+N_0)^2}\right), \tag{30}$$

$$\Phi_{exact} = \Phi_2 \tag{31}$$

The field ϕ can be expressed analytically for the exponential model. To avoid complications during subsequent numerical calculations, it is better to integrate $\sqrt{\Phi_{exact}}$ after selecting the model parameters.
The expression Φ_1 is very long and does not lead to an analytical expression for $\phi_1(N)$.
In the standard slow-roll approximation, the expression for the field's dependence on the e-folding number during inflation has a simple form

$$\phi_{sl} = 2\,N_0\,\sqrt{U_0 - 2\,Q_0\,\xi_0}\,\ln\left(N + N_0\right) + c_{sl}, \tag{32}$$

where c_{sl} is constant, which should be chosen in order to compare the behavior of ϕ and ϕ_{sl} in numerical analysis.

Using the values of the model parameters, we obtain:

$$\Phi = (\phi')^2 \approx \frac{1.75\,(N+1.5469)\,(N+0.73880)}{(N+1)^4}, \tag{33}$$

$$\phi = 0.5\arctan\left(\frac{0.37796(N-1.7202\cdot 10^{-9})}{S_q}\right) \tag{34}$$

$$+ \quad 1.3229\ln((N+1.1429+S_q)\cdot 10^9) - \frac{1.3229\,S_q}{(N+1)} - c_0, \tag{35}$$

where $S_q = \sqrt{(N+1.5468)\cdot(N+0.73886)}$, c_0 is a constant of integration. The choice of integration constant c_0 is not unique. Near $N \approx -0.738$ the field becomes complex and we choose $c_0 \approx 25.431$ to get $\phi|_{N\approx -0.738} = 0$.
We calculate the integration constant included to the slow-roll approximation of the field $c_{sl} = \phi - (2\,N_0\,\sqrt{U_0 - 2\,Q_0\,\xi_0}\ln(N+N_0))$ at the point $N = N_b$ and get: $c_{sl} \approx 1.75$. The choice of the integration constant allows us fix the same values of fields at the beginning of inflation. We solve the differential equation $\frac{d\phi}{dN} = \sqrt{\Phi}$ numerically assuming $\phi(N=N_b) = 7.145$ for all types of approximations and the exact solution.

Early we introduce the effective potential for EGB gravity [7]. The effective potential V_{eff} for it's equivalent presentation can play the role of potential for studying stability [8]. Here, we consider an equivalent presentation of the effective potential $\tilde{V}_{eff}$:

$$\tilde{V}_{eff} = -V_{eff}^{-1}, \quad \text{where} \quad V_{eff} = -U_0^2/V + \xi/3. \tag{36}$$

Here we should note, that the considering set of the model parameters is mostly toy and used to clear present the behavior of effective potential $\tilde{V}_{eff}$ which is analog to potential in the General Relativity models.

[7] E. O. Pozdeeva, M. Sami, A. V. Toporensky and S. Y. Vernov, Phys. Rev. D (2019) [arXiv:1905.05085 [gr-qc]].

[8] S. Vernov and E. Pozdeeva, Universe (2021) [arXiv:2104.11111 [gr-qc]].

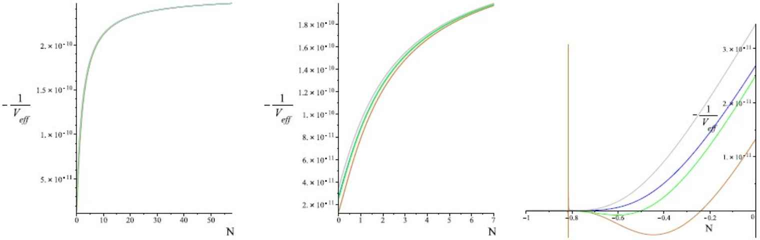

Fig. 1: The behavior of $\tilde{V}_{eff}$ during inflation for slow-roll approximations (the gray line corresponds to the standard slow-roll approximation, the blue line – to the approximation (2), the green line – to the approximation (1)) and the exact considerations (orange line) at the following values of the parameters: $N_0 = 1$, $N_b = 57.787$, $\xi_0 = 1.6788 \cdot 10^{10}/\pi^2$, $Q_0 \approx 1.8861 \cdot 10^{-12}\pi^2$, $U_0 = M_{Pl}^2/2$, $M_{Pl} = 1$.

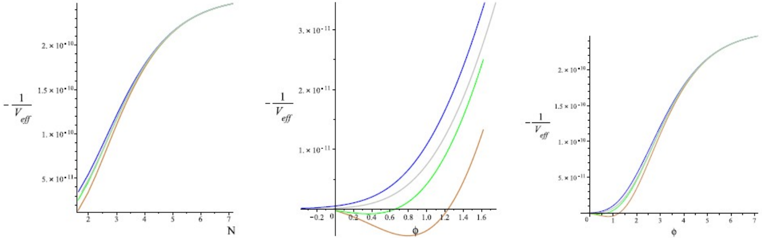

Fig. 3: The graphical behavior of effective potential $\tilde{V}_{eff} = -V_{eff}^{-1}$ during inflation for slow-roll approximations (the gray line corresponds to the standard slow-roll approximation, the blue line – to the approximation (2)), the green line – to the approximation (1)) and the exact considerations (orange line) at the following values of the parameters: $c_{sl} = 1.7556$, $N_0 = 1$, $N_b = 57.787$, $\xi_0 = 1.6569 \cdot 10^{10}/\pi^2$, $Q_0 \approx 1.8861 \cdot 10^{-12}\pi^2$, $U_0 = M_{Pl}^2/2$, $M_{Pl} = 1$. The left picture corresponds to evolution during inflation, the middle picture - to evolution after inflation, the right picture is the join evolution.

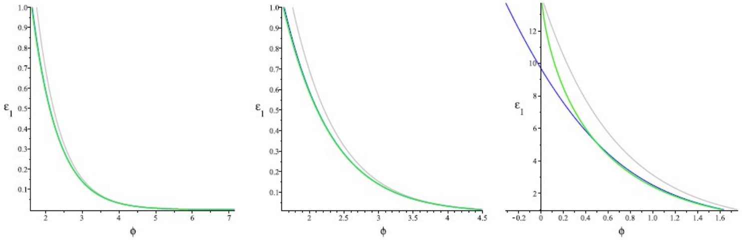

Fig. 9: The behavior $\epsilon_1(\phi)$ (the gray line corresponds to the standard slow-roll approximation, the blue line – to the approximation (2), the green line – to the approximation (1) and the exact behavior coincides with green line) at the following values of the parameters: $c_{sl} = 1.7556$, $N_0 = 1$, $N_b = 57.787$, $\xi_0 = 1.6569 \cdot 10^{10}/\pi^2$, $Q_0 \approx 1.8861 \cdot 10^{-12}\pi^2$, $U_0 = M_{Pl}^2/2$, $M_{Pl} = 1$. The left picture corresponds to the interval $N = 0..N_b$, the central picture corresponds to the interval $N = 0..7$ and the right picture corresponds to the interval $N = -0.73..0$.

Conclusion

• We proposed two extended slow-roll approximations in EGB gravity.
• We compared the standard slow-roll approximation and extended ones with exact analysis.
• The comparisons were carried out on models in terms of fields and e-folding numbers.
• Using the standard slow-roll approximation it is easy to reproduce the field dependence on e-folding numbers. However the low accuracy can leads to overestimation of A_s and a shortened duration of inflation.
• In EGB gravity it is better to do additional test using extended approximations.

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