Reconstruction and direct methods in bottom-up holography

QUARKS-2026

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18.05.2026

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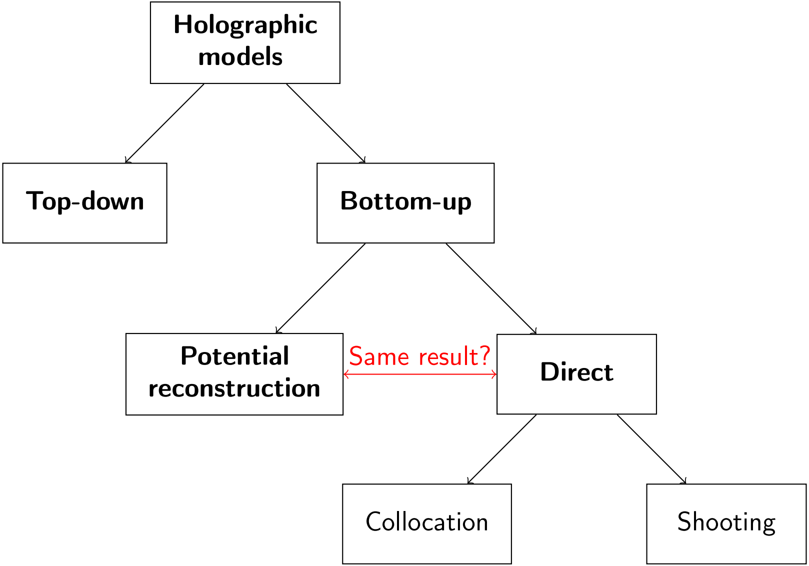
Outline

Introduction



Considered holographic model

Action of the model: $S = \frac{1}{16\pi G_5} \int d^5x \sqrt{-\mathfrak{g}} \left[R - \frac{\mathfrak{f}_0(\varphi)}{4} F^2 - \frac{1}{2} \partial_\mu \varphi \partial^\mu \varphi - \mathcal{V}(\varphi) \right]$

Poincare patch coordinates: $x^\mu = \{t, \vec{x}, \tilde{z}\}, \quad L = 1 \text{ GeV}^{-1}$

Ansatz: $ds^2 = e^{2A(\tilde{z})} \left[-g(\tilde{z}) dt^2 + d\vec{x}^2 \right] + e^{2B(\tilde{z})} \frac{d\tilde{z}^2}{g(\tilde{z})},$

$$\varphi = \varphi(\tilde{z}), \quad A_\mu = \left(A_t(\tilde{z}) \ , \vec{0}, 0 \right)$$

Conformal: $ds^2 = \frac{e^{2A(z)}}{z^2} \left[-g(z) dt^2 + d\vec{x}^2 + \frac{dz^2}{g(z)} \right]$

Domain wall: $ds^2 = e^{2A(r)} \left[-g(r) dt^2 + d\vec{x}^2 \right] + \frac{dr^2}{g(r)}.$

EOM in conformal coordinates

$$\varphi'' + \left(\frac{g'}{g} + 3A' - \frac{3}{z}\right)\varphi' + \left(\frac{z^2 e^{-2A}\left(A_t'\right)^2 \mathfrak{f}_{0;\varphi}}{2g} - \frac{e^{2A} V_\varphi}{z^2 g}\right) = 0$$

$$A_t'' + \left(\frac{\mathfrak{f}_{0;\varphi}\varphi'}{\mathfrak{f}_0} + A' - \frac{1}{z}\right) A_t' = 0$$

$$A'' - A'^2 + \frac{2}{z}A' + \frac{\varphi'^2}{6} = 0$$

$$g'' + \left(3A' - \frac{3}{z}\right) g' - e^{-2A} z^2 \mathfrak{f}_0 A_t'^2 = 0$$

$$A'' + 3A'^2 + \left(\frac{3g'}{2g} - \frac{6}{z}\right) A' - \frac{1}{z}\left(\frac{3g'}{2g} - \frac{4}{z}\right) + \frac{g''}{6g} + \frac{e^{2A}\mathcal{V}}{3z^2 g} = 0$$

EOM in domain wall coordinates

$$
\begin{aligned}
\varphi'' + \varphi'\left(4A' + \frac{g'}{g}\right) + \frac{e^{-2A}A_t'^2\mathfrak{f}_{0;\varphi} - 2\mathcal{V}_\varphi}{2g} &= 0\\
A_t'' + 2A'A_t' + \frac{A_t'\mathfrak{f}_{0;\varphi}\varphi'}{\mathfrak{f}_0} &= 0\\
g'' + 4A'g' - e^{-2A}\mathfrak{f}_0 A_t'^2 &= 0\\
6A'' + \varphi'^2 &= 0\\
2\mathcal{V} + e^{-2A}\mathfrak{f}_0 A_t'^2 + 6A'(4gA' + g') - g\varphi'^2 &= 0
\end{aligned}
$$

Thermodynamics and PT-1

Horizon at z_h or r_h: $g\big|_{\text{horizon}} = 0, \quad A_t\big|_{\text{horizon}} = 0.$
Conformal boundary at $z \to 0$ or $r \to \infty$.
1) Black hole thermodynamics and FOPT

Chemical potential: $\mu = A_t\big|_{\text{conf.bnd.}}$ Temperature: $T = \left|\frac{g' e^{A-B}}{4\pi}\right|_{\text{horizon}}$

Entropy density: $S = \frac{e^{3A}}{4G_5}\Big|_{\text{horizon}}$ Free energy density: $\boxed{F = -\int S \, \mathrm{d}T}$

Thermodynamics and PT-2

2) Confinement-deconfinement crossover

Wilson loops value from holography: $\langle W(\mathscr{C})\rangle = e^{-S_{NG}(\mathscr{C})}$,

$$S_{NG} = \frac{1}{2\pi\alpha'}\int \mathrm{d}\xi^1\mathrm{d}\xi^2\sqrt{-\det h_{ab}}, \quad \mathscr{C} = \tau \times R, \ \tau, R \gg L$$

Induced metric: $h_{ab} = g^{(S)}_{\mu\nu}\dfrac{\partial x^\mu}{\partial \xi^a}\dfrac{\partial x^\nu}{\partial \xi^b}$,

String frame metric from Einstein frame: $g^{(S)}_{\mu\nu} = g_{\mu\nu}e^{\sqrt{\frac{2}{3}}\varphi}$

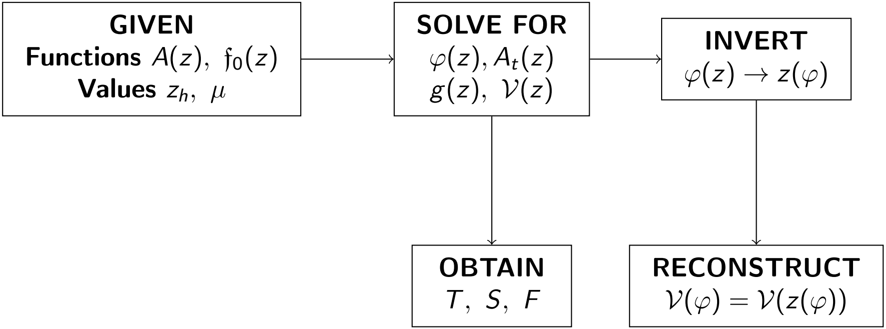
$$M(\tilde{z}) = \exp(A^{(S)}(\tilde{z}) + B^{(S)}(\tilde{z})),$$
$$\mathscr{F}(\tilde{z}) = g(\tilde{z})\exp(2[A^{(S)}(\tilde{z}) - B^{(S)}(\tilde{z})]).$$

The effective potential:

$$\boxed{V_{\text{eff}}(\tilde{z}) = M(\tilde{z})\sqrt{\mathscr{F}(\tilde{z})}.}$$

Potential reconstruction method

Work in $z \in [0,\ z_h]$.



Considered reconstruction model-1

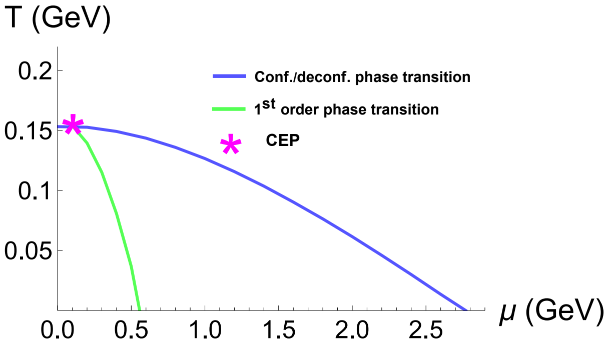
The considered reconstruction model(Aref'eva, Rannu, Slepov '20) takes the form

$$A(z) = -a \ln(1 + bz^2),$$
$$\mathfrak{f}_0(z) = e^{-cz^2 - A(z)}.$$

Parameter values

$$c = 0.227 \text{ GeV}^2 \text{ — fixed by Regge spectrum},$$
$$a = 4.046,\ b = 0.01613 \text{ GeV}^2 \text{ — fixed by CEP value}.$$

Considered reconstruction model-2



CEP is at $(T,\ \mu) = (0.1578,\ 0.04779)$ GeV.
Phase transition lines intersect at $(T,\ \mu) = (0.1538,\ 0.1043)$ GeV.

Direct method: Shooting calculations-1

Work in $r \in [r_h,\ \infty)$.

GIVEN
Functions $\mathcal{V}(\varphi),\ \mathfrak{f}_0(\varphi)$
Values $\varphi_h,\ a_1$

→

SOLVE FOR
$\varphi(r), A_t(r)$
$g(r),\ A(r)$

→

RECALCULATE
$z = -\int e^{-A(r)} \mathrm{d}r$

↓ (from SOLVE FOR)

OBTAIN
$T,\ S,\ F$

DeWolfe, Gubser et al. '10, Finazzo, Rougemont et al '14

Direct method: Shooting calculations-2

Input parameters:

- Characteristic energy scale (es) Λ_{es} — from the dilaton asymptotics.
- Dilaton dimension $\nu = \Delta_-$ from the potential UV expansion

$$\mathcal{V}(\varphi) = -12 + \frac{1}{2} m^2 \varphi^2 + \mathcal{O}\left(\varphi^3\right),$$
$$\Delta_\pm = 2 \pm \sqrt{4 + m^2}.$$

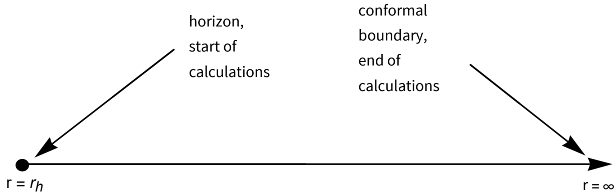
EOMs have symmetries of motion

$$\begin{array}{cccc} & & & r \to r + a, \\ r \to br, & g \to b^2 g, & A_t \to bA_t, & t \to b^{-1} t, \\ A \to A + \ln(c), & A_t \to cA_t, & t \to c^{-1} t, & \vec{x} \to c^{-1} \vec{x}. \end{array}$$

Direct method: Shooting calculations-3

Total unknowns			Total constraints			
Function values	Horizon position	Total	Horizon cond-s	Regularity of EOM	Symmetries	Total
8	1	9	2	2	3	7

We solve IVP from the horizon to the conformal boundary.
2 DOFs are dilaton at horizon φ_h and vector field derivative $A_t'(r_h) = a_1$.



Direct method: Collocation calculations-1

Work in $z \in [0,\ z_h]$.

GIVEN
Functions $\mathcal{V}(\varphi),\ \mathfrak{f}_0(\varphi)$
Values $z_h,\ \mu$

→

SOLVE FOR
$\varphi(z), A_t(z)$
$g(z),\ \mathcal{V}(z)$

→

OBTAIN
$T,\ S,\ F$

Shen et al. '25, Deng, Huang et al. '26

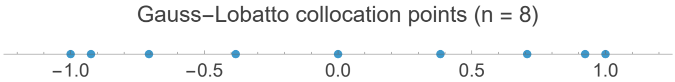
Coordinate transformation: $z \to 2\dfrac{z}{z_h} - 1,\ z \in [-1,\ 1],$

Expand all functions as $f(z) \simeq \sum_{k=0}^{n} f_{(k)} T_k(z)$

2nd type Chebyshev polynomials: $T_k(z = \cos\theta) = \cos(k\theta).$

Direct method: Collocation calculations-2

Gauss-Lobatto collocation points: $z_k = \cos\left(\frac{k}{n}\pi\right), k = 0, \ldots, n.$



Enforce ODE at each point $\rightarrow$ arrive at the algebraic system:

Total unknowns			Total constraints		
Different functions	Coefficients per function	Coefficients total	Differential equations	Collocation points	Equations total
4	n+1	4(n+1)	4	n+1	4(n+1)

Potential at fixed $(T,\ \mu)$-1

Values of parameters:

$$\nu = 1,\ \Delta = 3,$$
$$\Lambda_{es} = 1.53279\ \text{GeV}.$$

Search for potential of the form:

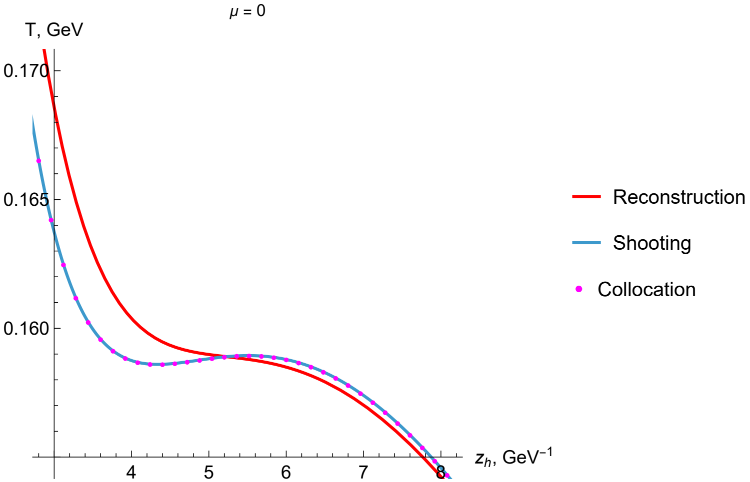
$$\mathcal{V}(\varphi, \{a_n\}) = -12\cosh(a_1\varphi) + (6a_1^2 - 1.5)\varphi^2 + \sum_{i=2}^{i=9} a_i\varphi^{2i}.$$

Idea to obtain the coefficients:

Reconstruct the potential from the potential reconstruction method at some fixed $(T,\ \mu)$, that is, fixed $(z_h,\ \mu)$.

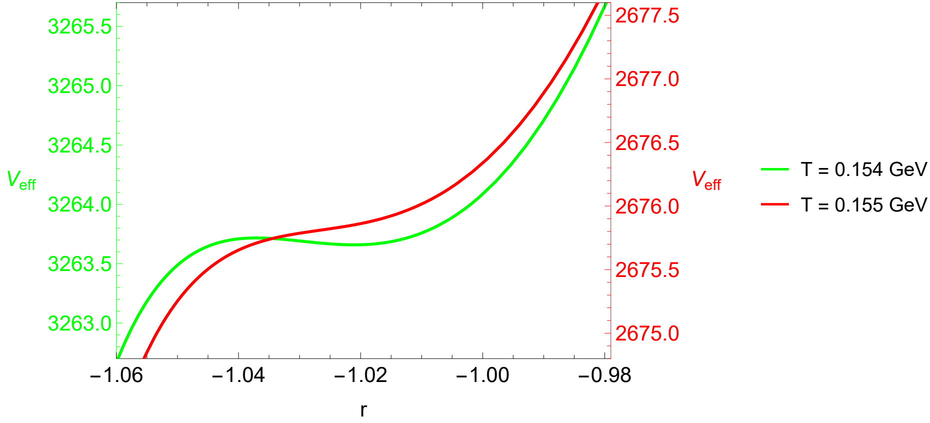
The resulting potential $\mathcal{V}_1(\varphi)$, calculated at $(T,\ \mu) = (0,\ 0)$ GeV, is in supplementary.

Potential at fixed $(T,\ \mu)$-2



There is a FOPT at $\mu = 0$ — **NO CEP/CEP AT $\mu = 0$.**

Potential at fixed $(T,\ \mu)$-3



There **is** a crossover phase transition.

Potential from difference minimization-1

Idea to obtain the coefficients without resulting FOPT:

Minimize difference of dependencies of $T(z_h)$ for reconstruction ($_{rec}$) and direct ($_{dir}$) approaches in some range $z_h \in [z_{h1},\ z_{h2}]$:

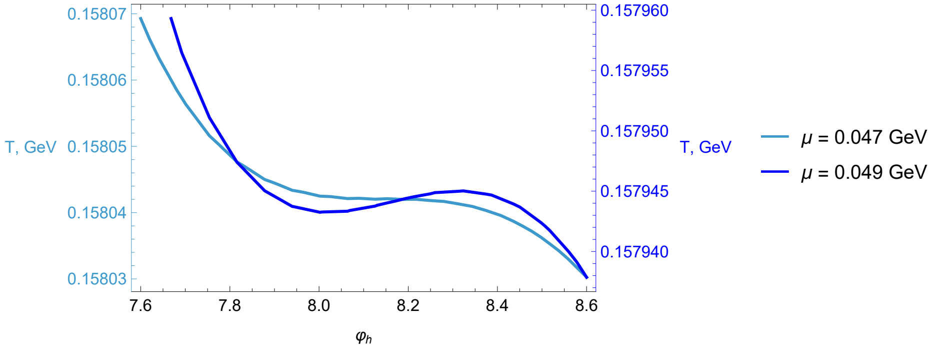
$$\Delta T_{int}(\{a_n\}) = \int\limits_{z_{h1}}^{z_{h2}} \mathrm{d}z_h \, \left(T_{rec}(z_h) - T_{dir}(z_h, \{a_n\})\right)^2.$$

As it turns out, the choice of $[z_{h1},\ z_{h2}]$ allows one to get rid of FOPT and to control the position of CEP.

For the domain $z_h \in [2,\ 8.375]$ GeV^{-1} the resulting CEP is near $(T,\ \mu) = (0.1578\text{ GeV},\ 0.048\text{ GeV})$ — close to the reconstruction values $(T,\ \mu) = (0.1578\text{ GeV},\ 0.04779\text{ GeV})$.

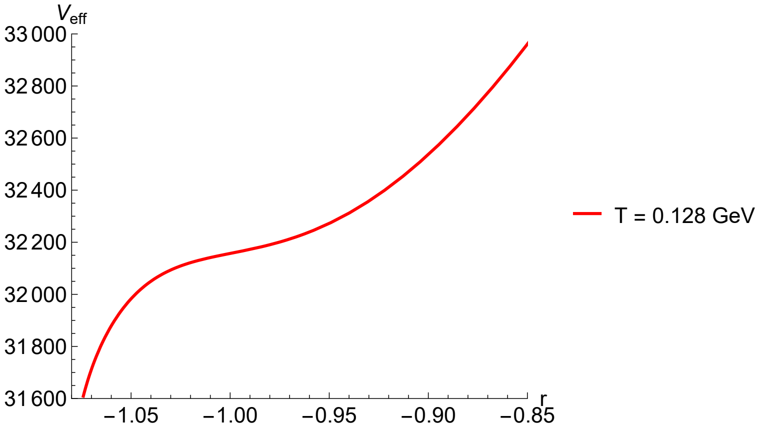
Corresponding potential is labeled as $\mathcal{V}_2(\varphi)$ and is in supplementary.

Potential from difference minimization-2



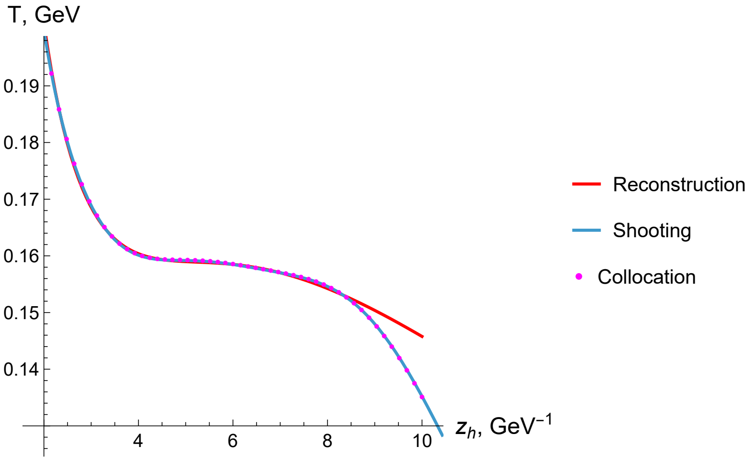
There **is** a CEP.

Potential from difference minimization-3

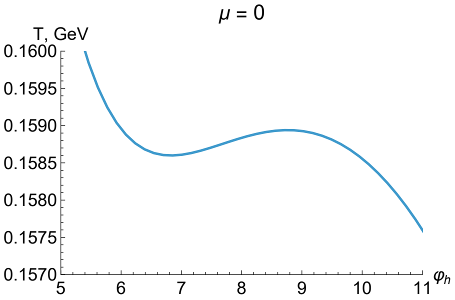


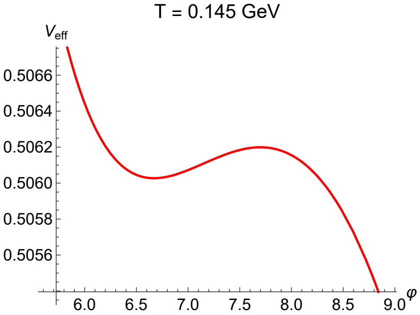
There **is no** confinement/deconfinement crossover.

Discrepancy analysis-1



Discrepancy analysis-2





Removing FOPT at $\mu = 0$ and getting a good CEP means changing the potential at the dilaton values responsible for crossover.

Reconstruction approach circumvents this, because the potential $\mathcal{V}(\varphi)$ changes for different $(T,\ \mu)$.

Conclusion

1. The potential reconstruction and direct approaches for bottom-up holography have been discussed.
2. The reconstruction model and it's basic properties have been outlined.
3. Two direct model have been created based on the reconstruction model, each of them successfully reproducing one phase transition.
4. Argument was made to explain why recreating both may be impossible.

Future prospects

1. More detailed analysis of crossover and Wilson loops behavior:
   - Calculation of the Cornell potential.
   - Calculation of the string tension.
2. Analysis of the running coupling in theory.
3. Application of ML (e.g. for better fits of the potentials).

Thank you for your attention!

This work was performed at the Steklov International Mathematical Center and supported by the Ministry of Science and Higher Education of the Russian Federation (agreement no. 075-15-2025-303).

Supplementary: recalculation between coordinates and BC

Recalculation between conformal and domain wall coordinates: $\mathrm{d}z = -e^{-A(r)}\mathrm{d}r$,

$g|_{\text{conf.bnd.}} = 1,$ — asymptotically AdS space,

$g|_{\text{horizon}} = 0,$ — the horizon position,

$A_t|_{\text{horizon}} = 0,$ — the horizon position,

$\varphi|_{\text{conf.bnd.}} = 0,$ — imposed by hand to obtain the UV fixed point,

$A_t|_{\text{conf.bnd.}} = \mu,$ — chemical potential,

A — depends on the coordinates.

Supplementary: fitted potentials and coupling function

$$\begin{aligned}
\mathfrak{f}_0(\varphi) = \exp(-0.068\varphi^2 + 0.0000185\varphi^4 - 4.5967\times10^{-7}\varphi^6 + 1.666\times10^{-9}\varphi^8 - \\
-3.841\times10^{-12}\varphi^{10} + 5.246\times10^{-15}\varphi^{12} - 3.892\times10^{-18}\varphi^{14} + 1.206\times10^{-21}\varphi^{16}), \\
\mathcal{V}_1(\varphi) = -12\cosh(0.767\varphi) + 2.03\varphi^2 + 0.0893\varphi^4 + 0.0014\varphi^6 + \\
+0.0000115\varphi^8 + 5.79\times10^{-8}\varphi^{10} + 1.87\times10^{-10}\varphi^{12} + \\
+5.18\times10^{-13}\varphi^{14} + 5\times10^{-16}\varphi^{16} + 1.76\times10^{-18}\varphi^{18}, \\
\mathcal{V}_2(\varphi) = -12\cosh(0.768\varphi) + 2.04\varphi^2 + 0.0715\varphi^4 + 0.0027\varphi^6 - \\
-6.9511\times10^{-6}\varphi^8 + 1.81\times10^{-7}\varphi^{10} + 4.25\times10^{-10}\varphi^{12} + \\
+1.28\times10^{-12}\varphi^{14} - 4.24\times10^{-14}\varphi^{16} + 1.63\times10^{-16}\varphi^{18}.
\end{aligned}$$