

Optimization of perturbation series for physical quantities using the QCD renormalization group:
necessary conditions and partial results for Bjorken Sum Rule and Adler D-function

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1. **Intro.** Examples of RG-invariants:

$$\text{Bjorken } C^{\text{Bjp}}(a_s) = 1 + a_s (c_1 + a_s c_2 + (a_s)^2 c_3 + (a_s)^3 c_4 + \dots)$$

and

$$\text{Adler } D_{ns}(a_s) = 1 + a_s (d_1 + a_s d_2 + (a_s)^2 d_3 + (a_s)^3 d_4 + \dots)$$

and the **BLM**-optimization of their PT-series.

2. Generalized **RG** transformation with a number of **free** parameters $\{\Delta_i\}$.
The constraints on the parameters $\{\Delta_i\}$ determine the “field of the game” for this **RG** transformation.
3. "Optimization" by truncating of the PT-series with the last coefficients,
 $c_{n-1,n} = 0$.
4. What is the decomposition **$\{\beta\}$ -expansion**? And what does it express?
Different transformations of PT-series on background of the constraints
5. **Conclusions**

Intro. What are Bjorken Sum Rule, C^{Bjp} , Adler D -function

– renorm-group invariant, single scale Q^2 , $Q^2 = \mu_R^2$, quantities C^{Bjp} , D taken in $\overline{\text{MS}}$:

Bjorken polarized Sum Rules in DIS

$$\frac{1}{6} \left| \frac{g_A}{g_V} \right| C^{\text{Bjp}}(a_s) + \text{h. tw.} = S_{\text{NS}}^{\text{Bjp}}(Q^2) = \int_0^1 [g_1^{\text{lp}}(x, Q^2) - g_1^{\text{ln}}(x, Q^2)] dx + \text{h. tw.}$$

Adler function

$$d_R D(a_s) = D_A = -12\pi^2 Q^2 \frac{d}{dQ^2} \Pi(Q^2); \quad Q^2 = -q^2$$

Crewther-Broadhurst-Kataev (CBK) relation

– a plausible **conjecture** [Crewther 1972,1997] inspired by conformal symmetry

$$D_{\text{ns}}(a_s) \cdot C^{\text{Bjp}}(a_s) = 1 + \beta(a_s) K(a_s), \text{ where } K(a_s) - \text{polynom in } a_s = \frac{\alpha_s}{4\pi}$$

Crucial 3-loop analysis [Broadhurst,Kataev,1993] in $\overline{\text{MS}}$ -scheme - **CBK relation**

[P.Baikov,K.Chetyrkin,J.Kühn, PRL2010]- confirmation in $O(a_s^4)$ **5 loops**.

Intro. First attempt at series optimization by varying μ^2 : the BLM

BLM (Brodsky, Lepage, Mackenzie) [PRD 28(1983)228], cited $\simeq 1300$

$$\text{RG equation: } \mu^2 \frac{d\bar{a}_s(\mu^2)}{d\mu^2} = -\bar{a}_s^2(\mu^2) (\beta_0 + \bar{a}_s \beta_1 + \dots), \quad a_s = \frac{\alpha_s}{4\pi}$$

$$\beta_0 = \frac{11}{3} C_A - \frac{4}{3} T_r n_f, \quad \bar{a}_s(\mu^2) = \frac{a_s(\nu^2)}{1 + a_s(\nu^2) \beta_0 \ln(\mu^2/\nu^2)}$$

Bjorken polarized Sum Rule $C^{\text{Bjp}}(Q^2/\mu^2, \mu^2)$,

$$C^{\text{Bjp}} \Big|_{Q^2=\mu^2} = 1 - 4 \left[a_s(\mu^2) + a_s^2(\mu^2) \left(\boxed{2 \cdot \beta_0} - \frac{11}{3} \right) - \frac{4}{3} n_f + \frac{23}{3} C_A - \frac{7}{3} C_F \right] + \dots$$
$$\left(\mu^2 \rightarrow \nu^2 = e^{-2} \mu^2 \right) = 1 - 4 \left[a_s(\nu^2) + a_s^2(\nu^2) (-11/3) + \dots \right]$$

Adler function $D_A(Q^2/\mu^2, \mu^2) [= R_{e^+e^- \rightarrow h}]$

$$\frac{D_A}{d_R} \Big|_{Q^2=\mu^2} = 1 + 4 \left[a_s(\mu^2) + a_s^2(\mu^2) \left(\boxed{0.69 \cdot \beta_0} + \frac{1}{3} - 0.69 \frac{2}{3} n_f + 2.87 C_A - C_F/2 \right) \right] + \dots$$
$$\left(\mu^2 \rightarrow \nu^2 \approx e^{-\ln(2)} \mu^2 \right) = 1 + 4 \left[a_s(\nu^2) + a_s^2(\nu^2) (1/3) + \dots \right]$$

♣ **Next LO coefficients become visibly smaller**

♠ $a_s(\nu^2)$ doesn't run here, ν^2 - a "true virtuality", characteristic of observable

2. Generalized RG transformation for invariant series C

$$\bar{a}(t) = \exp(-\Delta\beta(\bar{a})\partial_{\bar{a}}) \bar{a} \big|_{\bar{a}=a'} = a(\Delta, a') = a' - \frac{\Delta}{1!}\beta(a') + \left(\frac{\Delta}{2!}\beta(a')\partial_{a'}\right)(\beta(a')\Delta) + \dots$$

$$[\text{Kotlorz\&MS PRD2019}] (c_n, a_s, \mu^2) \xrightarrow{\text{RG}} (c'_n, a'_s, \mu'^2) \Rightarrow c'_n = \hat{\mathbf{B}}_{nj} c_j$$

Shift of the argument $t = \ln(\mu^2/\Lambda^2)$ with $\Delta \rightarrow \Delta(a')$

$$\ln(\mu^2/\mu'^2) = t - t' \equiv \Delta(a') = \Delta_0 + (a'\beta_0) \Delta_1 + (a'\beta_0)^2 \Delta_2 + \dots, \quad \boxed{\Delta_0 = c_2[1]/c_1}$$

Δ_j – free components

$\uparrow$ **BLM**

$$| a^1 \cdot c_1 \rightarrow a'^1 \cdot [c_1];$$

$$\boxed{c_2 = \beta_0 c_2[1] + c_2[0]}$$

$$\text{Each } | a^2 \cdot c_2 \rightarrow a'^2 \cdot [c'_2 = c_2 - 1\beta_0 \Delta_0];$$

$$\text{order } n | a^3 \cdot c_3 \rightarrow a'^3 \cdot [c'_3 = c_3 - c_2 2\beta_0 \Delta_0 - 1 \left(\beta_1 \Delta_0 - \beta_0^2 \Delta_0^2 + \beta_0^2 \underline{\Delta_1} \right)];$$

$$\text{acquires new } | a^4 \cdot c_4 \rightarrow a'^4 \cdot [c'_4 = c_4 - c_3 3\beta_0 \Delta_0 - c_2 \left(2\beta_1 \Delta_0 - 3\beta_0^2 \Delta_0^2 + 2\beta_0^2 \Delta_1 \right)$$

$$\text{parameter } \Delta_{n-2} | \quad \quad \quad - 1(\dots + \beta_0^3 \underline{\underline{\Delta_2}})]$$

Fitting components $\Delta_0, \Delta_1, \Delta_2, \dots$ of new normalization scale μ'^2 to adjust the elements $c'_2, c'_3, c'_4, \dots$ following to any **chosen optimization**.

2. General constraints to maintain the PT hierarchy, 1

General requirements **(A)**, **(B)**, **(C)** to a series $C(a_s) \xrightarrow{RG} C'(a'_s)$

$$t - t' \equiv \Delta = \Delta_0 + a'_s \beta_0 \Delta_1 + (a'_s \beta_0)^2 \Delta_2 \Rightarrow$$

$$\textbf{(A)} \quad |\Delta_0| > a'_s \beta_0 |\Delta_1| > (a'_s \beta_0)^2 |\Delta_2| > \dots$$

$$C(a_s) - 1 = c_1(a_s + (a_s)^2 c_2 + \dots) \xrightarrow{RG} c_1(a'_s + (a'_s)^2 c'_2 + \dots) \Rightarrow$$

$$\textbf{(B)} \quad 1/c_1 > a'_s > (a'_s)^2 |c'_2| > (a'_s)^3 |c'_3| > (a'_s)^4 |c'_4| > \dots$$

$$\textbf{(C)} \quad t' = t - \Delta \geq t_0 = 2.3 - \text{PT boundary at } \mu_0^2 = 1 \text{ GeV}^2.$$

The set of inequalities $\mathbf{A} \wedge \mathbf{B} \wedge \mathbf{C}$ (2D series) is universal for any physical quantity.

For **Bjorken** $C^{\text{Bjp}}(a_s)$ and **Adler** $D_{\text{NS}}(a_s)$ these constraints shape of

admissible domains for every value of $t = \ln\left(\frac{Q^2}{\Lambda_{qcd}^2}\right)$, [Kotlorz&MS PRD2019]

2. General constraints to maintain the PT hierarchy, 2

Besides the conditions (**A**), (**B**) on these Two series

one should solve the equation with respect to a'_s for every fixed t ,

$$a'_s \equiv \bar{a}_s(t') = a_s(t - \Delta(a'_s))$$

$t = 3, \quad 3.44, \quad 4, \quad 5, \dots, 8, \quad 9, \quad 10, \quad 11.32 :$

$$Q^2 = 2, (m_\tau = 1.77)^2, 5.5, 15, \dots, 301, 819, 2227, (M_Z = 91.19)^2 \text{ GeV}^2$$



each t corresponds to special **color**

$$C^{\text{Bjp}}(\Delta) = 1 + c_1 \cdot a_{\text{eff}}, \quad a_{\text{eff}} = a_s \left(1 + a_s c_2 + (a_s)^2 c_3 + (a_s)^3 c_4 + \dots \right)$$

$$\frac{D_{\text{NS}}(\Delta')}{d_R} = 1 + d_1 \cdot a'_{\text{eff}}, \quad a'_{\text{eff}} = a_s \left(1 + a_s d_2 + (a_s)^2 d_3 + (a_s)^3 d_4 + \dots \right)$$

2. 1D-2D domains for Bjorken $C^{\text{Bjp}}(a_s)$

$$C^{\text{Bjp}}(\Delta_0, \Delta_1, \Delta_2) = 1 + a_{\text{eff}} \cdot c_1$$

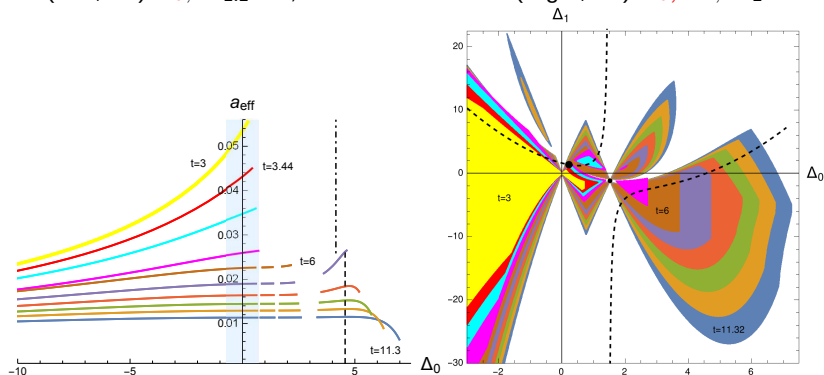
$t = 3, \quad 3.44, \quad 4, \quad 5, \dots, 8, \quad 9, \quad 10, \quad 11.32:$

$Q^2 = 2, (m_\tau = 1.77)^2, 5.5, 15, \dots, 301, 819, 2227, (M_Z = 91.19)^2 \text{ GeV}^2$

“channels” and “islands” coordinately appear in $\{\Delta\}$ admissible spaces.

(Left, **1D**) $\Delta_0, \Delta_{1,2} = 0;$

(Right, **2D**) $\Delta_0, \Delta_1, \Delta_2 = 0$



lines — — $c_4(\Delta_0, \Delta_1) = 0$; point $\bullet$ — $\{c_3(\Delta_0, \Delta_1) = c_4(\Delta_0, \Delta_1) = 0\}$,

blue strip corresponds to variation $[\mu^2/2 - 2\mu^2]$

The larger is t — the larger is size of the corresponding colored domain.

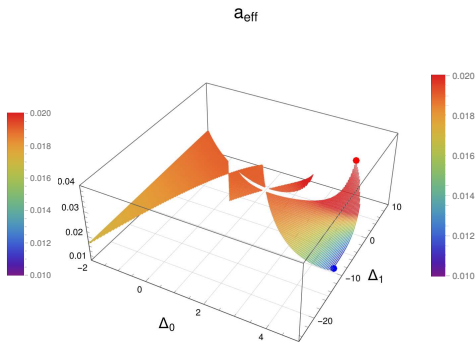
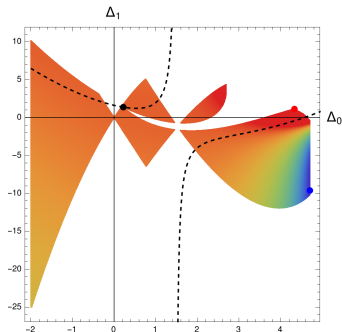
3. 2D domain of $\mathbf{C}^{\text{Bjp}}(a_s)$ at $t = 7(Q^2 \approx 111 \text{ GeV}^2)$. $\mathbf{c}'_3 = \mathbf{c}'_4 = \mathbf{0}$

at $\Delta_2 = 0, n_f = 5,$

$\bullet \rightarrow (\Delta_0^{(5)} = 0.220, \Delta_1^{(5)} = 1.368),$

$$C(t, \{\Delta\}) = 1 + c_1^{\text{ns}} \left(1 a'_s + c'_2(\Delta_0) a_s'^2 + 0 + 0 \right),$$

$$c'_2(\Delta_0) = c_2 - \beta_0 \cdot \Delta_0^{(4,5)} \xrightarrow{n_f=5} 11 \frac{2}{3} - \beta_0 \cdot 0.22$$



3. 2D domain of $\mathbf{C}^{\text{Bjp}}(a_s)$ at $t = 7$ ($Q^2 \approx 111 \text{ GeV}^2$). Extrema

↓ way to the min; ● **0.011**

$$\Delta' \rightarrow \{4.7, -9.62\} \downarrow$$

$$\bar{a}_s(t' = 2.872) = 0.0248$$

$$< \alpha_{\text{eff}}(t = 7) \approx \mathbf{0.0190} <$$

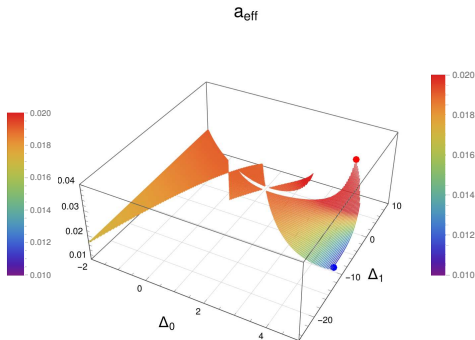
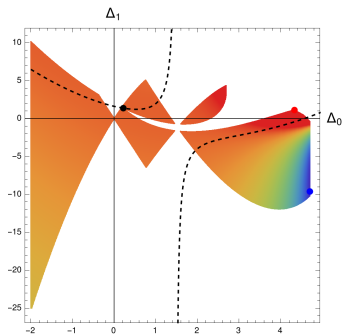
$$(\Delta_0 = 0, \Delta_1 = 0)$$

0.038 ●; way ↓ to the max,

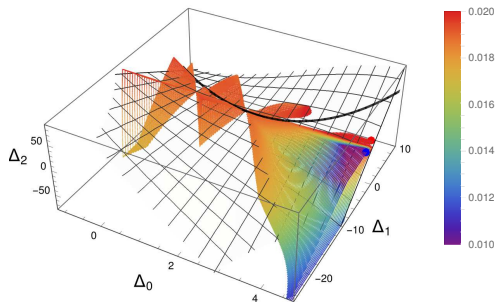
$$\Delta'' \rightarrow \{4.33, 1.12\} \downarrow$$

$$\bar{a}_s(t'' = 4.7) = 0.0431,$$

not reliable result



3. 3D domain of $\mathbf{C}^{\text{Bjp}}(a_s)$ at $t = 7$



3D: $\Delta_0, \Delta_1, \Delta_2$

Net – $\mathbf{c}_4 = 0$;

solid curve on the net –

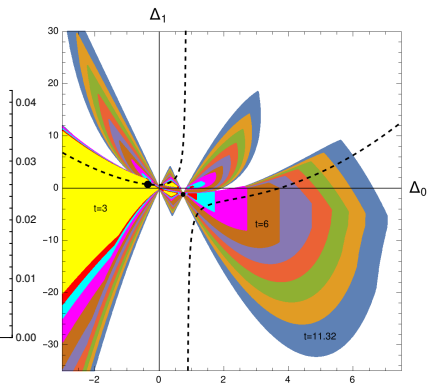
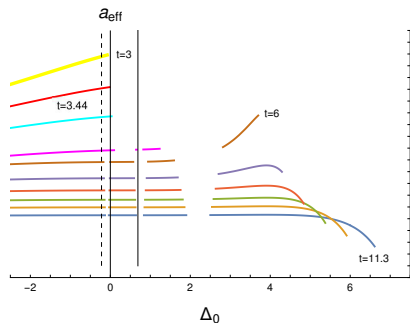
$\mathbf{c}_4 = \mathbf{c}_3 = 0$.

The red-●/blue-● points are
max/min of α_{eff} in the domain.

↓ way to the min; ● **0.00001** $< \alpha_{\text{eff}}(t = 7) \approx \mathbf{0.0190} < \mathbf{0.0428}$ ●; way ↓ to the max,
 $\Delta' \rightarrow \{4.69, -2.56, 7.11\}$ ↓ $\Delta'' \rightarrow \{4.57, 1.42, -3.27\}$
 $\bar{a}_s(t' = 4.59) = 0.041$ $\bar{a}_s(t'' = 4.68) = 0.0428$.

The results are not reliable, too close the **PT bound** $t', t'' \approx 4.7$

1D-2D domains for Adler $D_{ns}(a_s)$



lines — — — $d_4(\Delta_0, \Delta_1) = 0$; point $\bullet$ — $\{d_3(\Delta_0, \Delta_1) = d_4(\Delta_0, \Delta_1) = 0\}$,

4. Motivation for the revision of series representation

consider 1-scale Q^2 **RG-INVARIANT** quantities at $Q^2 = \mu^2, \overline{MS}$, e.g., D_{ns}

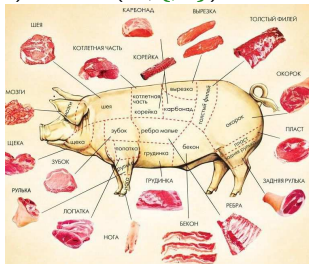
"Wild" approach: $\forall d_n$ - numbers,
taken **wholly**

$$D(a_s) \sim 1 + a_s d_1 + a_s^2 d_2 + a_s^3 d_3 + \dots$$



Delicate approach: $\forall d_n$ has an
inherent structure due to **a_s -renorm.**

$$D(a_s) \sim 1 + \hat{M}(a_s, \{\beta_i\}) \leftarrow \text{2D matrix}$$



$$d_2 = 31.77 - 1.84n_f;$$

$$d_3 = 1164.8 - 270.1n_f - 5.5n_f^2;$$

$$d_4 = 34765 - 8806.4n_f + 481.3n_f^2 \\ - 2.56n_f^3.$$

$$d_2 = \beta_0 d_2[1] + d_2[0]; [\text{base notation}]$$

$$d_3 = \beta_0^2 d_3[2] + \beta_1 d_3[0, 1] + \beta_0 d_3[1] + d_3[0]$$

$$d_4 = \beta_0^3 d_4[3] + \beta_2 d_3[0, 0, 1] + \dots$$

→ series becomes "thick"

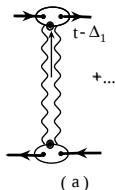
the decomposition is named **$\{\beta\}$ -expansion [MS2004-7]**
it shows the **dynamic** knowledge of D exhibited how **a_s -renorm.**

4. Alternative approaches inspired by $\{\beta\}$ -expansion

Evident usage of the $\{\beta\}$ -expansion is the different kinds of **optimization**: one can change the contributions of different origins of **a_s-renorm** playing with the choice of μ^2 . Let divide any c_n in two parts $c_n = c_n^{\text{rest}} + c_n^{\mu'}$. Every second part $c_n^{\mu'}$ we transfers into new scale μ' of coupling constant **a'**. For the finite sums of perturbation series one obtains

$$\mathbf{C} - \mathbf{1} = \sum_{n=1}^N \bar{a}_s^n(\mu^2) c_n \rightarrow \mathbf{C}' - \mathbf{1} = \sum_{n=1}^N a_s^n(\mu'^2) c_n^{\text{rest}}$$

PMC means that $c_n^{\text{rest}} = c_n[0]$ of the $\{\beta\}$ -expansion, while the sum of all the others elements of the expansion are transferred in new $\mu'^2 = \mu_{\text{PMC}}^2$.



$$c_2 = \underline{\beta_0 c_2[1]} + c_2[0] \rightarrow c_2[0] \text{ at } \mu^2 \rightarrow \mu_{\text{BLM}}^2 = \exp(-c_2[1]/c_1) \mu^2$$

$$\beta_0 = 11/3 C_A - 4/3 T_R n_f \quad \text{profit at } |\beta_0 c_2[1]| \gg |c_2[0]|$$

$$c_3 = \underline{\beta_0^2 c_3[2]} + \beta_1 c_3[0, 1] + \beta_0 c_3[1] + c_3[0] \rightarrow c_3[0]$$

$$c_n = \underline{\beta_0^{n-1} c_n[n-1]} + \dots + c_n[0] \rightarrow c_n[0]$$

NNA

PMC

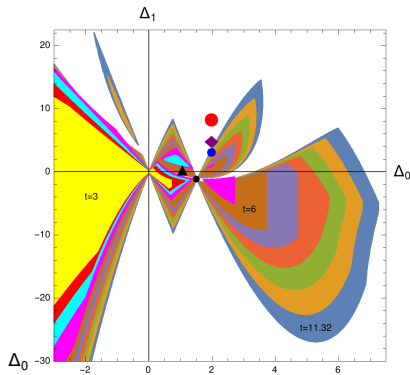
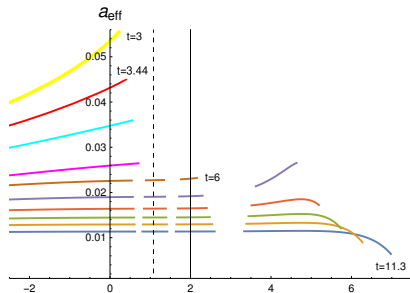
Principle Maximum Conformality in a sense of [S.Brodsky,X.-G.Wu,+ 2012-2026] [Cvetic&Kataev2016, Kataev&Molokoedov+2018-2023]: was inspired by the **CBK** [Baikov&MS2022] based on **QCDe** [Chetyrkin2022], [MS2004,2024]

4. PMC points on the background of domains for $C^{\text{Bjp}}(a_s)$

$$C^{\text{Bjp}}(\Delta_0, \Delta_1, \Delta_2) = 1 + a_{\text{eff}} \cdot c_1$$

(Left, **1D**) $\Delta_0, \Delta_{1,2} = 0$;

(Right, **2D**) $\Delta_0, \Delta_1, \Delta_2 = 0$



1D PMC: — 2, BLM; - - - $2 - \frac{11}{12}$, Brodsky+

2D PMC: ● — MS; ◆ — Kataev; ▲ — Brodsky+.

gNNA: ●; $\{c_{2,3,4} = 0\}$ — ..

4. PMC points on the background of domains for Adler D_{NS}

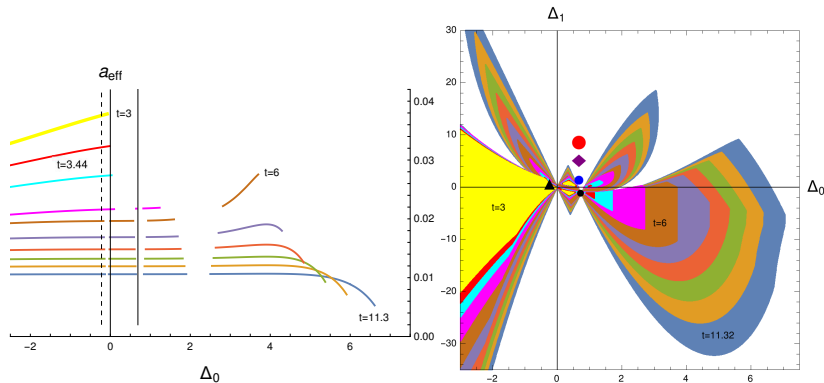
$t = 3, \quad 3.44, \quad 4, \quad 5, \dots, 8, \quad 9, \quad 10, \quad 11.32:$

$Q^2 = 2, (m_\tau = 1.77)^2, 5.5, 15, \dots, 301, 819, 2227, (M_Z = 91.19)^2 \text{ GeV}^2$

(Left, **1D**) $\Delta_0, \Delta_{1,2}=0;$

(Right, **2D**) $\Delta_0, \Delta_1, \Delta_2=0$

$$D_{\text{NS}}(\Delta_0, \Delta_1, \Delta_2)/d_F = 1 + \mathbf{a}_{\text{eff}} \cdot \mathbf{d}_1$$



1D PMC: vertical — 0.69, BLM; - - - 0.22, Brodsky+;

2D PMC: ● - MS; ◆ - Kataev; ▲ - Brodsky+. gNNA: ●. $\{c_{2,3,4} = 0\}$ - ●.

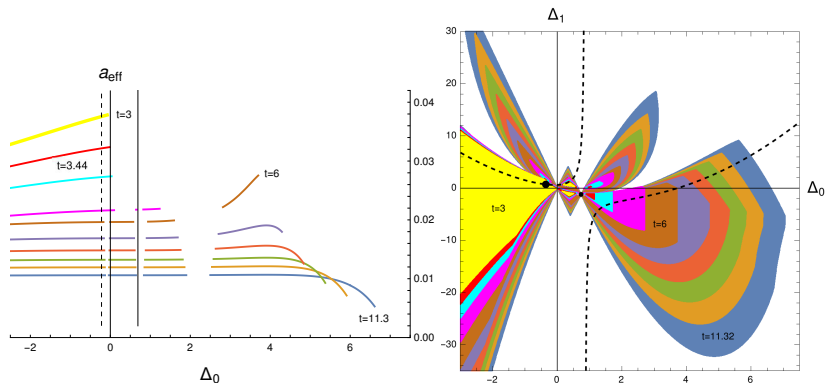
Criticism of PMC and superfluous of $\{\beta\}$ -expansion

1. Those kind of $\{\beta\}$ -expansion that was suggested by [Brotsky et al2012] is based on R_δ procedure and “special degeneration” looks **methodically questionable**. Practically its result doesn't agree with diagrammatic analysis for D_A , C^{Bjp} in orders $O(a_s^3)$, $O(a_s^3)$ [MS2024].
2. The special choice $d_n[0] = \gamma_n^{\text{photon}}$ for the case of D_A [Brotsky et al2012] looks completely **artificial** [Kataev&MS2015].
3. Renormalon-chain contributions, removed from c_n (they $\subset c_n^{\mu'}$) are transferred in new scale μ'^2 , that creates in its turn, **the $n!$ -problem** of grows there.
4. They are unknown the cases when **PMC**-procedure leads to practical improvement of the finite perturbative sums [Kataev et al 2023]. This is expected and natural because the **PMC** was suggested for another purpose.

CONCLUSIONS

1. The generalized **RG transformation**, which is equipped with a series of parameters $\{\Delta_i\}_0^{n-2}$ in order n , is invented for the **RG-invariant quantities**.
2. We suggest some necessary conditions for the admissible domains of $\{\Delta_i\}$, so called “**filters**”. In the frame of these conditions an optimization of PT-series with coefficients $c_4 = 0$ and $c_3 = c_4 = 0$ can be realized.
3. We review the results of various techniques to transform perturbation series of physical quantities such as the BSR C^{Bjp} and **Adler D-function**, against the background of the **filters**. Only a few of these results are acceptable in view of the **filters**.

STORE, 1D-2D domains for Adler $D_{ns}(a_s)$



lines — — $d_4(\Delta_0, \Delta_1) = 0$; point $\bullet$ — $\{d_3(\Delta_0, \Delta_1) = d_4(\Delta_0, \Delta_1) = 0\}$,