Optimization of perturbation series for physical quantities using the QCD renormalization group:

necessary conditions and partial results for Bjorken Sum Rule and Adler D-function

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Intro. What are Bjorken Sum Rule, $C^{\rm Bjp}$, Adler **D**-function

There are renorm-group invariant single scale Q^2, $Q^2 = \mu_R^2$, quantities $C^{\rm Bjp}$, D:

Bjorken polarized Sum Rules in DIS

$$\frac{1}{6}\left|\frac{g_A}{g_V}\right| C^{\rm Bjp}(a_s) + \text{h. tw.} = S_{\rm NS}^{\rm Bjp}(Q^2) = \int_0^1 \left[g_1^{lp}(x,Q^2) - g_1^{ln}(x,Q^2)\right] dx + \text{h. tw.}$$

Adler function

$$d_R \boldsymbol{D}(a_s) = D_A = -12\pi^2 Q^2 \frac{d}{dQ^2}\boldsymbol{\Pi}(Q^2); \quad Q^2 = -q^2$$

Crewther-Broadhurst-Kataev (CBK) relation

–a plausible **conjecture** **[Crewther 1972,1997]** inspired by conformal symmetry

$$D_{ns}(a_s)\cdot C^{\rm Bjp}(a_s) = \mathbb{1} + \beta(a_s)K(a_s), \text{where } K(a_s) - \text{polynom in } a_s = \frac{\alpha_s}{4\pi}$$

Crucial 3-loop analysis **[Broadhurst,Kataev,1993]** in $\overline{\textbf{MS}}$-scheme - **CBK** relation

———————————————————————————

[P.Baikov,K.Chetyrkin,J.Kühn, PRL2010]- confirmation in $O(a_s^4)$ **5 loops**.

Intro. First attempt at series optimization by varying μ^2: **the BLM**

BLM (Brodsky, Lepage, Mackenzie) [**PRD 28**(1983)228], cited $\simeq$**1300**

RG equation: $$\mu^2 \frac{d\bar{a}_s(\mu^2)}{d\mu^2} = -\bar{a}_s^2(\mu^2)\left(\beta_0 + \bar{a}_s\beta_1 + \ldots\right), \quad a_s = \frac{\alpha_s}{4\pi}$$

$$\beta_0 = \frac{11}{3}C_A - \frac{4}{3}T_r \boldsymbol{n_f}, \qquad \bar{a}_s(\mu^2) = \frac{a_s(\nu^2)}{1 + a_s(\nu^2)\beta_0 \ln\left(\mu^2/\nu^2\right)}$$

Bjorken polarized Sum Rule $\boldsymbol{C}^{\rm Bjp}\left(Q^2/\mu^2, \mu^2\right)$,

$$C^{\rm Bjp}\Big|_{Q^2=\mu^2} = 1 - 4\left[a_s(\mu^2) + a_s^2(\mu^2)\begin{pmatrix} \boxed{2\cdot\beta_0} - 11/3 \\ -\frac{4}{3}\boldsymbol{n_f} + \frac{23}{3}C_A - \frac{7}{3}C_F \end{pmatrix} + \ldots\right]$$

$$\left(\mu^2 \to \boldsymbol{\nu^2} = \boldsymbol{e^{-2}}\mu^2\right) = 1 - 4\left[\boldsymbol{a_s(\nu^2)} + \boldsymbol{a_s^2(\nu^2)}\,(-11/3) + \ldots\right]$$

Adler function $\boldsymbol{D}_A\left(\boldsymbol{Q^2/\mu^2, \mu^2}\right)$ $[= \boldsymbol{R_{e^+e^-\to h}}]$

$$\frac{D_A}{d_R}\Big|_{Q^2=\mu^2} = 1 + 4\left[a_s(\mu^2) + a_s^2(\mu^2)\begin{pmatrix} \boxed{0.69\cdot\beta_0} + 1/3 \\ -0.69\frac{2}{3}\boldsymbol{n_f} + 2.87C_A - C_F/2 \end{pmatrix} + \ldots\right]$$

$$\left(\mu^2 \to \boldsymbol{\nu^2} \approx \boldsymbol{e^{-\ln(2)}}\mu^2\right) = 1 + 4\left[\boldsymbol{a_s(\nu^2)} + \boldsymbol{a_s^2(\nu^2)}\,(1/3) + \ldots\right]$$

♣ **Next LO coefficients become visibly smaller**

♠ $\boldsymbol{a_s(\nu^2)}$ doesn't run **here**, $\boldsymbol{\nu^2}$ - **a"true virtuality", characteristic of observable**

OUTLINE

1. **Intro.** Examples of RG-invariants:
   Bjorken $C^{\mathrm{Bjp}}(a_s)$ and **Adler** $\boldsymbol{D_{ns}(a_s)}$
   and **BLM**-optimization of their PT-series.

2. Generalized **RG** transformation with a number of parameters $\{\boldsymbol{\Delta}_i\}$.
   The constraints on the parameters $\{\boldsymbol{\Delta}_i\}$ determine
   the “field of the game” for this **RG** transformation.

3. "Optimization" by truncating of the PT-series with the last coefficients,
   $\boldsymbol{c_{n-1,n} = 0}$.

4. What is the decomposition **$\{\beta\}$-expansion**? And what does it express?
   Different transformations of PT-series on background of the constraints

5. **Conclusions**

STORE

2. Generalized RG transformation for invariant series C

$$\bar{a}(t) = \exp\left(-\Delta\beta(\bar{a})\partial_{\bar{a}}\right)\bar{a}\,|_{\bar{a}=a'} = a(\Delta, a') = a' - \frac{\Delta}{1!}\beta(a') + \left(\frac{\Delta}{2!}\beta(a')\partial_{a'}\right)(\beta(a')\Delta) + \ldots$$

[Kotlorz&MS PRD2019] $(c_n, a_s, \boldsymbol{\mu^2}) \xrightarrow{RG} (c'_n, a'_s, \boldsymbol{\mu'^2}) \Rightarrow c'_n = \hat{\boldsymbol{B}}_{nj} c_j$

Shift of the argument $t = \ln(\mu^2/\Lambda^2)$ **with** $\Delta \to \boldsymbol{\Delta}(a')$

$$\ln(\mu^2/\mu'^2) = t - t' \equiv \boldsymbol{\Delta}(a') = \Delta_0 + (a'\beta_0)\,\boldsymbol{\Delta_1} + (a'\beta_0)^2\boldsymbol{\Delta_2} + \ldots, \quad \boxed{\Delta_0 = c_2[1]/c_1}$$

$\boldsymbol{\Delta_j}$ – free components

↑ **BLM**

$$\boxed{c_2 = \beta_0 c_2[1] + c_2[0]}$$

	$a^1 \cdot c_1$	$\to \; a'^1 \cdot [c_1];$
Each	$a^2 \cdot c_2$	$\to \; a'^2 \cdot [c'_2 = c_2 - 1\beta_0\Delta_0];$
order n	$a^3 \cdot c_3$	$\to \; a'^3 \cdot \left[c'_3 = c_3 - c_2 2\beta_0\boldsymbol{\Delta_0} - 1\left(\beta_1\boldsymbol{\Delta_0} - \beta_0^2\boldsymbol{\Delta_0^2} + \beta_0^2\underline{\boldsymbol{\Delta_1}}\right)\right];$
acquires new	$a^4 \cdot c_4$	$\to \; a'^4 \cdot \left[c'_4 = c_4 - c_3 3\beta_0\boldsymbol{\Delta_0} - c_2\left(2\beta_1\boldsymbol{\Delta_0} - 3\beta_0^2\boldsymbol{\Delta_0^2} + 2\beta_0^2\boldsymbol{\Delta_1}\right)\right.$
parameter $\boldsymbol{\Delta_{n-2}}$		$\left. - 1(\ldots + \beta_0^3\underline{\underline{\boldsymbol{\Delta_2}}})\right]$

Fitting components $\boldsymbol{\Delta_0, \Delta_1, \Delta_2, \ldots}$ of new normalization scale $\boldsymbol{\mu'^2}$ to adjust the elements $\boldsymbol{c'_2, c'_3, c'_4}, \ldots$ following to any **chosen optimization**.

2. General constraints to maintain the PT hierarchy, 1

General requirements $(\boldsymbol{A}), (\boldsymbol{B}), (\boldsymbol{C})$ to a series $C(a_s) \stackrel{RG}{\rightarrow} C'(a'_s)$

$$t - t' \equiv \boldsymbol{\Delta} = \boldsymbol{\Delta_0} + a'_s\beta_0\boldsymbol{\Delta_1} + (a'_s\beta_0)^2\boldsymbol{\Delta_2} \Rightarrow$$

$$(\boldsymbol{A}) \quad |\boldsymbol{\Delta_0}| > a'_s\beta_0|\boldsymbol{\Delta_1}| > (a'_s\beta_0)^2|\boldsymbol{\Delta_2}| > \ldots$$

$$C(a_s) - 1 = c_1(a_s + (a_s)^2c_2 + \ldots) \xrightarrow{RG} c_1(a'_s + (a'_s)^2c'_2 + \ldots) \Rightarrow$$

$$(\boldsymbol{B}) \quad 1/c_1 > a'_s > (a'_s)^2|c'_2| > (a'_s)^3|c'_3| > (a'_s)^4|c'_4| > \ldots$$

$$(\boldsymbol{C}) \quad t' = t - \boldsymbol{\Delta} \geqslant \boldsymbol{t_0} = \boldsymbol{2.3} - \textbf{PT boundary at } \boldsymbol{\mu_0^2 = 1}\,\text{GeV}^2.$$

The set of inequalities $\boldsymbol{A} \wedge \boldsymbol{B} \wedge \boldsymbol{C}$ ($2\boldsymbol{D}$ series) is universal for any physical quantity. For **Bjorken** $\boldsymbol{C}^{\textbf{Bjp}}(a_s)$ and **Adler** $\boldsymbol{D}_{\text{NS}}(\boldsymbol{a_s})$ these constrains shape of **admissible domains** for every value of $\boldsymbol{t} = \ln\left(\frac{Q^2}{\Lambda^2_{qcd}}\right)$, **[Kotlorz&MS PRD2019]**

2. General constraints to maintain the PT hierarchy, 2

Besides the conditions $(\boldsymbol{A})$, $(\boldsymbol{B})$ on these Two series

one should solve the equation with respect to a'_s for every fixed t,

$$a'_s \equiv \bar{a}_s(t') = a_s(t - \Delta(a'_s))$$

$$t = 3, \quad 3.44, \quad 4, \quad 5, \ldots, \quad 8, \quad 9, \quad 10, \quad 11.32:$$
$$Q^2 = 2, (m_\tau = 1.77)^2, 5.5, 15, \ldots, 301, \; 819, \; 2227, \; (M_Z = 91.19)^2 \text{ GeV}^2$$

$$C^{\text{Bjp}}(\Delta) = 1 + c_1 \cdot a_{\text{eff}}, \; a_{\text{eff}} = a_s\left(1 + a_s c_2 + (a_s)^2 c_3 + (a_s)^3 c_4 + \ldots\right)$$

$$\frac{D_{\text{NS}}(\Delta')}{d_R} = 1 + d_1 \cdot a'_{\text{eff}}, \; a'_{\text{eff}} = a_s\left(1 + a_s d_2 + (a_s)^2 d_3 + (a_s)^3 d_4 + \ldots\right)$$

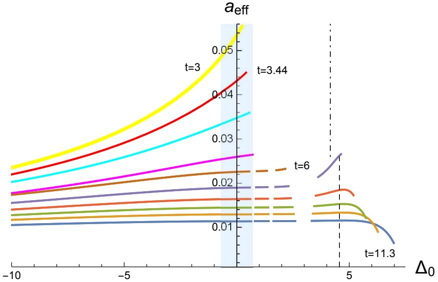
2. 1D-2D domains for Bjorken $\boldsymbol{C}^{\text{Bjp}}(a_s)$

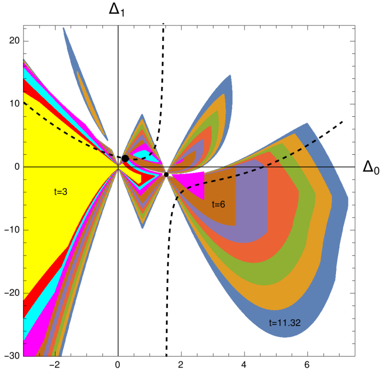
$$\boldsymbol{C}^{\text{Bjp}}(\boldsymbol{\Delta_0}, \boldsymbol{\Delta_1}, \boldsymbol{\Delta_2}) = 1 + \boldsymbol{a}_{\text{eff}} \cdot c_1$$

$$t \;=\; \mathbf{3}, \quad \mathbf{3.44}, \quad \mathbf{4}, \quad \mathbf{5}, \ldots, \; 8, \quad 9, \quad 10, \quad 11.32:$$
$$Q^2 = \mathbf{2}, (\boldsymbol{m_\tau} = \mathbf{1.77})^{\mathbf{2}}, \mathbf{5.5}, \mathbf{15}, \ldots, 301, \; 819, \; 2227, \; (M_Z = 91.19)^2 \text{ GeV}^2$$

"channels" and "islands" coordinately appear in $\{\boldsymbol{\Delta}\}$ admissible spaces.

(Left, **1D**) $\boldsymbol{\Delta_0}$, $\boldsymbol{\Delta_{1.2}} = \mathbf{0}$; (Right, **2D**) $\boldsymbol{\Delta_0}, \boldsymbol{\Delta_1}$, $\boldsymbol{\Delta_2} = \mathbf{0}$





lines $- - -\boldsymbol{c_4}(\Delta_0, \Delta_1) = 0$; point $\bullet$ – $\{\boldsymbol{c_3}(\Delta_0, \Delta_1) = \boldsymbol{c_4}(\Delta_0, \Delta_1) = 0\}$,

blue strip corresponds to variation $[\mu^2/2 - 2\mu^2]$

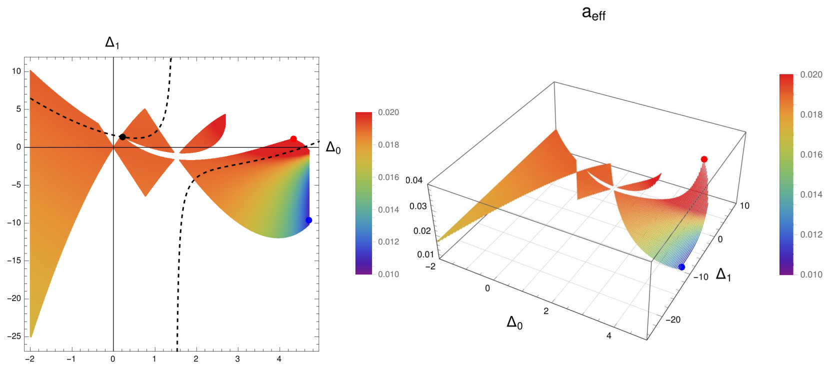
The larger is t – the larger is size of the corresponding domain.

3. 2D domain of $C^{\rm Bjp}(a_s)$ at $t = 7(Q^2 \approx 111\ \text{GeV}^2)$. $c'_3 = c'_4 = 0$

$$\text{at } \Delta_2 = 0, n_f = 5, \qquad \bullet \to (\Delta_0^{(5)} = 0.220, \Delta_1^{(5)} = 1.368),$$

$$C\,(t, \{\Delta\}) = 1 + c_1^{\rm ns}\left(1\,a'_s + c'_2(\Delta_0)\,a'^2_s + 0 + 0\right),$$

$$c'_2(\Delta_0) = c_2 - \beta_0 \cdot \Delta_0^{(4,5)} \xrightarrow{n_f=5} 11\frac{2}{3} - \beta_0 \cdot 0.22$$

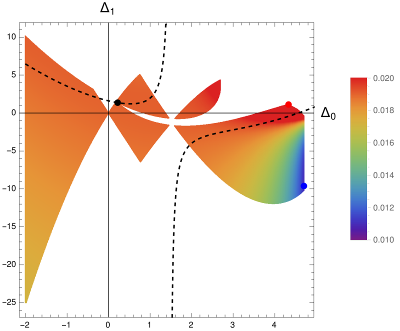


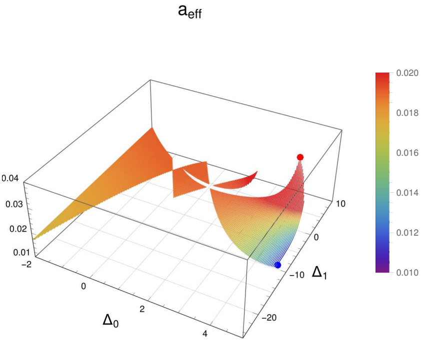
3. 2D domain of $\boldsymbol{C}^{\rm Bjp}(a_s)$ at $t = 7$ ($Q^2 \approx 111\ \text{GeV}^2$). Extrema

$\downarrow$ way to the min; $\bullet$ **0.011** $< \boldsymbol{\alpha_{\rm eff}(t = 7) \approx 0.0190} <$ **0.038** $\bullet$; way $\downarrow$ to the max,

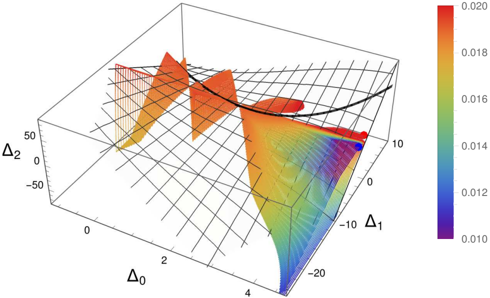
$\Delta' \to \{4.7, -9.62\} \downarrow$ $\quad (\Delta_0 = 0, \Delta_1 = 0) \quad$ $\Delta'' \to \{4.33, 1.12\} \downarrow$

$\bar{a}_s(t' = 2.872) = 0.0248$ $\quad\quad$ $\bar{a}_s(t'' = 4.7) = 0.0431$,





3. 3D domain of $C^{\rm Bjp}(a_s)$ at $t = 7$



3D: Δ_0, Δ_1, Δ_2

The red-●/blue-● points are max/min of $\alpha_{\rm eff}$ in the domain.

Net – $c_4 = 0$;

solid curve on the net – $c_4 = c_3 = 0$.

$\downarrow$ way to the min; ● **0.00001** $< \alpha_{\rm eff}(t = 7) \approx$ **0.0190** $<$ **0.0428** ●; way $\downarrow$ to the max,

$\Delta' \to \{4.69,\ -2.56,\ 7.11\} \downarrow$ $\quad\downarrow \Delta'' \to \{4.57, 1.42, -3.27\}$

$\bar{a}_s(t' = 4.59) = 0.041$ $\quad\bar{a}_s(t'' = 4.68) = 0.0428$.

4. Motivation for the revision of series representation

consider 1-scale Q^2 RG-INVARIANT quantities at $Q^2 = \mu^2$, $\overline{\text{MS}}$, e.g., D_{ns}

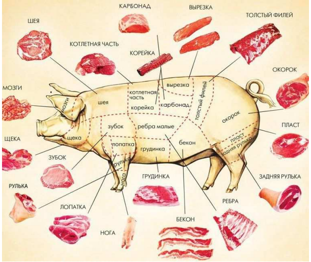
"**Wild**" approach: $\forall d_n$ - numbers, taken **wholly**

$$D(a_s) \sim 1 + a_s d_1 + a_s^2 d_2 + a_s^3 d_3 + \ldots$$

$$\begin{aligned} d_2 &= 31.77 - 1.84 n_f; \\ d_3 &= 1164.8 - 270.1 n_f - 5.5 n_f^2; \\ d_4 &= 34765 - 8806.4 n_f + 481.3 n_f^2 \\ &\quad - 2.56 n_f^3. \end{aligned}$$

Delicate approach: $\forall d_n$ has an inherent structure due to a_s-renorm.

$$D(a_s) \sim 1 + \hat{M}(a_s, \{\beta_i\}) \leftarrow \text{2D matrix}$$



$$\begin{aligned} d_2 &= \beta_0 d_2[1] + d_2[0]; \text{[base notation]} \\ d_3 &= \beta_0^2 d_3[2] + \beta_1 d_3[0,1] + \beta_0 d_3[1] + d_3[0] \\ d_4 &= \beta_0^3 d_4[3] + \beta_2 d_3[0,0,1] + \ldots \\ &\to \text{series becomes "thick"} \end{aligned}$$

the decomposition is named **$\{\beta\}$-expansion [MS2004-7]**

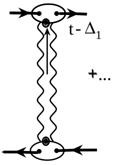
it shows the **dynamic** knowledge of D exhibited how **a_s-renorm.**

4. Alternative approaches inspired by $\{\beta\}$-expansion

Evident usage of the $\{\beta\}$-**expansion** is the different kinds of **optimization**: one can change the contributions of different origins of a_s-**renorm** playing with the choice of μ^2. Let devide any c_n in two parts $c_n = c_n^{\text{rest}} + c_n^{\mu'}$
Every second part $c_n^{\mu'}$ we transfers into new scale μ' of coupling constant a'
For the finite sums of perturbation series one obtains

$$C - 1 = \sum_{n=1}^{N} \bar{a}_s^n(\mu^2)\, c_n \to C' - 1 = \sum_{n=1}^{N} a_s^n(\mu'^2)\, c_n^{\text{rest}}$$

PMC means that $c_n^{\text{rest}} = c_n[0]$ of the $\{\beta\}$-**expansion**, while the sum of all the others elements of the expansion are transferred in new $\mu'^2 = \mu^2_{PMC}$.



(a)

$$c_2 = \underline{\beta_0 c_2[1]} + c_2[0] \to c_2[0] \text{ at } \mu^2 \to \mu^2_{BLM} = \exp(-c_2[1]/c_1)\mu^2$$

$$\beta_0 = 11/3C_A - 4/3T_R n_f \Big| \text{ profit at } |\ \beta_0 c_2[1]\ | \gg |\ c_2[0]\ |$$

$$c_3 = \underline{\beta_0^2 c_3[2]} + \beta_1 c_3[0,1] + \beta_0 c_3[1] + c_3[0] \to c_3[0]$$

$$c_n = \underline{\beta_0^{n-1} c_n[n-1]} + \ldots + c_n[0] \to c_n[0]$$

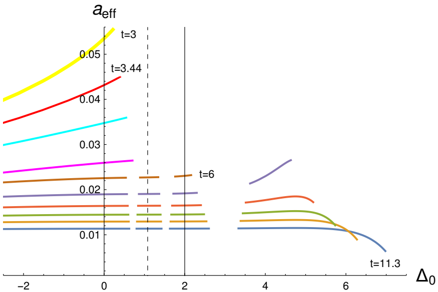
<u>**NNA**</u> **PMC**

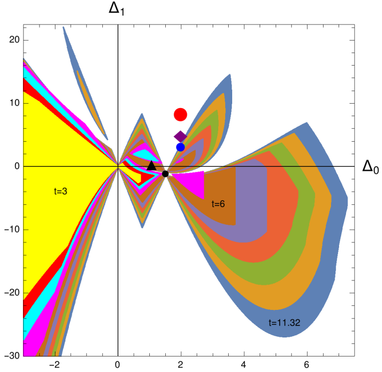
PrincipeMaximumConformality in a sense of **[S.Brodsky,X.-G.Wu,+ 2012-2026]**
[Cvetic&Kataev2016, Kataev&Molokoedov+2018-2023]: was inspired by the **CBK**
[Baikov&MS2022] based on **QCDe** **[Chetyrkin2022]**, **[MS2004,2024]**

4. PMC points on the background of domains for $C^{\mathrm{Bjp}}(a_s)$

$$C^{\mathrm{Bjp}}(\Delta_0, \Delta_1, \Delta_2) = 1 + a_{\mathrm{eff}} \cdot c_1$$

(Left, 1D) Δ_0, $\Delta_{1.2} = 0$; (Right, 2D) Δ_0, Δ_1, $\Delta_2 = 0$





1D PMC: —— 2, BLM; - - - - $2 - \frac{11}{12}$, Brodsky+

2D PMC: ● – MS; ♦ – Kataev; ▲ – Brodsky+. gNNA: ●; $\{c_{2,3,4} = 0\}$ – .

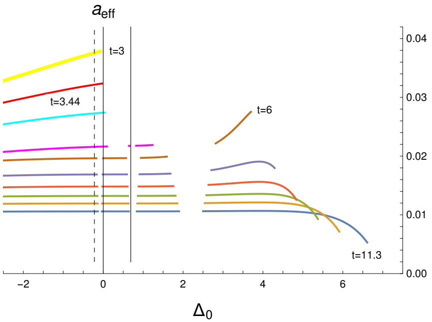
4. PMC points on the background of domains for Adler D_{NS}

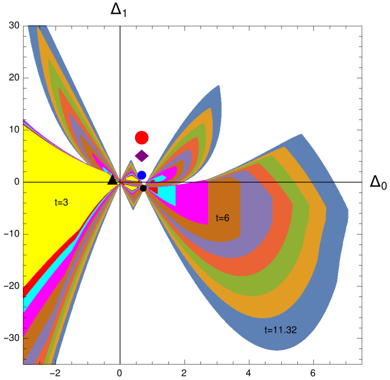
$t\ =\ \mathbf{3},\quad \mathbf{3.44},\quad \mathbf{4},\quad \mathbf{5},\ldots,\ 8,\quad 9,\quad 10,\quad 11.32:$
$Q^2 = \mathbf{2}, (\boldsymbol{m_\tau} = \mathbf{1.77})^{\mathbf{2}}, \mathbf{5.5}, \mathbf{15}, \ldots, 301,\ 819,\ 2227,\ (M_Z = 91.19)^2\ \mathrm{GeV}^2$

(Left, 1D) $\boldsymbol{\Delta_0}$, $\boldsymbol{\Delta_{1.2}=0}$;

(Right, 2D) $\boldsymbol{\Delta_0}$, $\boldsymbol{\Delta_1}$, $\boldsymbol{\Delta_2=0}$

$$D_{\mathrm{NS}}(\boldsymbol{\Delta_0}, \boldsymbol{\Delta_1}, \boldsymbol{\Delta_2})/d_F = 1 + \boldsymbol{a}_{\mathrm{eff}} \cdot d_1$$





1D PMC: vertical —— 0.69, BLM; - - - −0.22, Brodsky+;

2D PMC: ● - MS; ♦ – Kataev; ▲ – Brodsky+. gNNA: ●. $\{c_{2,3,4} = 0\}$ – ●.

Criticizm of PMC and superfluous of $\{\beta\}$-expansion

1. Those kind of $\{\beta\}$**-expansion** that was suggested by **[Brodsky et al2012]** is based on $\boldsymbol{R_\delta}$ procedure and **"special degeneration"** looks **methodically questionable**. Practically its result doesn't agree with diagrammatic analysis for $\boldsymbol{D_A}$, $\boldsymbol{C}^{\text{Bjp}}$ in orders $O(a_s^3), O(a_s^3)$ **[MS2024]**.
2. The special choice $\boldsymbol{d_n}[\mathbf{0}] = \gamma_{\boldsymbol{n}}^{\text{photon}}$ for the case of $\boldsymbol{D_A}$ **[Brodsky et al2012]** looks completely **artificial** **[Kataev&MS2015]**.
3. Renormalon-chain contributions, removed from $\boldsymbol{c}_n$ (they $\subset \boldsymbol{c}_{\boldsymbol{n}}^{\mu'}$) are transferred in new scale μ'^2, that creates in its turn, **the n!-problem** of grows there.
4. They are unknown the cases when **PMC**-procedure leads to practical improvement of the finite perturbative sums **[Kataev et al 2023]**. This is expected and natural because the **PMC** was suggested for another purpose.

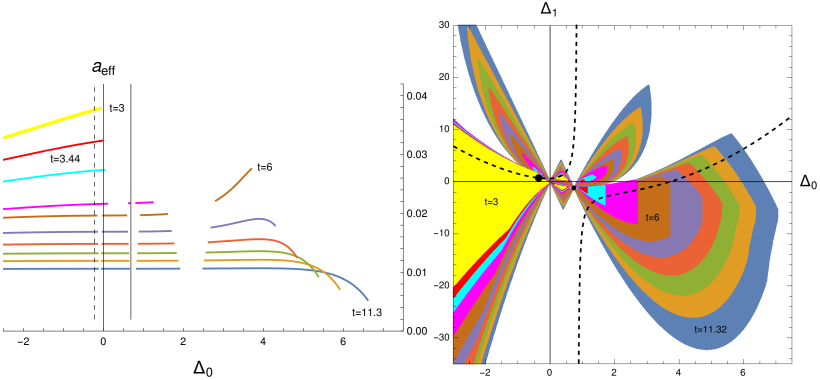
CONCLUSIONS

1. The generalized **RG transformation**, which is equipped with a series of parameters $\{\Delta_i\}_0^{n-2}$ in order n, is invented for the **RG-invariant quantities**.

2. We suggest some necessary conditions for the admissible domains of $\{\Delta_i\}$, so called **"filters"**. In the frame of these conditions an optimization of PT-series with coefficients $c_4 = 0$ and $c_3 = c_4 = 0$ can be realized.

3. We review the results of various techniques to transform perturbation series of physical quantities such as the BSR C^{Bjp} and **Adler D-function**, against the background of the **filters**. Only a few of these results are acceptable in view of the **filters**.

STORE, 1D-2D domains for Adler $D_{ns}(a_s)$



lines $- - -\boldsymbol{d_4}(\Delta_0, \Delta_1) = 0$; point $\bullet - \{\boldsymbol{d_3}(\Delta_0, \Delta_1) = \boldsymbol{d_4}(\Delta_0, \Delta_1) = 0\}$,