Lessons from Quantum Strings for Quantum Gravity

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Based on:

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- Y. M. JHEP 07 (2018) 104 [arXiv:1802.07541]
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Nonrenormalizability of Quantum Gravity

Quadratic divergences Λ^2 in Quantum Gravity are like those in 4-Fermi theory where the idea of peritization by Feinberg-Pais (1963) was to choose the 4-Fermi coupling $G \sim 1/\Lambda^2$ to compensate them. It never worked in $d = 4$ because of additional logarithmic divergences

Emergent action of bosonic string involves higher derivatives of the metric tensor $\Longrightarrow$ nonrenormalizable self-interaction but Λ^2 is compensated by the coupling $1/\Lambda^2$ with no logarithmic divergences thanks to conformal invariance of (regularized) theory which can be exactly solved about the mean-field ground state using the methods of CFT

Important new results (NOT seen within perturbative regularization):

- Lilliputian continuum limit of regularized bosonic string in $d \geq 2$ is STRINGY (bypassing existing no-go theorem)
- Nambu-Goto and Polyakov strings are told apart by the higher-derivative terms
- Exact solution of Nambu-Goto string in $d = 4$ is consistent (in contrast to Polyakov string) and supposedly described by the (4,3) minimal model

2. Mean-field ground state of bosonic string

Nambu-Goto and Polyakov strings

Nambu-Goto string (imaginary Lagrange multiplier λ^{ab}) independent metric tensor g_{ab} Alvarez (1981)

$$K_0 \int \mathrm{d}^2\omega \, \sqrt{\det \partial_a X \cdot \partial_b X} = K_0 \int \mathrm{d}^2\omega \, \sqrt{g} + \frac{K_0}{2} \int \mathrm{d}^2\omega \, \lambda^{ab} \left(\partial_a X \cdot \partial_b X - g_{ab} \right)$$

World-sheet coordinates $\omega_1, \omega_2 \in \omega_L \times \omega_\beta$ rectangle.

Ground state $\lambda^{ab} = \bar{\lambda} \sqrt{g} g^{ab}$ classically $\bar{\lambda} = 1 \Longrightarrow$ Polyakov string

$$\mathcal{S} = \frac{K_0}{2} \int \mathrm{d}^2\omega \, \sqrt{g} g^{ab} \partial_a X \cdot \partial_b X$$

Equivalence: classically Polyakov (1981), one loop Fradkin-Tseytlin (1982).

Closed bosonic string winding once around compactified dimension of circumference β, propagating (Euclidean) time L with topology of cylinder or torus (bagel). No tachyon if β is large enough.

Gaussian path integral over X^μ_{q} by splitting $X^\mu = X^\mu_{\mathrm{cl}} + X^\mu_{\mathrm{q}}$: $\Longrightarrow$ Emergent (or effective) action

$$\begin{aligned} S[g_{ab}, \lambda^{ab}] \; &= \; K_0 \int \mathrm{d}^2\omega \, \sqrt{g} + \frac{K_0}{2} \int \mathrm{d}^2\omega \, \lambda^{ab} \left(\partial_a X_{\mathrm{cl}} \cdot \partial_b X_{\mathrm{cl}} - g_{ab} \right) \\ &+ \frac{d}{2} \mathrm{tr} \log \left(-\frac{1}{\sqrt{g}} \partial_a \lambda^{ab} \partial_b \right) + \text{ ghosts} \end{aligned}$$

Covariant Pauli-Villars regularization

Schwinger proper-time regularization of the trace

$$\mathrm{tr}\log\mathcal{O}\big|_{\mathrm{reg}} = -\int_{a^2}^{\infty}\frac{d\tau}{\tau}\mathrm{tr}\,e^{-\tau\mathcal{O}}, \qquad \Lambda^2 = \frac{1}{4\pi a^2}$$

Pauli-Villars regularization of the trace

$$\det(\mathcal{O})\big|_{\mathrm{reg}} \equiv \frac{\det(\mathcal{O})\det(\mathcal{O}+2M^2)}{\det(\mathcal{O}+M^2)^2}, \qquad \Lambda^2 = \frac{M^2}{2\pi}\log 2$$

$$\mathrm{tr}\log\mathcal{O}\big|_{\mathrm{reg}} = -\int_0^{\infty}\frac{d\tau}{\tau}\mathrm{tr}\,e^{-\tau\mathcal{O}}\left(1-e^{-\tau M^2}\right)^2, \quad \left\langle\left|e^{-\tau\mathcal{O}}\right|\right\rangle = \frac{1}{4\pi\tau}+\ldots$$

Diaz, Troost, van Nieuwenhuizen, Van Proeyen (1989)

Covariant Pauli-Villars regulator Y (preserves conformal invariance)

$$\mathcal{S}^{(\mathrm{reg})} = \int\left(\lambda^{ab}\,\partial_a\bar{Y}\cdot\partial_b Y + \boxed{M^2\sqrt{g}}\,\bar{Y}\cdot Y + \frac{1}{2}\lambda^{ab}\,\partial_a Z\cdot\partial_b Z + \boxed{M^2\sqrt{g}}\,Z^2\right)$$

Two anticommuting Grassmann Y and $\bar{Y}$ of mass squared M^2 and one Z of mass squared $2M^2$ with normal statistics

Advantages over the proper-time regularization:
Feynman's diagrams apply for Pauli-Villars regularization
Gel'fand-Yaglom technique to compare with DeWitt-Seeley expansion

Mean-field ground state

Ambjørn, Y.M. (2017)

For diagonal and constant $\lambda^{ab} = \bar{\lambda}\delta^{ab}$ and $\rho_{ab} = \bar{\rho}\delta_{ab}$

$$
\begin{aligned}
S_{\text{eff}} &= \frac{K_0}{2}\bar{\lambda}\left(\frac{L^2}{\omega_L^2} + \frac{\beta^2}{\omega_\beta^2}\right)\omega_L\omega_\beta + K_0(1-\bar{\lambda})\bar{\rho}\,\omega_L\omega_\beta \\
&\quad - \left(\frac{d}{2\bar{\lambda}} - 1\right)\Lambda^2\bar{\rho}\,\omega_L\omega_\beta - \frac{\pi(d-2)}{6}\frac{\omega_L}{\omega_\beta}
\end{aligned}
$$

with boundary terms omitted for $L \gg \beta$.
The minimum is reached at (quantum ground state)

$$
\begin{aligned}
\bar{\lambda} &= \frac{1}{2}\left(1 + \frac{\Lambda^2}{K_0} + \sqrt{\left(1 + \frac{\Lambda^2}{K_0}\right)^2 - \frac{2d\Lambda^2}{K_0}}\right) \\
\bar{\rho} &\propto \frac{\bar{\lambda}}{\sqrt{(1 + \frac{\Lambda^2}{K_0})^2 - \frac{2d\Lambda^2}{K_0}}} \\
S_{\text{mf}} &= K_0\bar{\lambda}L\sqrt{\beta^2 - \frac{\pi(d-2)}{3K_0\bar{\lambda}}} \qquad \text{(Alvarez-Arvis)} \\
\frac{\omega_\beta}{\omega_L} &= \frac{1}{L}\sqrt{\beta^2 - \frac{\pi(d-2)}{3K_0\bar{\lambda}}} \qquad \text{modular parameter}
\end{aligned}
$$

Mean-field ground state (cont.)

The approximation describes a mean field (variational like Peierls (1930s)) that accounts for an infinite set of bubble diagrams of pertubation theory about the classical vacuum.
Then λ^{ab} and ρ_{ab} do not fluctuate which becomes exact at large d

It is like $O(N)$ sigma-model at large N where the Lagrange multiplier does not fluctuate (summing the bubble graphs) and the large-N vacuum is very close to the physical vacuum even for $N = 3$.
Symmetry restoration $O(N-1) \to O(N)$ is NOT seen perturbatively.
Renormalizability in d=3 is NOT seen perturbatively

The square root is well-defined for $d \geq 2$ if

$$K_0 > K_* = \left(d - 1 + \sqrt{d^2 - 2d}\right) \Lambda^2 \stackrel{d\to\infty}{\to} 2d\Lambda^2$$

Perturbation theory is recovered by expanding in $1/K_0 \sim \hbar$. Then $\bar{\lambda}$ ranges between 1 (classical) and (quantum) value

$$\bar{\lambda}_* = \frac{1}{2}\left(d - \sqrt{d^2 - 2d}\right) \stackrel{d\to\infty}{\to} \frac{1}{2}$$

3. Two scaling regimes: Gulliver's vs. Lilliputian

Particle-like scaling limit (Gulliver's)

The ground state energy (Alvarez-Arvis)

$$E_0(\beta) = K_0\bar{\lambda}\sqrt{\beta^2 - \frac{\pi(d-2)}{3K_0\bar{\lambda}}}$$

does not scale because $K_0 > K_* \sim \Lambda^2$ for $\bar{\lambda}$ to be real ($> \bar{\lambda}_*$).

Only one (lowest) state can scale if

$$\beta^2 \to \frac{\pi(d-2)}{3K_*\bar{\lambda}_*}, \qquad \bar{\lambda}_* = \frac{1}{2}\left(d - \sqrt{d^2 - 2d}\right)$$

from above for not to have a tachyon.

Scaling regime does not exist for excited states and thus is particle-like similar to lattice regularizations of a string, where only the lowest mass scales to finite, excitations scale to infinity (no-go theorem)

Durhuus, Fröhlich, Jonsson (1984), Ambjørn, Durhuus (1987)

Particle-like scaling limit (Gulliver's)

The exited state energy (Alvarez-Arvis)

$$E_0(\beta) = K_0\bar{\lambda}\sqrt{\beta^2 - \frac{\pi(d-2)}{3K_0\bar{\lambda}} + \frac{8N}{K_0\bar{\lambda}}}$$

does not scale because $K_0 > K_* \sim \Lambda^2$ for $\bar{\lambda}$ to be real ($> \bar{\lambda}_*$).

Only one (lowest) state can scale if

$$\beta^2 \to \frac{\pi(d-2)}{3K_*\bar{\lambda}_*}, \qquad \bar{\lambda}_* = \frac{1}{2}\left(d - \sqrt{d^2 - 2d}\right)$$

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Durhuus, Fröhlich, Jonsson (1984), Ambjørn, Durhuus (1987)

Lilliputian string-like scaling limit

Let us "renormalize" the units of length

$$L_R = \sqrt{\frac{\bar\lambda}{\bar\lambda - \bar\lambda_*}}\, L, \qquad \beta_R = \sqrt{\frac{\bar\lambda}{\bar\lambda - \bar\lambda_*}}\, \beta$$

to obtain finite effective action

$$S_{\mathrm{mf}} = K_R\, L_R \sqrt{\beta_R^2 - \frac{\pi(d-2)}{3K_R}}, \qquad K_R = K_0(\bar\lambda - \bar\lambda_*)$$

The renormalized string tension K_R scales to finite

$$K_0 \to K_* + \frac{K_R^2}{2\Lambda^2\sqrt{d^2-2d}}, \qquad K_* = \left(d - 1 + \sqrt{d^2 - 2d}\right)\Lambda^2 \stackrel{d\to\infty}{\to} 2d\Lambda^2$$

reproducing the Alvarez-Arvis spectrum of continuum string

The average area is also finite

$$\langle Area\rangle = L_R \frac{\left(\beta_R^2 - \frac{\pi(d-2)}{6K_R}\right)}{\sqrt{\beta_R^2 - \frac{\pi(d-2)}{3K_R}}}$$

$\Rightarrow$ minimal area for large β_R and diverges if $\beta_R^2 \to \pi(d-2)/3K_R$

The Lilliputian world

Like for the zeta-function regularization except for nonlinearities, but

$$\text{length}_0 = \sqrt{\frac{\bar{\lambda} - \bar{\lambda}_*}{\bar{\lambda}}}\,\text{length}_R \propto \frac{\sqrt{K_R}}{\Lambda}\,\text{length}_R$$

in target space which is of order of the cutoff ($\Rightarrow$ Lilliputian)

Nevertheless, the cutoff (in parameter space) $\boxed{\Delta\omega = 1/(\Lambda \sqrt[4]{g})}$ fixes maximal number of modes in the mode expansion to be

$$n_{\max} \sim \Lambda \sqrt[4]{g}\omega_\beta$$

Classically $\quad \sqrt[4]{g}\omega_\beta = \beta \quad$ reproducing Brink-Nielsen (1973) but

Quantumly $\quad \sqrt[4]{g}\omega_\beta \propto \dfrac{\beta}{\sqrt{\bar{\lambda} - \bar{\lambda}_*}} = \dfrac{\sqrt{K_0}\,\beta}{\sqrt{K_R}} \quad$ is much larger

- Continuum because infinitely smaller distances can be probed (classical music can be played on the Lilliputian strings)

- Gulliver's tools are too coarse to resolve the Lilliputian world (this is why lattice string regularizations of 1980's never reproduce canonical quantization)

The KING of BROBDINGNAG, and GULLIVER.

—Vide. Swift's Gulliver: Voyage to Brobdingnag.

Pub.d June 26th 1803. by H. Humphrey 27 St. James's Street.

4. Instability of classical ground state

Semiclassical energy

Brink, Nielsen (1973)

Semiclassical (or one-loop) correction due to zero-point fluctuations

$$S_{1l} = \left[K_0 - \frac{(d-2)}{2} \Lambda^2 \right] L\beta - \frac{\pi(d-2)L}{6\beta}$$

bulk term Casimir energy

To make it finite, the renormalized string tension is introduced

$$K_R = K_0 - \frac{(d-2)}{2} \Lambda^2$$

which is kept finite as $\Lambda \to \infty$. Then it is assumed it works order by order of the perturbative expansion about the classical ground state, so K_R can be made finite by fine tuning K_0

We see however from the mean-field formula

$$S_{\rm mf} = K_0 \bar{\lambda} L \sqrt{\beta^2 - \frac{\pi(d-2)}{3K_0\bar{\lambda}}}$$

that $S_{\rm mf}$ cannot be finite with changing K_0 (except for $\beta \approx \beta_{\rm min}$). Thus the one-loop correction simply lowers for $d > 2$ the energy of the classical ground state which may indicate its instability

Effective potential

To check stability of the ground state, add the source term

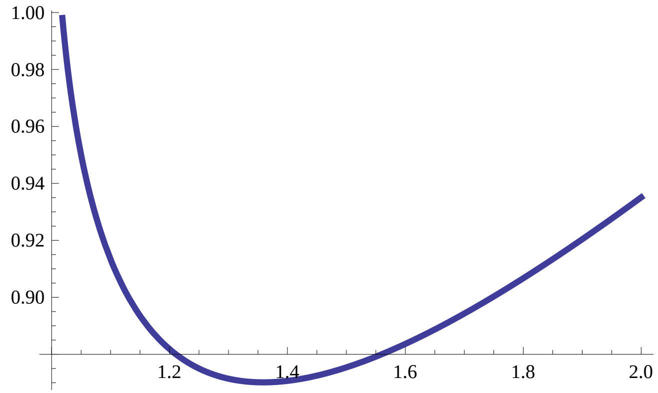
$$S_{\rm src} = \frac{K_0}{2}\int d^2\omega\, j^{ab}\rho_{ab}$$

defining the field

$$\rho_{ab}(j) = -\frac{2}{K_0}\frac{\delta}{\delta j^{ab}}\log Z.$$

Minimizing for constant $j^{ab} = j\delta^{ab}$ we find Ambjørn, Y.M. (2017)
for "effective potential" given by the Legendre transformation

$$\Gamma(\bar\rho) = -\frac{1}{K_0 L\beta}\log Z - j(\bar\rho)\bar\rho$$



In the mean-field approximation

$$\Gamma(\bar\rho) = \left(1+\frac{\Lambda^2}{K_0}\right)\bar\rho - \sqrt{\frac{2d\Lambda^2}{K_0}\bar\rho(\bar\rho-1)}$$

Classical vacuum $\bar\rho = 1$ is unstable and globally stable minimum is

$$\bar\rho(0) = \bar\rho_{\rm m.f.} \qquad \text{if } K_0 > K_* \text{ (same value as before)}$$

5. Fluctuations about mean field

Generalized conformal anomaly

Path-integrating out X^μ_q

$$S[g_{ab},\lambda^{ab}] = K_0 \int \left(\sqrt{g} - \frac{1}{2}\lambda^{ab} g_{ab}\right) + S_X[g_{ab},\lambda^{ab}]$$

$$\mathcal{S}_X = \frac{d}{96\pi}\int \left[-\frac{12\sqrt{g}}{a^2\sqrt{\det \lambda^{ab}}} + \sqrt{g}\, R\frac{1}{\Delta}R - \left(\beta\lambda^{ab} g_{ab} R + 2\lambda^{ab}\nabla_a\partial_b \frac{1}{\Delta}R\right)\right]$$

Higher orders in Schwinger's proper-time ultraviolet cutoff τ dropped. $\beta = 1$ for the Nambu-Goto string but kept arbitrary for generality.

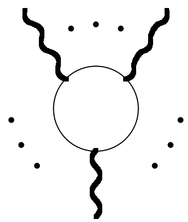
The action is derived from the DeWitt-Seeley expansion of

$$\mathcal{O} = -(\sqrt{g})^{-1}\partial_a \lambda^{ab}\partial_b = -h^{ab}\partial_a\partial_b + A^a\partial_a$$

$$\left\langle \left| \mathrm{e}^{-\tau\mathcal{O}} \right| \right\rangle = \frac{1}{4\pi\tau} + \frac{1}{4\pi}\left(\frac{1}{6}R(h) + E(A,h)\right) + O(\tau)$$

Alternatively Coleman-Weinberg's effective action with Pauli-Villars

$$\frac{d}{2}\,\mathrm{tr}\ln\left[-\frac{1}{\sqrt{g}}\partial_a\lambda^{ab}\partial_b\right]_{\mathrm{reg}} = \sum_n \frac{1}{n}$$



wavy lines correspond to fluctuations $\delta\lambda^{ab}$ or δg_{ab} about ground state

Conformal gauge and flat background

Emergent action becomes local in conformal gauge

$$g_{ab} = \hat{g}_{ab}\,\mathrm{e}^{\varphi}$$

where $\hat{g}_{ab}$ is background (or fiducial) metric tensor.
Usual ghosts and their usual contribution to effective action

Euclidean CFT: conformal coordinates z and $\bar{z}$ in flat background
$g_{zz} = g_{\bar{z}\bar{z}} = 0$, $g_{z\bar{z}} = g_{\bar{z}z} = 1/2$ ($\beta = 1$ for the Nambu-Goto string)

$$\begin{aligned} \mathcal{S}[\varphi,\lambda^{ab}] \;=\; & K_0 \int \mathrm{e}^{\varphi}(1-\lambda^{z\bar{z}}) + \frac{1}{24\pi}\int \{\frac{3\,\mathrm{e}^{\varphi}}{a^2}(2-\frac{d}{\sqrt{\det \lambda^{ab}}}) \\ & +(d-26)\varphi\partial\bar{\partial}\varphi + d[2(1+\beta)\lambda^{z\bar{z}}\partial\bar{\partial}\varphi + \lambda^{zz}\nabla\partial\varphi + \lambda^{\bar{z}\bar{z}}\bar{\nabla}\bar{\partial}\varphi]\} \end{aligned}$$

$\nabla = \partial - \partial\varphi$ is covariant derivative in conformal gauge so it describes a theory with interaction (no such interaction if only $\lambda^{z\bar{z}} = \lambda^{ab}\hat{g}_{ab}$)

Polyakov's book: λ^{ab} has mass $\sim 1/a$ and is localized and decouples.
Thus only φ fluctuates (stable fluctuations for $d < 26$) $\Longrightarrow$ Liouville.
Private life at distances $\sim a$ is macroscopically observable Y.M. (2021)

Equivalence with four-derivative Liouville action

Path integral over $\delta\lambda^{ab}$ has a saddle point justified by small a^2 at

$$\delta\lambda^{ab} = \frac{1}{3}\sqrt{g}a^2\left(g^{ac}g^{bd}\nabla_c\partial_d\varphi + \frac{(\beta-1)}{4}g^{ab}\Delta\varphi\right) + \mathcal{O}(a^4)$$

Thus we arrive at four-derivative Liouville action (conformal gauge)

$$\mathcal{S}[\varphi] = \frac{1}{16\pi b_0^2}\int\sqrt{\hat{g}}[\hat{g}^{ab}\partial_a\varphi\partial_b\varphi + \varepsilon\,\mathrm{e}^{-\varphi}\hat{\Delta}\varphi\left(\hat{\Delta}\varphi - G\hat{g}^{ab}\,\partial_a\varphi\partial_b\varphi\right)]$$

with $G = -1/3$ for the Nambu-Goto string

$$b_0^2 = \frac{6}{26-d}, \quad G = -\frac{1}{1+(1+\beta)^2/2}, \quad \varepsilon = \frac{2d\bar{\lambda}^3}{3G(d-26)}a^2 \sim \Lambda^{-2}$$

which was exactly solved previously Y.M. (2023)

Classically higher-derivative terms vanish for smooth $\varepsilon R \ll 1$.
Quantumly quartic derivative provides UV cutoff but also interaction with coupling ε $\Rightarrow$ uncertainties $\varepsilon \times \varepsilon^{-1}$ which revive $\Longrightarrow$ anomalies.
Yet higher terms which are primary scalars like R^n do not change – universality. ($g^{ab}\,\partial_a\varphi\partial_b\varphi$ is not primary and does change)

Smallness of ε is compensated by change of the metric (shift of φ)

Improved Energy-Momentum Tensor

Callan-Coleman-Jackiw (1970)

Symmetric minimal energy-momentum tensor (by applying $\delta/\delta\hat{g}^{ab}$)

$$-4b_0^2 T_{ab}^{(\mathrm{min})} = \partial_a\varphi\partial_b\varphi - \frac{1}{2}g_{ab}\partial^c\varphi\partial_c\varphi - \mu_0^2 g_{ab} - \varepsilon\partial_a\varphi\partial_b\Delta\varphi - \varepsilon\partial_a\Delta\varphi\partial_b\varphi$$
$$+\varepsilon g_{ab}\partial^c\varphi\partial_c\Delta\varphi + \frac{\varepsilon}{2}g_{ab}(\Delta\varphi)^2 - G\varepsilon\partial_a\varphi\partial_b\varphi\Delta\varphi + G\frac{\varepsilon}{2}\partial_a\varphi\partial_b(\partial^c\varphi\partial_c\varphi)$$
$$+G\frac{\varepsilon}{2}\partial_a(\partial^c\varphi\partial_c\varphi)\partial_b\varphi - G\frac{\varepsilon}{2}g_{ab}\partial^c\varphi\partial_c(\partial^d\varphi\partial_d\varphi)$$

(same as Belinfante EMT) is conserved but not traceless

"Improved" e-m tensor accounts for nonminimal interaction with background gravity coming from $\sqrt{g}R = \sqrt{\hat{g}}\left(\hat{R} - \hat{\Delta}\varphi\right)$

$$-4b_0^2 T_{ab} = -4b_0^2 T_{ab}^{(\mathrm{min})} - 2(\partial_a\partial_b - g_{ab}\partial^c\partial_c)(\varphi - \varepsilon\Delta\varphi + G\frac{\varepsilon}{2}g^{ab}\partial_a\varphi\partial_b\varphi)$$
$$+2G\varepsilon(\partial_a\partial_b - g_{ab}\partial^c\partial_c)\frac{1}{\Delta}\partial^d(\partial_d\varphi\Delta\varphi)$$

is conserved and traceless (!) thanks to diffeomorphism invariance

“Massive” Conformal Field Theories

Improved EMT is traceless in spite of ε is dimensionful.
Analogously massive Pauli-Villars regulators also contribute to EMT

$$\mathcal{S}^{(\text{reg})} = \frac{1}{16\pi b_0^2} \int \sqrt{g} \left[g^{ab}\partial_a Y \partial_b Y + M^2 Y^2 + \varepsilon(\Delta Y)^2 - G\varepsilon g^{ab}\partial_a Y \partial_b Y \Delta\varphi \right]$$

$$\begin{aligned} -4b_0^2 T_{ab}^{(\text{reg})} \;&=\; \partial_a Y \partial_b Y - \frac{1}{2} g_{ab} \partial^c Y \partial_c Y - \frac{M^2}{2} g_{ab} Y^2 - \varepsilon \partial_a Y \partial_b \Delta Y - \varepsilon \partial_a \Delta Y \partial_b Y \\ &+ \varepsilon g_{ab} \partial^c Y \partial_c \Delta Y + \frac{\varepsilon}{2} g_{ab} (\Delta Y)^2 - G\varepsilon \partial_a Y \partial_b Y \Delta\varphi + G\frac{\varepsilon}{2} \partial_a \varphi \partial_b (\partial^c Y \partial_c Y) \\ &+ G\frac{\varepsilon}{2} \partial_a (\partial^c Y \partial_c Y) \partial_b \varphi - G\frac{\varepsilon}{2} g_{ab} \partial^c \varphi \partial_c (\partial^d Y \partial_d Y) - G\varepsilon (\partial_a \partial_b - g_{ab} \partial^c \partial_c)(\partial^c Y \partial_c Y) \end{aligned}$$

Nevertheless total improved EMT $T_{ab} + T_{ab}^{(\text{reg})}$ is traceless thanks to classical equation of motion for φ. This is a general property because

$$T_a^a \propto \hat{g}^{ab} \frac{\delta \mathcal{S}[g]}{\delta \hat{g}^{ab}} = -\frac{\delta \mathcal{S}[g]}{\delta \varphi} \quad \text{for} \quad g_{ab} = \hat{g}_{ab}\, \mathrm{e}^{\varphi}$$

Pauli-Villars regulators do not break conformal symmetry despite they are massive (in contrast to QFT without diffeomorphism invariance)

6. CFT á la KPZ-DDK

Review of KPZ-DDK

Knizhnik-Polyakov-Zamolodchikov (1988), David (1988), Distler-Kawai (1989)

Liouville action in fiducial (or background) metric $\widehat{g}_{ab}$

$$S_L = \frac{1}{8\pi b^2} \int \sqrt{\widehat{g}} \left(\frac{1}{2} \widehat{g}^{ab} \partial_a \varphi \partial_b \varphi + q \widehat{R} \varphi \right) + \mu^2 \int \sqrt{\widehat{g}}\, \mathrm{e}^{\varphi}$$

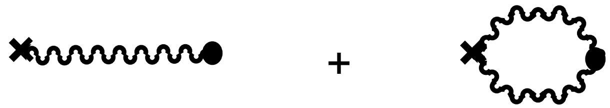
renormalized parameters b^2 and q called "intelligent" one (ione) loop

$$b^2 = b_0^2 + \mathcal{O}(b_0^4), \quad q = 1 + \mathcal{O}(b_0^2), \quad b_0^2 = \frac{6}{26-d}$$

Energy-momentum pseudotensor

$$T_{zz}^{(\varphi)} = -\frac{1}{4b^2} \left(\partial_z \varphi \partial_z \varphi - 2q \partial_z^2 \varphi \right) \qquad \sqrt{g} R = \sqrt{\widehat{g}}\, (q \widehat{R} - \widehat{\Delta} \varphi)$$

Background independence:



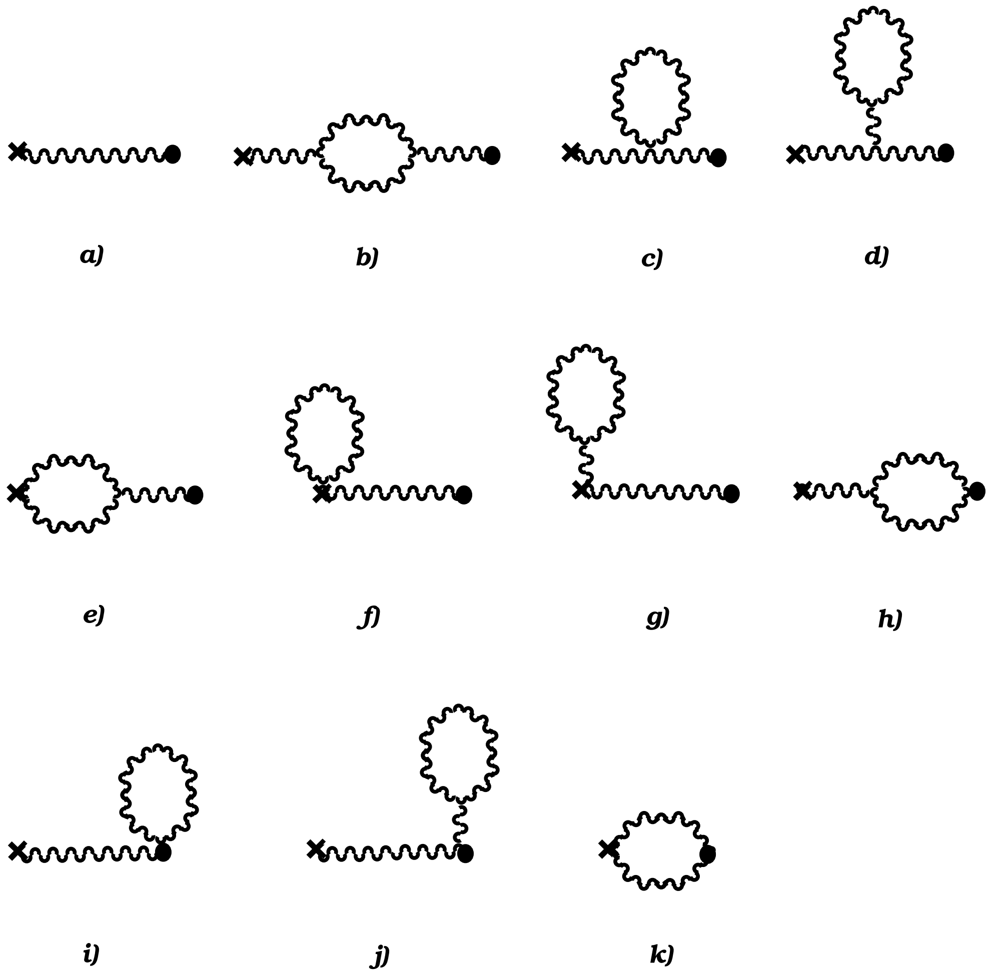
conformal weight $\quad d^{(\mathrm{e}^{\varphi})} = q - b^2 = 1$

total central charge $\quad d - 26 + 6 \frac{q^2}{b^2} + 1 = 0$

$$\Longrightarrow \qquad b = \sqrt{\frac{25-d}{24}} - \sqrt{\frac{1-d}{24}}, \qquad q = 1 + b^2$$

KPZ-DDK for the four-derivative Liouville action

One-loop operator products $T_{zz}(z)\, \mathrm{e}^{\varphi(0)}$ and $T_{zz}(z) T_{zz}(0)$



Conformal weight of $\mathrm{e}^{\varphi(0)}$: $\quad 1 = q - b^2$.

In central charge of φ nonlocal term revives: $\quad c^{(\varphi)} = \frac{6q^2}{b^2} + 1 + 6Gq$

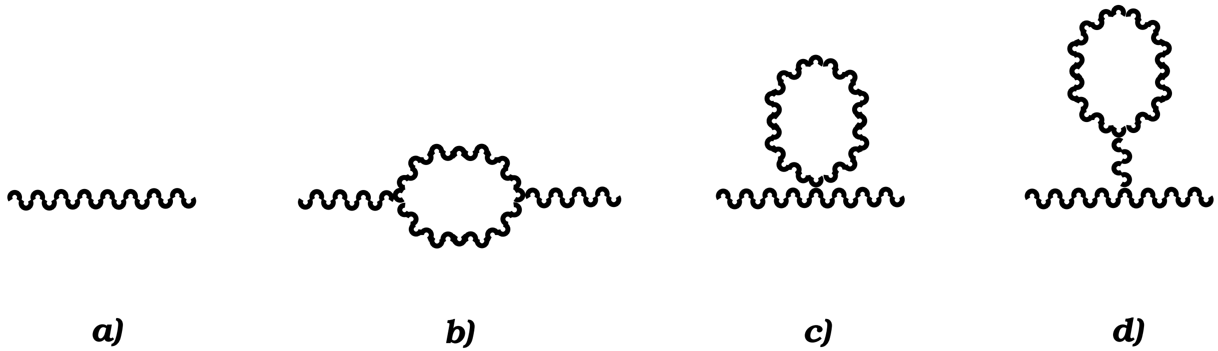
7. Algebraic check of DDK

Salieri:

"I checked the harmony with algebra.
Then finally proficient in the science,
I risked the rare delights of creativity."

A. Pushkin, *Mozart and Salieri*

One-loop propagator



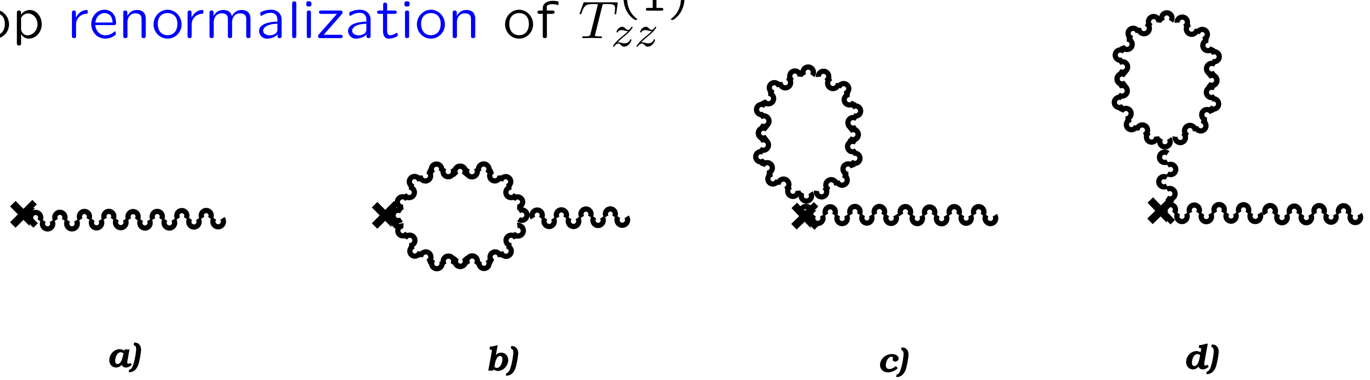
$$
\begin{aligned}
b) \;=\; & -\frac{1}{4}\int \frac{\mathrm{d}^2 k}{(2\pi)^2}\Bigg\{\frac{\varepsilon^2 k^2 (k-p)^2}{(1+\varepsilon k^2)[1+\varepsilon(k-p)^2]} \\
& -2\frac{(\varepsilon k^2 (k-p)^2 - M^2)^2}{(k^2+M^2+\varepsilon k^4)[(k-p)^2+M^2+\varepsilon(k-p)^4]} \\
& +\frac{(\varepsilon k^2 (k-p)^2 - 2M^2)^2}{(k^2+2M^2+\varepsilon k^4)[(k-p)^2+2M^2+\varepsilon(k-p)^4]}\Bigg\}|\varphi(p)|^2 \\
\to\; & b)|_{\text{reg div}} - \frac{p^2}{96\pi}|\varphi(p)|^2
\end{aligned}
$$

One-loop renormalization of b^2 where $T(\varepsilon M^2) \sim \sqrt{\varepsilon} M =$ tadpole $d)$

$$
\frac{1}{b^2} = \frac{1}{b_0^2} - \left(\frac{1}{6} - 4 + T + 2G\int \mathrm{d}k^2 \frac{\varepsilon}{(1+\varepsilon k^2)} - \frac{1}{2}GT\right) + \mathcal{O}(b_0^2)
$$

One-loop renormalization of T_{zz}

One-loop renormalization of $T_{zz}^{(1)}$



$$\frac{q}{b^2} = \frac{1}{b_0^2} - \frac{1}{6} + 2 - \frac{1}{2}T - \frac{1}{2}G - G\int \mathrm{d}k^2 \frac{\varepsilon}{(1+\varepsilon k^2)} + \frac{1}{4}GT + \mathcal{O}(b_0^2)$$

or squaring and multiplying by b^2

$$\frac{q^2}{b^2} = \left(\frac{q}{b^2}\right)^2 \times b^2 = \frac{1}{b_0^2} - \frac{1}{6} - G + \mathcal{O}(b_0^2)$$

This precisely confirms the above shift of the central charge by $6G$ obtained by conformal field theory technique of DDK

Tremendous cancellation due to diffeomorphism invariance making "intelligent" one (ione) loop to be exact: (like Duistermaat-Heckman?)

$$1 = q - b^2, \qquad -\frac{6}{b_0^2} + \frac{6q^2}{b^2} + 1 + 6Gq = 0$$

8. Method of singular products as pragmatic mixture of CFT and QFT

Generator of conformal transformation

Generator of conformal transformation for nonquadratic e-m tensor

$$\widehat{\delta}_\xi \equiv \int_{D_1} \left(\xi' \frac{\delta}{\delta\varphi} + \xi \partial\varphi \frac{\delta}{\delta\varphi} \right) \stackrel{\text{w.s.}}{=} \int_{C_1} \frac{\mathrm{d}z}{2\pi\mathrm{i}} \xi(z) T_{zz}(z)$$

where D_1 includes singularities of $\xi(z)$ and C_1 bounds D_1.

Equivalence of two forms is proved by integrating the total derivative

$$\bar{\partial} T_{zz} = -\pi \partial \frac{\delta S}{\delta\varphi} + \pi \partial\varphi \frac{\delta S}{\delta\varphi}$$

and using the (quantum) equation of motion

$$\frac{\delta S}{\delta\varphi} \stackrel{\text{w.s.}}{=} \frac{\delta}{\delta\varphi}$$

Actually, the form of $\widehat{\delta}_\xi$ in the middle is primary.

It takes into account a tremendous cancellation of the diagrams, while there are subtleties associated with singular products

Method of singular products: some resuls

Y.M. (2023)

- The proof of "intelligent" one loop:

Exact conformal weight

$$\hat{\delta}_\xi\, \mathrm{e}^{\varphi} \stackrel{\text{w.s.}}{=} d^{(\,\mathrm{e}^{\varphi})}\xi'\, \mathrm{e}^{\varphi} + \xi\partial\, \mathrm{e}^{\varphi} \qquad d^{(\,\mathrm{e}^{\varphi})} = q - b^2$$

$$\text{conformal weight of } \mathrm{e}^{\varphi} = 1 \quad \Longrightarrow \quad d^{(\,\mathrm{e}^{\varphi})} = 1$$

Exact central charge $c^{(\varphi)}$ of φ

$$\left\langle \hat{\delta}_\xi T_{zz}(\omega) \right\rangle = \frac{c^{(\varphi)}}{12}\xi'''(\omega) \qquad c^{(\varphi)} = \frac{6q^2}{b^2} + 1 + 6Gq$$

$$\text{total central charge } = 0 \quad \Longrightarrow \quad d - 26 + c^{(\varphi)} = 0$$

- Almost the proof of Universality for Polyakov string – the results do not depend on higher-order terms in the effective action (shown for four- and six-derivative actions)

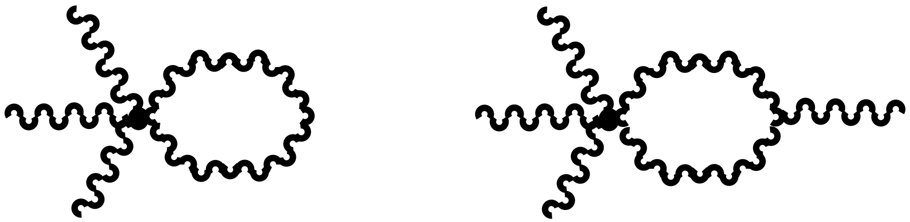
Heuristic understanding of universality

For general action $\quad (F(x)=(1+x)/8\pi b_0^2$ for four-derivative)

$$S^{\rm gen}[\varphi]=-\frac{1}{2}\int\sqrt{\hat g}\varphi\hat\Delta F(-\varepsilon\,{\rm e}^{-\varphi}\hat\Delta)\varphi,\qquad F(0)=\frac{1}{8\pi b_0^2}$$

propagator: $\quad k^2F(\varepsilon k^2)$ and triple vertex: $\quad \varepsilon k^4F'(\varepsilon k^2)$.

One-loop renormalization ${\rm e}^{\varphi}\Rightarrow{\rm e}^{\alpha\varphi}\quad$ (wavy lines represent φ)



a) *b)*

$$b)=\frac{{\rm e}^{\varphi}}{2}\times\varphi\int\frac{{\rm d}^2k}{(2\pi)^2}\frac{\varepsilon k^4F'(\varepsilon k^2)}{[k^2F(\varepsilon k^2)]^2}=\frac{1}{8\pi F(0)}{\rm e}^{\varphi}\varphi={\rm e}^{\varphi}b_0^2\varphi$$

independently on the choice of F like anomalies in QFT

9. Relation to minimal models

Exact solution for four-derivative action

Solution to two modified DDK equations

$$\begin{aligned} b^{-1} &= \sqrt{\frac{d_+ - d}{24}} + \sqrt{\frac{d_- - d}{24}} \\ d_\pm &= 13 - 6G \pm 12\sqrt{1+G} \\ q &= 1 + b^2 \end{aligned}$$

where $d = 26 - 6/b_0^2$ to comply with the Liouville action.
KPZ barriers are shifted to $d_\pm$ which depend on $G \in [-1, 0]$.
For $G = -1/3$ (the Nambu-Goto string) $\Longrightarrow$ $d_- = 15 - 4\sqrt{6} \approx 5.2 > 4$

The string susceptibility equals

$$\gamma_{\text{str}} = (h-1)\frac{q}{b^2} + 2 = (h-1)\frac{25 - d - 6G + \sqrt{(d-d_+)(d-d_-)}}{12} + 2$$

It is real for $d < d_-$ with $d_- > 1$ increasing from 1 at $G = 0$ to 19 at $G = -1$ for $0 \geq G \geq -1$ required for stability as it follows from the identity (modulo boundary terms)

$$\int \mathrm{e}^{-\varphi}\left[(\partial\bar\partial\varphi)^2 - G\partial\varphi\bar\partial\varphi\partial\bar\partial\varphi\right] = \int \mathrm{e}^{-\varphi}\left[(1+G)(\partial\bar\partial\varphi)^2 - G\nabla\partial\varphi\bar\nabla\bar\partial\varphi\right]$$

BPZ null-vectors and Kac's spectrum

Like in usual Liouville theory the operators   á la Dotsenko-Fateev (1984)

$$V_\alpha = \mathrm{e}^{\alpha\varphi}, \qquad \alpha = \frac{1-n}{2} + \frac{1-m}{2b^2}$$

are the BPZ null-vectors for integer n and m obeying

$$\left(L_{-1}^2 + b^2 L_{-2}\right)\mathrm{e}^{-\varphi/2} = 0, \quad \left(L_{-1}^2 + b^{-2} L_{-2}\right)\mathrm{e}^{-b^{-2}\varphi/2} = 0, \quad \ldots$$

Their conformal weights

$$\Delta_\alpha = \alpha + (\alpha - \alpha^2) b^2$$

coincide with Kac's spectrum

$$\Delta_{m,n}(c) = \frac{c-1}{24} + \frac{1}{4}\left((m+n)\sqrt{\frac{1-c}{24}} + (m-n)\sqrt{\frac{25-c}{24}}\right)^2$$

for

$$c = 1 + 6(b + b^{-1})^2 = 26 - d + G\frac{\left[25 - d - 6G + \sqrt{(d-d_+)(d-d_-)}\right]}{2(1+G)}$$

Minimal models from four-derivative action

To describe minimal models we choose like in usual Liouville theory

$$c = 25 + 6\frac{(p-q)^2}{pq} \quad \Longrightarrow \quad G = \frac{(1 - d - 6\frac{(p-q)^2}{pq})q}{6(q+p)}$$

with coprime $q > p$

If $G = 0$ this would imply

$$d = 1 - 6\frac{(p-q)^2}{pq}$$

for central charge of matter but now d is a free parameter obeying

$$1 - 6\frac{(p-q)^2}{pq} \leq d \leq 19 - 6\frac{p}{q} \quad \Longleftarrow \quad 0 \geq G \geq -1$$

Contrary to the usual Liouville theory now Kac's $c \neq c^{(\varphi)} = 26 - d$

Remarkably, $G = -1/3$ is associated in $d = 4$ with $p = 3$, $q = p+1 = 4$
unitary minimal model like critical Ising model on a random lattice

Conclusion (for Quantum Strings)

- "Massive" CFTs exist and solved by (almost) usual CFT technique except for nonlocality in improved EMT $\approx 100\%$

- Nambu-Goto and Polyakov strings are told apart by higher-derivative terms which revive quantumly like anomalies in QFT $\approx 100\%$

- Emergence of the four-derivative Liouville action alludes to (4,3) minimal model like critical Ising model on a random lattice $\approx 75\%$

Conclusion (for Quantum Gravity)

- Quadratic divergences of regularized bosonic string are manageable (nonperturbatively). No logarithmic divergences emerge thanks to conformal invariance in massive case

- Lilliputian scaling is caused by singular behavior of metric tensor at the critical point

- More opportunities than in usual QFT thanks to diffeomorphism invariance

- Generalized conformal anomaly may suggest an extension of Riegert-Fradkin-Tseytlin action

- Large-d limit of string is vector (like $O(N)$-sigma model) while large-d limit of gravity is matrix (like QCD) Strominger (1981)