Slightly off-shell 5-point amplitudes in SYM

Roman Lee[a]

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Budker Institute of Nuclear Physics

[a] In collaboration with A. Belitsky, L. Bork, A. Onishchenko, and V. Smirnov, based on [2509.12056], [2510.16471], [2605.17057], and [2605.17056].

Outline

Motivation

Motivation

Disclaimer

I am not an expert in $\mathcal{N}=4$ SYM, I am rather an expert in multiloop calculations methods. So I will present a **new approach** to the evaluation of slightly off-shell integrals which is definitely very effective for $\mathcal{N}=4$ SYM calculations, but may also be useful for realistic setup.

- Planar $\mathcal{N}=4$ SYM is a classical playground for QFT calculations. Perturbative calculations provide input for all-loop conjectures (like Bern-Dixon-Smirnov ansatz).
- Opportunity to remind about method of regions (MofR), which is ubiquitous in multiloop calculations and yet may be underestimated in other applications.

Dual conformal integrals

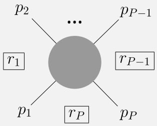
Slightly off-shell amplitudes in planar SYM

- Amplitudes in $\mathcal{N} = 4$ SYM enjoy **dual conformal invariance (DCI)**. However they contain infrared divergences, therefore some regularization is needed.
- Standard choice is the dimensional regularization $d = 4 - 2\epsilon$. However, it breaks DCI, although in somewhat controlled way.
- Another option is to consider slightly off-shell amplitudes. In particular, we can consider amplitudes small equal off-shellness of external legs, which is called . Such a regularization preserves DCI.
- While in conventional dimensional regularization the collinear singularities are governed by Γ_{cusp}, in small-off-shellness asymptotics the leading logs are governed by $\Gamma_{oct} = 2\log(\cosh(2\pi g))/\pi^2$.
- Within this approach, we want to calculate the small-off-shellness asymptotics of these amplitudes up to power corrections.

Dual conformal symmetry

P-point amplitude A_P in planar SYM

To formulate the dual conformal invariance, one introduces **dual variables** r_i, s.t. $p_i = r_i - r_{i-1}$.



- Then the A_P is invariant under conformal group acting on r_i and generated by

shifts: $r_i \to r_i + a$, rot.&dilat.: $r_i \to \Lambda r_i$, inversion: $r_i \to r_i / r_i^2$

- As a result, the amplitudes depend only on

DCI cross-ratios

$$\frac{r_{ij}^2 r_{kl}^2}{r_{il}^2 r_{kj}^2}\,, \qquad \text{where } r_{kl} = r_k - r_l = \sum_{i=l+1}^{k} p_i.$$

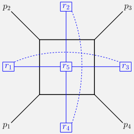
Simple exercise: check that these cross-ratios are DCI invariant.

Perturbatitive expansion

In perturbation theory the amplitudes are expressed via multiloop integrals which also inherit DCI. E.g., up to two loop the amplitude A_4 is expressed via integrals

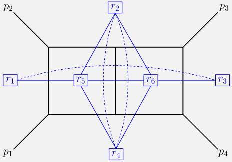
1-loop box

$$\int d^4 r_5 \frac{r_{13}^2 r_{24}^2}{r_{15}^2 r_{25}^2 r_{35}^2 r_{45}^2}$$



2-loop DCI box

$$\int d^4 r_5 d^4 r_6 \frac{r_{13}^2 r_{24}^4}{r_{15}^2 r_{25}^2 r_{26}^2 r_{36}^2 r_{45}^2 r_{46}^2 r_{56}^2}$$

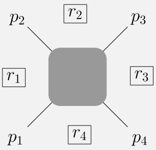


Example: $P = 4$

Simplest example: 4-point amplitude $A_{P=4} = A_4$

A_4 depends on $s = (p_1 + p_2)^2 = r_{24}^2$, $t = (p_2 + p_3)^2 = r_{31}^2$, and $m_i^2 = p_i^2 = r_{i,i-1}^2$. Due to DCI symmetry, these invariants appear only via two cross-ratios

$$u_1 = \frac{m_1^2 m_3^2}{st}, \quad u_2 = \frac{m_2^2 m_4^2}{st}$$



If we define $M_P = A_P / A_P^{tree}$ we have a nice all-loop conjecture for M_4:

$$\log M_4 = -\frac{1}{16}\Gamma_{oct}(g)\log^2(u_1 u_2) - \frac{g^2}{4}\log^2(u_1/u_2) - \frac{1}{2}D(g) + O(m^2)$$

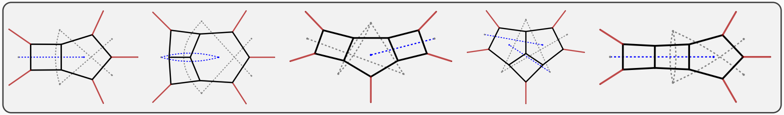
where

$$\Gamma_{oct}(g) = \frac{2}{\pi^2}\log\cosh(2\pi g) = 4g^2 - 16\zeta_2 g^4 + 256\zeta_4 g^6 + \dots$$

$$D(g) = \frac{1}{4}\log\frac{\sinh(4\pi g)}{4\pi g} = 4\zeta_2 g^2 - 32\zeta_4 g^4 + \frac{1024}{3}\zeta_6 g^6 + \dots$$

Five-point amplitude A_5

- Can we discover a similar all-loop conjecture for five-point amplitude A_5?
- We need to be able to calculate several first terms of perturbation theory. E.g., in two and three loops the following topologies contribute:



- NB:dashed lines denote numerators.
- Till recently even the first diagram, slightly off-shell DCI pentabox integral was not calculated!
- Why don't we apply the method of regions (MofR)?

NB: for DCI pentabox integral one can take all virtualities equal, $\boxed{p_1^2 = m^2}$, without loss of generality.

Method of regions

Method of regions

- Method of regions is a tool for calculating asymptotics of integrals as functions of parameters.
- In order to separate the contribution of each region it uses factor of the form $p(x)^\nu$, either existing or introduced, in the integrand.

Example: calculate asymptotics of complete elliptic integral

$$\mathrm{K}(1-a) = \int_0^\infty \frac{dx}{\sqrt{(1+x^2)(a+x^2)}} \xrightarrow{a\to+0} ?$$

1. Introduce factor $x^{2\epsilon}$ in the integrand.
2. Find regions: $x \sim a^0,\ x \sim a^{1/2}$.
3. For each region $x \sim a^q$ rescale $x \to a^q x$ and expand wrt a under $\int$.
4. Take the integrals, assuming $x^{2\epsilon}$ provides convergence.
5. Add up contributions, calculate limit $\epsilon \to 0$.

Method of regions (contd.)

In this oversimplified example we can write down a general term of expansion in each region. We have

$$C(x \sim a^0) = \sum_{n=0}^{\infty} \frac{a^n \Gamma\left(n - \epsilon + \frac{1}{2}\right) \Gamma\left(-n + \epsilon\right)}{2\Gamma\left(\frac{1}{2} - n\right)\Gamma(n+1)}$$

$$C(x \sim \sqrt{a}) = \sum_{n=0}^{\infty} \frac{a^{n+\epsilon} \Gamma\left(n + \epsilon + \frac{1}{2}\right) \Gamma\left(-n - \epsilon\right)}{2\Gamma\left(\frac{1}{2} - n\right)\Gamma(n+1)}$$

Note that highlighted parts turn into ∞ at $\epsilon = 0$.

Adding contributions and taking the limit $\epsilon \to 0$ term-wise, we obtain

Full asymptotic series

$$\mathrm{K}(1-a) \simeq \sum_n a^n \frac{\Gamma\left(n + \frac{1}{2}\right)^2}{\pi \Gamma(n+1)^2} \left[\psi(n+1) - \psi\left(n + \frac{1}{2}\right) - \frac{1}{2}\log a\right]$$

NB: in fact, this series has finite convergence radius ($= 1$).

Method of regions (contd.)

How can we find the regions for $a \to 0$ asymptotics of the integral

$$\int_{\mathbb{R}_+^n} \prod_{i=1}^{n} dx_i \prod_k p_k(\boldsymbol{x}, a)^{\mu_k} ?$$

Newton polytope of the polynomial p

If $p(\boldsymbol{x}) = \sum c_k \boldsymbol{x}^{\boldsymbol{n_k}}$ then $\mathrm{Newt}(p(\boldsymbol{x})) = \{\sum \alpha_k \boldsymbol{n_k} | \alpha_k \geqslant 0, \sum \alpha_k \leqslant 1\}$. In other words, $\mathrm{Newt}(p(\boldsymbol{x}))$ is a convex hull of exponents of its terms.

Lower faces of $\mathrm{Newt}(p(\boldsymbol{x}, a))$ ⇔ regions

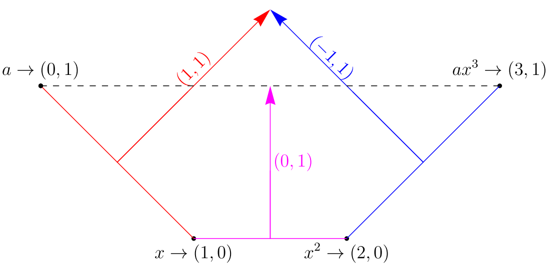
For the polynomial $p = \prod_k p_k$ with positive (or sufficiently generic) coefficients c_k each lower[a] face F of $\mathrm{Newt}(p(\boldsymbol{x}, a))$ corresponds to a separate region: let $\boldsymbol{N}_F = (\boldsymbol{q}, 1)$ be an inner normal vector to F, then the region is given by rescaling $\boldsymbol{x} \to a^{\boldsymbol{q}} \boldsymbol{x}$.

[a] down-up axis corresponds to the exponents of a.

Method of regions (contd.)

Example: calculate $a \to 0$ asymptotics of the integral

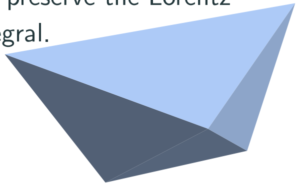
$$\int_0^\infty \frac{dx}{(a + x + x^2 + ax^3)^\nu} \xrightarrow{a\to+0} ?$$



Region I: $x \to a^1 x$ Region II: $x \to a^0 x$ Region III: $x \to a^{-1} x$

Method of regions (contd.)

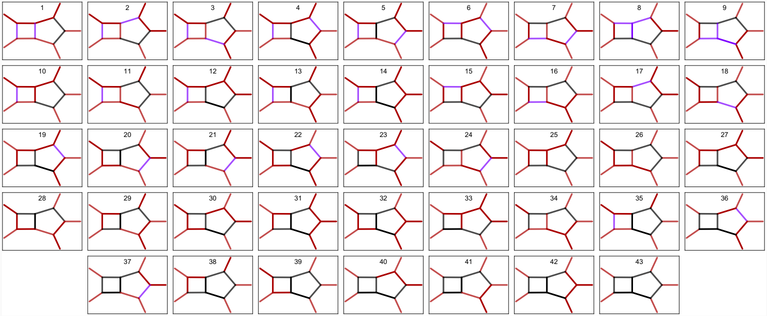
- In multiloop calculations the **dimensional regularization** is often sufficient to separate the regions.
- Sometimes, in order to separate the regions it is necessary to introduce also the analytic regularization.
- Finding the full set of regions can be automatized (with some known exceptions) with `asy` [Pak and Smirnov, 2011, Semenova et al., 2019], or `GetRegions` (`github.com/rnlg/Get-regions`).
- Both dimensional and analytic regularizations preserve the Lorentz invariance of the original scalar multiloop integral.



Method of regions for pentabox integral

NB: I learned about this problem on August, 12. The first idea was, of course, to try the MofR.

- There are 43 regions



where the color denotes scalings of Feynman parameters as m^0,m^{-2},m^{-4}.

Method of regions for pentabox integral (cont.)

- “Only” 32 regions contribute in the leading order in m.
- The contributions of 9 regions was expressed via Γ-functions.
- The contributions of 10 regions was expressed via ${}_pF_q$-functions.
- The contributions of 7 regions was expressed via 2-fold MB.
- The contributions of 2 regions was expressed via 3-fold MB.
- The contributions of 2 regions was expressed via 4-fold FP.
- The contributions of 2 regions was expressed via 5-fold FP.

So, it became clear that the calculation will not be simple.

Method of regions for pentabox integral (cont.)

On August 19 the paper [arXiv:2508.14298] by Belitsky&Smirnov appeared where they followed the same approach and succeeded!
But their result was quite complicated:

Belitsky&Smirnov result

$$PB = 3\log^4 m^2 - \left[\tfrac{5}{2}\log(s_{12}s_{23}s_{34}s_{51}) - 2\log(s_{45})\right]\log^3 m^2 + \boxed{\text{1.2Kb}}\log^2 m^2 + \boxed{\text{32Kb}}\log m^2 + \boxed{\text{2.4Mb}}$$

The coefficients were expressed via GPL up to weight 5
example: $G(1-s_{51}/s_{34}, 1-s_{51}/s_{34}, 1, 0, 1-s_{51}/s_{34}|1)$.

But it is expected that, at least when in the amplitude, this result should give rise to logs only!

NB: It does not necessarily mean that pentabox itself is expressed via logs.

Method of regions for pentabox integral (cont.)

Maybe, there is a better way to calculate this integral?

Idea

- Dimensional regularization breaks DCI, so the contribution of each region is not DCI.
- We always want the regularization to retain as much symmetry as possible.
- Maybe, if we introduce the regularization which preserves DCI, the calculation will be easier?

DCI regularization

DCI integrals in $d = 4$

Let us consider the P-point L-loop integral in 4 dimensions

$$I_L(r_1, \dots r_P) = \int \prod_{l=P+1}^{P+L} \frac{d^4 r_l}{\pi^2} \prod_{i=1}^{M} D_i^{-n_i}, \qquad D_i = (r_{k_i} - r_{m_i})^2 \,.$$

What are conditions for this integral to be dual conformal invariant?

- It is obviously invariant under the Poincare transformations $r_k \to a + \Lambda r_k$.
- Inversion: $(r_k - r_m)^2 \to (r_k - r_m)^2/(r_k^2 r_m^2)$, $dr_l \to dr_l/(r_l^2)^4$.
- Then we have to require that all additional factors cancel.

If we introduce the indicator function θ_{li} which is 1 if r_l appears in D_i and 0 otherwise, we obtain

DCI condition (condtion that integral is DCI)

$$\sum_i n_i \theta_{li} = \begin{cases} 0, & l \leqslant P \\ 4, & l > P \end{cases}$$

DCI regularization

Consider now the regularized integral, both dimensionally and analytically,

$$I_L^{\mathsf{reg}}(r_1, \ldots r_P) = \int \prod_{l=P+1}^{P+L} \frac{d^d r_l}{\pi^{d/2}} \prod_{i=1}^{M} D_i^{-\nu_i},$$

where $d = 4 - 2\epsilon$ and $\nu_i = n_i + \alpha_i$. For this integral to remain DCI we have to require similar conditions on ϵ and α_i:

DCI regularization (condition that the regularization preserves DCI)

$$\sum_i \alpha_i \theta_{li} = \begin{cases} 0, & l \leqslant P \\ -2\epsilon, & l > P \end{cases}$$

In particular, from these conditions we conclude that we can not avoid analytic continuation.

NB: to avoid unnecessary complications we put $\alpha_i = 0$ for numerators depending on loop momenta.

But what do we gain from this regularization?

Pedagogical example: slightly off-shell one-loop box.

Generic regularization

$$B^{\mathsf{reg}} = \int \frac{d^d r_5}{\pi^{d/2}} \frac{\left[r_{13}^2\right]^{1-\alpha_5} \left[r_{24}^2\right]^{1-\alpha_6} \left[r_{21}^2\right]^{-\alpha_7} \left[r_{32}^2\right]^{-\alpha_8}}{\left[r_{15}^2\right]^{1+\alpha_1} \left[r_{25}^2\right]^{1+\alpha_2} \left[r_{35}^2\right]^{1+\alpha_3} \left[r_{45}^2\right]^{1+\alpha_4}},$$

- Dimensional regularization: $\alpha = 0$
- DCI regularization:

$$\alpha_4 + \alpha_6 = 0,\ \alpha_1 + \alpha_5 + \alpha_7 = 0,\ \alpha_3 + \alpha_5 + \alpha_8 = 0,$$
$$\alpha_2 + \alpha_6 + \alpha_7 + \alpha_8 = 0,\ \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 = -2\epsilon.$$

[1] `github.com/rnlg/Get-regions`

Pedagogical example: slightly off-shell one-loop box.

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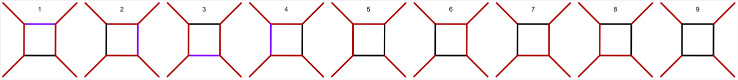
- Dimensional regularization: $\alpha = 0$
- DCI regularization:

$$\alpha_4 + \alpha_6 = 0,\ \alpha_1 + \alpha_5 + \alpha_7 = 0,\ \alpha_3 + \alpha_5 + \alpha_8 = 0,$$
$$\alpha_2 + \alpha_6 + \alpha_7 + \alpha_8 = 0,\ \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 = -2\epsilon.$$

- With `GetRegions`[1] we find 9 regions:



[1] `github.com/rnlg/Get-regions`

Pedagogical example: slightly off-shell one-loop box.

#	dim reg.
1	$s^\epsilon \Gamma(1-\epsilon)\Gamma(\epsilon)^2 \left(p_1^2 p_2^2\right)^{-\epsilon}$
2	$t^\epsilon \Gamma(1-\epsilon)\Gamma(\epsilon)^2 \left(p_2^2 p_3^2\right)^{-\epsilon}$
3	$s^\epsilon \Gamma(1-\epsilon)\Gamma(\epsilon)^2 \left(p_3^2 p_4^2\right)^{-\epsilon}$
4	$t^\epsilon \Gamma(1-\epsilon)\Gamma(\epsilon)^2 \left(p_4^2 p_1^2\right)^{-\epsilon}$
5	$\Gamma(-\epsilon)^2\Gamma(\epsilon) \left(p_1^2\right)^{-\epsilon}/\Gamma(-2\epsilon)$
6	$\Gamma(-\epsilon)^2\Gamma(\epsilon) \left(p_2^2\right)^{-\epsilon}/\Gamma(-2\epsilon)$
7	$\Gamma(-\epsilon)^2\Gamma(\epsilon) \left(p_3^2\right)^{-\epsilon}/\Gamma(-2\epsilon)$
8	$\Gamma(-\epsilon)^2\Gamma(\epsilon) \left(p_4^2\right)^{-\epsilon}/\Gamma(-2\epsilon)$
9	B_9^{dim}

$$B_9^{\text{dim}} = \frac{st\pi\Gamma(-\epsilon)}{\sin(\pi\epsilon)\Gamma(-2\epsilon)}\left[2\pi\cot(\pi\epsilon)\left(\tfrac{s+t}{st}\right)^\epsilon - \tfrac{ts^{-1-\epsilon}}{1+\epsilon}\,{}_2F_1\left(1,1;2+\epsilon;-\tfrac{t}{s}\right) - \tfrac{st^{-1-\epsilon}}{1+\epsilon}\,{}_2F_1\left(1,1;2+\epsilon;-\tfrac{s}{t}\right)\right]$$

Pedagogical example: slightly off-shell one-loop box.

#	dim reg.	dci.reg
1	$s^\epsilon \Gamma(1-\epsilon)\Gamma(\epsilon)^2 \left(p_1^2 p_2^2\right)^{-\epsilon}$	$\frac{\Gamma(1-\alpha_1-\epsilon,\alpha_1+\alpha_2+\epsilon,\alpha_1+\alpha_4+\epsilon)}{\Gamma(1+\alpha_1,1+\alpha_2,1+\alpha_4)} u_1^{\alpha_2+\alpha_3+\epsilon}$
2	$t^\epsilon \Gamma(1-\epsilon)\Gamma(\epsilon)^2 \left(p_2^2 p_3^2\right)^{-\epsilon}$	$\frac{\Gamma(1-\alpha_2-\epsilon,\alpha_1+\alpha_2+\epsilon,\alpha_2+\alpha_3+\epsilon)}{\Gamma(1+\alpha_1,1+\alpha_2,1+\alpha_3)}$
3	$s^\epsilon \Gamma(1-\epsilon)\Gamma(\epsilon)^2 \left(p_3^2 p_4^2\right)^{-\epsilon}$	$\frac{\Gamma(1-\alpha_3-\epsilon,\alpha_3+\alpha_4+\epsilon,\alpha_2+\alpha_3+\epsilon)}{\Gamma(1+\alpha_2,1+\alpha_3,1+\alpha_4)} u_2^{\alpha_1+\alpha_2+\epsilon}$
4	$t^\epsilon \Gamma(1-\epsilon)\Gamma(\epsilon)^2 \left(p_4^2 p_1^2\right)^{-\epsilon}$	$\frac{\Gamma(1-\alpha_4-\epsilon,\alpha_3+\alpha_4+\epsilon,\alpha_1+\alpha_4+\epsilon)}{\Gamma(1+\alpha_1,1+\alpha_3,1+\alpha_4)} \times u_1^{\alpha_2+\alpha_3+\epsilon} u_2^{\alpha_1+\alpha_2+\epsilon}$
5	$\Gamma(-\epsilon)^2\Gamma(\epsilon)\left(p_1^2\right)^{-\epsilon}/\Gamma(-2\epsilon)$	0
6	$\Gamma(-\epsilon)^2\Gamma(\epsilon)\left(p_2^2\right)^{-\epsilon}/\Gamma(-2\epsilon)$	0
7	$\Gamma(-\epsilon)^2\Gamma(\epsilon)\left(p_3^2\right)^{-\epsilon}/\Gamma(-2\epsilon)$	0
8	$\Gamma(-\epsilon)^2\Gamma(\epsilon)\left(p_4^2\right)^{-\epsilon}/\Gamma(-2\epsilon)$	0
9	B_9^{dim}	0

$$B_9^{\text{dim}} = \frac{st\pi\Gamma(-\epsilon)}{\sin(\pi\epsilon)\Gamma(-2\epsilon)}\left[2\pi\cot(\pi\epsilon)\left(\tfrac{s+t}{st}\right)^\epsilon - \tfrac{ts^{-1-\epsilon}}{1+\epsilon}\,{}_2F_1\left(1,1;2+\epsilon;-\tfrac{t}{s}\right) - \tfrac{st^{-1-\epsilon}}{1+\epsilon}\,{}_2F_1\left(1,1;2+\epsilon;-\tfrac{s}{t}\right)\right]$$

Pedagogical example: slightly off-shell one-loop box.

#	dim reg.	dci.reg
1	$s^\epsilon\Gamma(1-\epsilon)\Gamma(\epsilon)^2\left(p_1^2p_2^2\right)^{-\epsilon}$	$\frac{\Gamma(1-\alpha_1-\epsilon,\alpha_1+\alpha_2+\epsilon,\alpha_1+\alpha_4+\epsilon)}{\Gamma(1+\alpha_1,1+\alpha_2,1+\alpha_4)}u_1^{\alpha_2+\alpha_3+\epsilon}$
2	$t^\epsilon\Gamma(1-\epsilon)\Gamma(\epsilon)^2\left(p_2^2p_3^2\right)^{-\epsilon}$	$\frac{\Gamma(1-\alpha_2-\epsilon,\alpha_1+\alpha_2+\epsilon,\alpha_2+\alpha_3+\epsilon)}{\Gamma(1+\alpha_1,1+\alpha_2,1+\alpha_3)}$
3	$s^\epsilon\Gamma(1-\epsilon)\Gamma(\epsilon)^2\left(p_3^2p_4^2\right)^{-\epsilon}$	$\frac{\Gamma(1-\alpha_3-\epsilon,\alpha_3+\alpha_4+\epsilon,\alpha_2+\alpha_3+\epsilon)}{\Gamma(1+\alpha_2,1+\alpha_3,1+\alpha_4)}u_2^{\alpha_1+\alpha_2+\epsilon}$
4	$t^\epsilon\Gamma(1-\epsilon)\Gamma(\epsilon)^2\left(p_4^2p_1^2\right)^{-\epsilon}$	$\frac{\Gamma(1-\alpha_4-\epsilon,\alpha_3+\alpha_4+\epsilon,\alpha_1+\alpha_4+\epsilon)}{\Gamma(1+\alpha_1,1+\alpha_3,1+\alpha_4)}\times u_1^{\alpha_2+\alpha_3+\epsilon}u_2^{\alpha_1+\alpha_2+\epsilon}$
5	$\Gamma(-\epsilon)^2\Gamma(\epsilon)\left(p_1^2\right)^{-\epsilon}/\Gamma(-2\epsilon)$	0
6	$\Gamma(-\epsilon)^2\Gamma(\epsilon)\left(p_2^2\right)^{-\epsilon}/\Gamma(-2\epsilon)$	0
7	$\Gamma(-\epsilon)^2\Gamma(\epsilon)\left(p_3^2\right)^{-\epsilon}/\Gamma(-2\epsilon)$	0
8	$\Gamma(-\epsilon)^2\Gamma(\epsilon)\left(p_4^2\right)^{-\epsilon}/\Gamma(-2\epsilon)$	0
9	B_9^{dim}	0

$$B_9^{\text{dim}} = \frac{st\pi\Gamma(-\epsilon)}{\sin(\pi\epsilon)\Gamma(-2\epsilon)}\Bigg[2\pi\cot(\pi\epsilon)\left(\tfrac{s+t}{st}\right)^\epsilon - \tfrac{ts^{-1-\epsilon}}{1+\epsilon}\,{}_2F_1\left(1,1;2+\epsilon;-\tfrac{t}{s}\right) - \tfrac{st^{-1-\epsilon}}{1+\epsilon}\,{}_2F_1\left(1,1;2+\epsilon;-\tfrac{s}{t}\right)\Bigg]$$

All contributions in dci regularization are expressed in terms of Γ-functions, no ${}_2F_1$. Many contributions vanished.

Applications

Back to pentabox

The most general **DCI** regularization of pentabox integral reads

$$PB = PB(\epsilon, \alpha_1, \ldots, \alpha_6) = \int \frac{d^d r_6}{\pi^{d/2}} \frac{d^d r_7}{\pi^{d/2}}$$

$$\times \frac{r_{14}^{2(\alpha_3-\alpha_{246})} r_{25}^{2(1+\alpha_{45})} r_{31}^{2(\alpha_{1246}-\alpha_3)} r_{42}^{2(1+\alpha_2-\alpha_5)} r_{53}^{2(1+\alpha_3-\alpha_{124})} r_{17}^{2}}{r_{56}^{2(1+\alpha_3-\alpha_{12})} r_{57}^{2(1+\alpha_5)} r_{61}^{2(1+\alpha_1)} r_{62}^{2(1+\alpha_2)} r_{72}^{2(1+\alpha_4)} r_{73}^{2(1+\alpha_6)} r_{74}^{2(1+\alpha_3-\alpha_{456})} r_{67}^{2(1-\alpha_3-2\epsilon)}}$$

With this regularization we have

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With this regularization we have

- Zero regions expressed via n-fold FP.

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- Zero regions expressed via n-fold MB.

Back to pentabox

The most general **DCI** regularization of pentabox integral reads

$$PB = PB(\epsilon, \alpha_1, \ldots, \alpha_6) = \int \frac{d^d r_6}{\pi^{d/2}} \frac{d^d r_7}{\pi^{d/2}}$$

$$\times \frac{r_{14}^{2(\alpha_3-\alpha_{246})} r_{25}^{2(1+\alpha_{45})} r_{31}^{2(\alpha_{1246}-\alpha_3)} r_{42}^{2(1+\alpha_2-\alpha_5)} r_{53}^{2(1+\alpha_3-\alpha_{124})} r_{17}^{2}}{r_{56}^{2(1+\alpha_3-\alpha_{12})} r_{57}^{2(1+\alpha_5)} r_{61}^{2(1+\alpha_1)} r_{62}^{2(1+\alpha_2)} r_{72}^{2(1+\alpha_4)} r_{73}^{2(1+\alpha_6)} r_{74}^{2(1+\alpha_3-\alpha_{456})} r_{67}^{2(1-\alpha_3-2\epsilon)}}$$

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Success

The calculation became elementary! (up to some heuristic change of variables in FP and special treatment of hard contribution).

Result

Adding up and calculating the limit $\epsilon, \alpha_i \to 0$, we obtain

Result for DCI pentabox integral

$$PB = \frac{1}{2} L_1 \left(L_3 L_2^2 + 2 L_3 L_4 L_2 + L_4 L_5^2 + 2 L_3 L_4 L_5 \right)$$
$$+ \frac{1}{2} \big(4 L_1 L_2 + 4 L_1 L_5 + 4 L_3 L_4 + 2 L_1 L_3 + 2 L_1 L_4 - 2 L_4 L_2 - 2 L_3 L_5$$
$$- L_2^2 - L_5^2 \big) \zeta_2 + \left(L_3 + L_4 - 2 L_1 \right) \zeta_3 + 5 \zeta_4,$$

where $L_i = \log \frac{m^2 s_i}{s_{i-2} s_{i+2}}$

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5-point amplitude

$$M_5 = A_5 / A_5^{tree} = 1 + M_5^{(1)} + M_5^{(2)} + \dots$$

- $M_5^{(1)}$ is expressed via box with one (strongly) off-shell leg.
- $M_5^{(2)}$ is expressed via double-box with one (strongly) off-shell leg and dci pentabox integral PB.

All-loop ansatz for $\log M_5$

Amplitude up to two loops

$$\log M_5 = -\left(g^2 - 4\zeta_2 g^4\right) \log v_1 \log v_2 - g^4 \zeta_2 \log \frac{v_1}{v_2} \log \frac{v_2}{v_3}$$
$$- \zeta_2 g^2 + \frac{27}{4}\zeta_4 g^4 + O(g^6) + \text{cyclic permutations},$$

where $\boxed{v_i = \frac{r^2_{i+1,i-1} r^2_{i+2,i-2}}{r^2_{i+1,i-2} r^2_{i-1,i+2}}}$.

All-loop conjecture

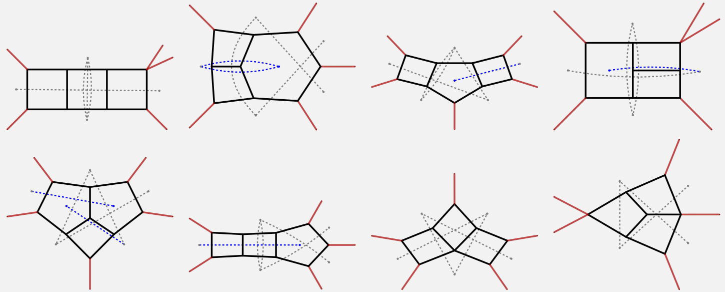
$$\log M_5 = -\tfrac{1}{4}\Gamma_{oct} \log v_1 \log v_2 - \tfrac{1}{8}\left(\Gamma_{cusp} - \Gamma_{oct}\right) \zeta_2 \log \frac{v_1}{v_2} \log \frac{v_2}{v_3}$$
$$+ \text{cyclic permutations} + D_5(g),$$

where $D_5(g) = -5\zeta_2 g^2 + \frac{135}{4}\zeta_4 g^4 + \ldots$

Three loops

It is obvious that our all-loop conjecture is based on very thin ice.[2]

How about 3 loops?



- [2605.17057]— calculation of masters
- [2605.17056]— construction of the amplitude

[2] I made a bet with my coauthor that our conjecture is not correct in the finite terms.

Calculation of master integrals

In addition to DCI regularization more tricks were needed:

Additional techniques

- Pulling out dependence on DCI cross ratios.
- Successive asymptotics to simplify the integrands.

 Magic identities.

 Detecting finite integrals.

 IBP reduction in parametric representation.

 Using direct integration with `HyperInt`.

Example: result for $\mathcal{I}_2^{(3)} = \mathcal{I}_6^{(3)}$

$$
\begin{aligned}
\mathcal{I}_2^{(3)} = \mathcal{I}_6^{(3)} = {} & \frac{1}{12} L_1 \left(L_3^2 L_2^3 + 3 L_3^2 L_4 L_2^2 + 3 L_3^2 L_4^2 L_2 + L_4^2 L_5^3 + 3 L_3 L_4^2 L_5^2 + 3 L_3^2 L_4^2 L_5\right) \\
& + \frac{1}{6}(L_1 L_2^3 - L_3 L_2^3 + 6 L_1 L_3 L_2^2 + 3 L_1 L_4 L_2^2 - 3 L_3 L_4 L_2^2 + 3 L_1 L_3^2 L_2 + 3 L_1 L_4^2 L_2 - 3 L_3 L_4^2 L_2 \\
& + 12 L_1 L_3 L_4 L_2 + L_1 L_5^3 - L_4 L_5^3 + 3 L_3^2 L_4^2 + 3 L_1 L_3 L_4^2 + 3 L_1 L_3 L_5^2 + 6 L_1 L_4 L_5^2 - 3 L_3 L_4 L_5^2 + 3 L_1 L_3^2 L_4 \\
& + 3 L_1 L_3^2 L_5 + 3 L_1 L_4^2 L_5 - 3 L_3^2 L_4 L_5 + 12 L_1 L_3 L_4 L_5)\zeta_2 - \frac{1}{6}(L_2^3 + 3 L_4 L_2^2 + 3 L_4^2 L_2 + L_5^3 + 3 L_1 L_3^2 \\
& + 3 L_1 L_4^2 - 3 L_3 L_4^2 + 3 L_3 L_5^2 - 3 L_3^2 L_4 + 12 L_1 L_3 L_4 + 3 L_3^2 L_5)\zeta_3 + \frac{1}{4}(-10 L_2^2 + 42 L_1 L_2 - 10 L_3 L_2 \\
& - 20 L_4 L_2 + 7 L_3^2 + 7 L_4^2 - 10 L_5^2 + 35 L_1 L_3 + 35 L_1 L_4 + 20 L_3 L_4 + 42 L_1 L_5 - 20 L_3 L_5 - 10 L_4 L_5)\zeta_4 \\
& - (4 L_1 + L_2 - 5 L_3 - 5 L_4 + L_5)\zeta_2 \zeta_3 - 4 L_1 \zeta_5 + \frac{1}{8}\left(16 \zeta_3^2 + 77 \zeta_6\right)
\end{aligned}
$$

- Uniform transcendental function.
- Polynomial in L_i of 6th degree.

Example: result for $\mathcal{I}_5^{(3)}$

$$
\begin{aligned}
\mathcal{I}_5^{(3)} = {} & \frac{1}{12}(L_5^2L_1^3+L_2L_5L_1^3+3L_4L_5^2L_1^2+3L_2L_4L_5L_1^2+6L_3L_4L_5^2L_1+3L_2L_3^2L_4L_1+L_2L_3^3L_4 \\
& +6L_2L_3L_4L_5L_1)L_2+\frac{1}{6}(-L_2L_3^3+L_4L_3^3-3L_1L_2L_3^2+3L_1L_4L_3^2+6L_2L_4L_3^2+6L_1L_2^2L_3-3L_1L_5^2L_3 \\
& -3L_2L_5^2L_3+3L_4L_5^2L_3+3L_2^2L_4L_3+6L_1L_2L_4L_3-3L_2^2L_5L_3+12L_2L_4L_5L_3+3L_1^2L_2^2+3L_1L_2L_5^2 \\
& +6L_1L_4L_5^2+3L_1L_2^2L_4+3L_1L_2^2L_5+6L_1^2L_2L_5+12L_1L_2L_4L_5)\zeta_2+\frac{1}{6}(6L_5^2L_3-L_3^3-3L_1L_3^2+3L_2^2L_3 \\
& +6L_1L_2L_3-6L_1L_4L_3-12L_2L_4L_3+6L_1L_5L_3-6L_4L_5L_3-6L_1L_2^2+3L_1L_5^2+3L_2L_5^2-3L_4L_5^2 \\
& +6L_2^2L_4+6L_1L_2L_4+3L_2^2L_5)\zeta_3+\frac{1}{4}(3L_2^2+41L_1L_2+20L_3L_2+L_4L_2+28L_5L_2-10L_3^2-4L_5^2 \\
& +12L_1L_3+9L_1L_4+9L_3L_4+9L_1L_5-22L_3L_5+31L_4L_5)\zeta_4+(7L_3-4L_1-7L_2+2L_4-4L_5)\zeta_2\zeta_3 \\
& -2(3L_1+L_2+2L_3-2L_5)\zeta_5+c_1\zeta_6+c_2\zeta_3^2+O(v_i)
\end{aligned}
$$

We failed to find rational numbers $c_{1,2}$ in constant term $c_1\zeta_6+c_2\zeta_3^2$.

Amplitude

Substituting the found masters into the amplitude, we obtain

$$\log M_5 = \sum_i \Gamma_0(g) L_i^2 + \Gamma_1(g) L_i L_{i+1} + \Gamma_2(g) L_i L_{i+2} + D_5(g),$$

where

$$\Gamma_0(g) = \zeta_2 g^4 - 16\zeta_4 g^6 + O(g^8) \quad \Gamma_1(g) = -g^2 + 2\zeta_2 g^4 - \tfrac{59}{2}\zeta_4 g^6 + O(g^8)$$

$$\Gamma_2(g) = \zeta_2 g^4 - \tfrac{37}{2}\zeta_4 g^6 + O(g^8) \quad D_5(g) = -5\zeta_2 g^2 + \frac{135}{4}\zeta_4 g^4 + O(g^6)$$

NB: $\Gamma_0(g) + \Gamma_1(g) + \Gamma_2(g) = -\frac{1}{4}\Gamma_{oct}(g)$ including new $\propto g^6$ terms.

Conclusion and Outlook

Non-DCI integrals

Realistic calculations in QCD: non-DCI integrals. Can the DCI-inspired regularization help? **Probably, yes.**

Non-DCI pentagon

$$PB_{\text{non-DCI}} = \frac{1}{s_1^2 s_2 s_5}\left[\frac{1}{4}L_3^2 L_4^2 + \frac{1}{2}(L_3+L_4)^2\zeta_2 + (L_3+L_4)\zeta_3 + \frac{21}{4}\zeta_4\right]$$
$$+\frac{1}{s_1 s_3 s_4 s_5}\left[L_1 L_2 L_3\left(L_4+\frac{L_2}{2}\right)+\left(2L_1L_2 + L_4(L_3-L_2+L_1)-\frac{L_2^2}{2}\right)\zeta_2+(L_3-L_1)\zeta_3+\frac{5\zeta_4}{2}\right]$$
$$+\frac{1}{s_1 s_2 s_3 s_4}\left[L_1 L_4 L_5\left(L_3+\frac{L_5}{2}\right)+\left(2L_1L_5 + L_3(L_4-L_5+L_1)-\frac{L_5^2}{2}\right)\zeta_2+(L_4-L_1)\zeta_3+\frac{\zeta_4}{2}\right]$$
$$+\frac{1}{s_1^2 s_4 s_5}\left[\frac{1}{4}L_2(2L_4+L_2)L_3^2+\left(2L_3L_2+L_4L_2+L_3L_4+\frac{1}{2}L_2^2\right)\zeta_2-(L_3+L_4)\zeta_3+\frac{21}{4}\zeta_4\right]$$
$$+\frac{1}{s_1^2 s_2 s_3}\left[\frac{1}{4}L_5(2L_3+L_5)L_4^2+\left(2L_4L_5+L_3L_5+L_3L_4+\frac{1}{2}L_5^2\right)\zeta_2-(L_3+L_4)\zeta_3+\frac{21}{4}\zeta_4\right],$$

Conclusion and Outlook

- Dimensional regularization obscures DCI at intermediate steps.
- The DCI-preserving regularization drastically simplifies calculations.
- Several nontrivial examples have already been considered, including two-loop and three-loop pentagon integrals.
- The exponentiation of five-point slightly off-shell amplitude in $\mathcal{N} = 4$ SYM is checked up to 3 loops. Leading singularities are governed by $\Gamma_{oct}(g)$ as conjectured.
- A new ansatz for $\log M_5$ has been suggested. It seems to be capable of surviving further checks at higher loops.
- Future work: consider more applications to non-DCI integrals, consider the 6-point two-loop integrals.

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Thank you!

References

Alexey Pak and Alexander Smirnov. Geometric approach to asymptotic expansion of feynman integrals. *The European Physical Journal C*, 71(4):1626, 2011.

Tatiana Yu Semenova, Alexander V. Smirnov, and Vladimir A. Smirnov. On the status of expansion by regions. *Eur. Phys. J. C*, 79(2): 136, 2019. doi: 10.1140/epjc/s10052-019-6653-3.