Nonlocal gravitational effective action and its implications in cosmology

based on `arXiv:2605.12447`

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May 20, 2026

Motivation

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- However, these fields can play the important role at large energies, e.g. during the **Cosmological Inflation**.
- What is imprint of the conformally invariant fields on the inflationary observables like CMB spectrum?

Weyl anomaly

- Weyl-Invariant fields

$$S[\phi, g] \quad \mapsto \quad S[\phi^\sigma, g^\sigma] = S[\phi, g], \qquad (g^\sigma)_{\mu\nu} = e^{2\sigma} g_{\mu\nu}, \quad \sigma = \sigma(x)$$

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- Example: free massless non-minimally coupled scalar

$$S[\phi, g] = \frac{1}{2} \int d^D x \, \sqrt{g} \Big[(\nabla_\mu \phi)^2 + \xi R \, \phi^2 \Big], \qquad \xi = \tfrac{1}{4} \tfrac{D-2}{D-1}$$

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$$W[g] = -\log \int \mathcal{D}\phi \, e^{-S[\phi, g]} = \tfrac{1}{2} \log \mathrm{Det} \big[-\Box + \xi R \big],$$

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- Anomalous term Γ_a is known explicitly

$$\Gamma_{\mathrm{a}}[\sigma, g] = \tfrac{1}{16\pi^2} \int d^4 x \sqrt{g} \left\{ \left[\alpha C^2 + \beta \mathcal{E}_4 \right] \sigma + 2\beta \sigma \Delta_4 \sigma \right\}$$

$$- \tfrac{1}{32\pi^2} \left(\tfrac{\gamma}{6} + \tfrac{\beta}{9} \right) \int d^4 x \left(\sqrt{g^\sigma} (R^\sigma)^2 - \sqrt{g} R^2 \right),$$

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$$S = S_{\mathrm{EH}}[g^{\sigma}] + \Gamma[g] + \Gamma_{\mathrm{a}}[\sigma, g], \qquad \sigma = \log a(t)$$

  - expand $\Gamma[g] = \Gamma[\bar{g} + h]$ about simple background $\bar{g}$
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  - AdS/CFT context (Hawking, Hertog, Reall 2001)
  - Neglecting non-local part (Barvinsky, Kamenshchik 2006)
- Choice of background $\bar{g}$:
  - R^D (Barvinsky, Vilkovisky 1990)
  - $S^1 \times R^d$ (**this talk**)
  - $R \times S^d$, $S^1 \times S^d$ (preliminary result)

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- Using **heat kernel** methods and **zeta-regularization**

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- For scalar operator $F = -\Box + P$ (Barvinsky, Vilkovisky 1990)

$$\mathrm{Tr}\,K(\tau) = \frac{1}{(4\pi\tau)^{D/2}}\int d^D x\sqrt{g}\Big[1+\tau\big(\tfrac{1}{6}R-P\big) + \frac{1}{2}\tau^2\Big(R_{\mu\nu}f_1(-\tau\Box)R_{\mu\nu} + R\,f_2(-\tau\Box)\,R + P\,f_3(-\tau\Box)\,R + P\,f_4(-\tau\Box)\,P\Big)\Big]$$

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- Effective action $\varphi_I(-\Box) = c_I(-\Box)^{\frac{D}{2}-2}(\log(-\Box/\mu^2) + d_I)$

$$W[g] = \int d^Dx\sqrt{g}\Big[R_{\mu\nu}\varphi_1(-\Box)R_{\mu\nu} + R\,\varphi_2(-\Box)\,R + P\,\varphi_3(-\Box)\,R + P\,\varphi_4(-\Box)\,P\Big]$$

Calculation procedure

- Expansion of operator $F = -\Box + P$ induces those of heat trace

$$F = \sum_n F_n, \qquad F_n \sim h^n \quad \to \quad K(\tau) = e^{-\tau F} = \sum_n K_n(\tau),$$

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- First few terms of expansion about flat metric

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\mathrm{Tr}\, K_0(\tau) &= \frac{1}{(4\pi\tau)^{D/2}} \int d^D x\, r(\tau), \\
\mathrm{Tr}\, K_1(\tau) &= \frac{1}{(4\pi\tau)^{D/2}} \int d^D x\, \big(T_{\mu\nu}(\tau) h_{\mu\nu}(x) + T(\tau)\, P(x)\big) \\
\mathrm{Tr}\, K_2(\tau) &= \frac{1}{(4\pi\tau)^{D/2}} \frac{1}{2} \int d^D x\, \big(h_{\mu\nu}(x)\, \mathit{\Pi}_{\mu\nu,\rho\sigma}(\tau, \partial)\, h_{\rho\sigma}(x) \\
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\end{aligned}$$

- Here $\mathit{\Pi}$ and T — $SO(d)$-invariant tensors, linear in **basic form-factors**

$$f(\tau,\partial) = \int_0^1 d\alpha \, e^{\alpha(1-\alpha)\tau\partial_\mu^2} \frac{\sqrt{4\pi\tau}}{\beta} \sum_n e^{-\tau(\omega_n + i\alpha\partial_0)^2}, \qquad r(\tau) \equiv f(\tau,0)$$

Search for nice form

- The answer is **diffeomorphism-invariant**, i.e. under

$$\delta h_{\mu\nu} = \partial_\mu \xi_\nu + \partial_\nu \xi_\mu + \xi^\rho \partial_\rho h_{\mu\nu} + \partial_\mu \xi^\rho h_{\rho\nu} + \partial_\nu \xi^\rho h_{\mu\rho}.$$

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- **Stationary** case ($\omega = \frac{1}{2}\ln g_{00}$):

$$\begin{aligned}
\operatorname{Tr} K(\tau)\Big|_{\text{stationary}} = \frac{1}{(4\pi\tau)^{D/2}} \int d^D x \sqrt{g} \Bigg[& \frac{1}{(g_{00})^{\tau\partial_\tau}} r(\tau) \\
& + \tau \frac{1}{(g_{00})^{\tau\partial_\tau}} \left(\frac{1}{6} R - P + \frac{1}{3}\tau\partial_\tau \frac{R_{00}}{g_{00}} + \frac{1}{3}\tau\partial_\tau(\tau\partial_\tau + 1)(\nabla\omega)^2 \right) r(\tau) \\
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& \qquad + R\, \bar{f}_5\, {}^{(d)}R + P\, \bar{f}_6\, {}^{(d)}R + {}^{(d)}R_{ij}\, \bar{f}_7\, {}^{(d)}R_{ij} + {}^{(d)}R\, \bar{f}_8\, {}^{(d)}R \Big) \Bigg]
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- Leads to **Tolman temperature** $T_{\text{Tolman}}(x) = T / \sqrt{g_{00}(x)}$

$$(g_{00})^{-\tau\partial_\tau} r(\tau) = r(\tau)\Big|_{T = T_{\text{Tolman}}}.$$

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$$\xi^\mu_0 = \delta^\mu_0, \qquad \xi^\mu_1 = \Xi^\mu_{\nu\rho}(\partial) h_{\nu\rho}, \qquad \xi^\mu_2 = \frac{1}{2} \Xi^\mu_{\nu\rho,\sigma\kappa}(\partial^x, \partial^y) h_{\nu\rho}(x) h_{\sigma\kappa}(y)\Big|_{y=x}.$$

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  - Stationary limit: $\xi^\mu[h(\boldsymbol{x})] = \delta^\mu_0$
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  - Transformation law: $\delta_\zeta \xi^\mu[h] = \zeta^\rho \partial_\rho \xi^\mu[h] - \partial_\rho \zeta^\mu \, \xi^\rho[h]$
- This imply (at linear order)

$$k^\rho \, \Xi^{\mu}_{\rho\sigma}(k) = \frac{1}{2} k_0 \, \delta^\mu_\sigma, \qquad \Pi^{\mu}_{\rho\sigma}(k)\Big|_{k_0=0} = 0$$

Covariant form in general case

- Generalized Killing vectors allows to covariantize stationary answer

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  1. Covariantize stationary answer using ξ^μ, **add Killing structures**
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- Additional structures

$$\left(\nabla_{(\mu}\xi_{\nu)}\right)^2, \qquad \left(\nabla_\mu\xi^\mu\right)^2 \qquad \xi^\mu\,\nabla_\xi\xi^\nu\nabla_{(\mu}\xi_{\nu)},$$

$$\xi^\mu\,\nabla^\nu\xi^2\,\nabla_{(\mu}\xi_{\nu)}, \qquad \xi^\mu\xi^\nu\nabla_{(\mu}\xi_{\nu)}\,\nabla_\rho\xi^\rho, \qquad \left(\xi^\mu\xi^\nu\nabla_{(\mu}\xi_{\nu)}\right)^2.$$

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Answer

- Covariant answer for general (non-stationary) perturbations ($f_I = f_I(\tau, \nabla_\mu)$)

$$
\begin{aligned}
\mathrm{Tr}K(\tau) = \frac{1}{(4\pi\tau)^{D/2}} \int d^D x \sqrt{g} \Bigg[& \frac{1}{(g_{\xi\xi})^{\tau\partial_\tau}} r(\tau) \\
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& \qquad + \frac{1}{3} \left[(\nabla_\mu \omega)^2 - 2\xi^\mu \, \nabla^\nu \xi^2 \, \nabla_{(\mu}\xi_{\nu)} + \ldots \right] \tau\partial_\tau (\tau\partial_\tau + 1) \Bigg) r(\tau) \\
& + \frac{1}{2}\tau^2 \Bigg(R_{\mu\nu} \, f_1 \, R_{\mu\nu} + R \, f_2 \, R + P \, f_3 \, R + P \, f_4 \, P \\
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& + \tau \frac{1}{(g_{\xi\xi})^{\tau\partial_\tau}} \Bigg(\frac{1}{6} R - P + \frac{1}{3} \left[\frac{R_{\xi\xi}}{g_{\xi\xi}} + \left(\nabla_{(\mu}\xi_{\nu)}\right)^2 + \dots \right] \tau\partial_\tau \\
& \qquad + \frac{1}{3} \left[(\nabla_\mu \omega)^2 - 2\xi^\mu \, \nabla^\nu \xi^2 \, \nabla_{(\mu}\xi_{\nu)} + \dots \right] \tau\partial_\tau(\tau\partial_\tau + 1) \Bigg) r(\tau) \\
& + \frac{1}{2} \tau^2 \Bigg(R_{\mu\nu} \, f_1 \, R_{\mu\nu} + R \, f_2 \, R + P \, f_3 \, R + P \, f_4 \, P \\
& \qquad + R \, f_5 \, {}^{(d)}R + P \, f_6 \, {}^{(d)}R + {}^{(d)}R_{ij} \, f_7 \, {}^{(d)}R_{ij} + {}^{(d)}R \, f_8 \, {}^{(d)}R \Bigg) \Bigg].
\end{aligned}$$

- “Tolman temperature” in non-stationary geometries

$$T_{\mathrm{Tolman}}(x) = T / \sqrt{g_{\xi\xi}(x)}$$

Effective action

- Splitting of the form-factors into vacuum and thermal parts

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- Effective action have the same form as heat trace upon substitution

$$\tau^p f_\beta(\tau, \nabla) \quad \mapsto \quad \varphi^p_\beta(\nabla) = \int_0^\infty \frac{d\tau}{\tau} \frac{1}{\tau^{D/2}} \tau^p f_\beta(\tau, \nabla)$$

Thermal form factors

- Thermal form factors are linear combinations of $(k_\mu = (\omega_m, \boldsymbol{k}) \leftrightarrow i\nabla_\mu)$

$$\varphi^p_\beta(k) = \partial^q_{z^2}\, \Omega\, \Phi_q(z), \qquad z = \tfrac{\beta k}{2} = \tfrac{\beta}{2}\sqrt{-\Box}, \quad q = \frac{D}{2} - p$$

where "stationary" form factor reads

$$\Phi_q(z) = (-)^q \frac{(2\pi)^{2q+1}}{(2q)!}\frac{1}{2iz}\big(\zeta'(-2q, -\tfrac{iz}{2\pi}) - \zeta'(-2q, \tfrac{iz}{2\pi})\big)$$

$$- \frac{\pi^2}{2(2q)!} z^{2q-1} + \sum_{l=0}^{q} \frac{2}{(2l+1)!}\zeta(2q-2l)\big(\log(\tfrac{z}{2\pi}) + H_{2q} - H_{2l+1}\big) z^{2l}$$

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- Account higher Matrsubara harmonics ω_m induces inclusion of operator

$$\Omega\Phi(k) = \frac{i}{2}\frac{\omega_m}{|\boldsymbol{k}|}\Big[\Phi(i\omega_m + |\boldsymbol{k}|) - \Phi(-i\omega_m + |\boldsymbol{k}|)\Big] \\ + \frac{1}{2}\Big[\Phi(i\omega_m + |\boldsymbol{k}|) + \Phi(-i\omega_m + |\boldsymbol{k}|)\Big]$$

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- Invariant form is written in terms of non-linear completion of Bardeen invariants Φ, Ψ

Discussion

Bottom line

- Finite-temperature effective action for non-minimally coupled scalar field in **non-stationary** metric perturbations is constructed.
- This extends **vacuum** (Barvinsky, Vilkovisy 1990) and **stationary** results (Gusev, Zelnikov 2000; Elías, Mazzitelli, Trombetta, 2017; Valle, Vazquez-Mozo 2025)
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What is next?

- Euclid $\mapsto$ Schwinger-Keldysh: analytic continuation using methods of (Barvinsky, NK, 2024).
- Higher spin and fermion fields & general backgrounds.

Thank you for your attention!