Averaging over N and quantum chaos: view from random matrix theory

Mikhail Khramtsov
Steklov Mathematical Institute

based on 2606.xxxxx with Ben Craps,
Juan Hernandez and Maria Knysh

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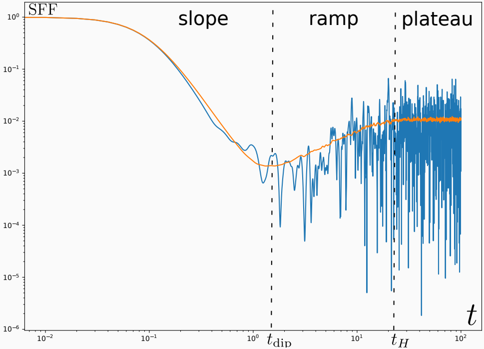
Introduction: spectral form factor and chaos

The diagnostic of fine-grained quantum chaos - spectral form factor

$$S(\beta, T) = Z(\beta + iT)Z(\beta - iT) = \mathrm{Tr}\ \mathrm{e}^{-\beta H - iHT}\,\mathrm{Tr}\ \mathrm{e}^{-\beta H + iHT}$$
$$= \sum_{n,m} \mathrm{e}^{-\beta(E_m+E_n)}\mathrm{e}^{iT(E_m-E_n)} = \int dE_1 dE_2 \rho(E_1)\rho(E_2) e^{-\beta(E_1+E_1)+iT(E_1-E_2)}.$$



Empirics for chaotic systems:

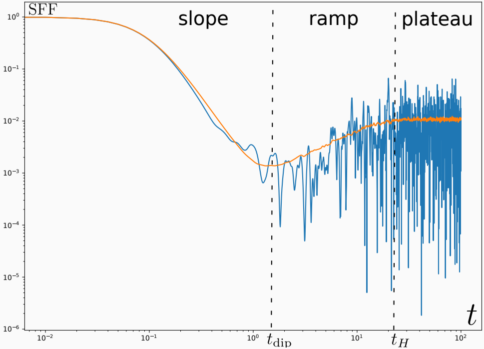
- Slope is smooth, self-averaging
- Ramp & plateau are erratic, non self-averaging

Motivation 1: chaos in strongly coupled systems

Bohigas-Giannoni-Schmit conjecture: *spectral correlations in a quantized system, that is classically chaotic, are determined by the universal laws of a Gaussian random matrix ensemble.*

$\langle S(\beta, T)\rangle = \int \mathcal{D}H \exp\left(-\frac{L}{2}\operatorname{Tr} H^2\right) \operatorname{Tr}\ \mathrm{e}^{-\beta H - iHT} \operatorname{Tr}\ \mathrm{e}^{-\beta H + iHT}$.

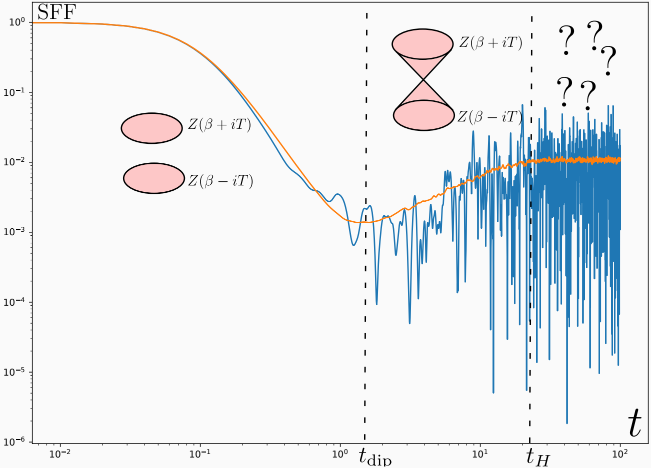
This talk: Gaussian Unitary Ensemble, $H = H^\dagger \in GUE(L)$.



For an N-qubit system, $L = 2^N$.
The ensemble averaged SFF is well-behaved at arbitrarily late times.
Question: how to defeat the fluctuations in regular quantum systems, without the ensemble?

Motivation 2: ensemble averaging from gravity path integral

Using holography, $\langle Z(\beta + iT) Z(\beta - iT)\rangle = \int_{\text{b, cs}} \mathcal{D}g_{\mu\nu} e^{-\frac{1}{G_N} I[g]}$, with $G_N \sim \frac{1}{N}$. Evaluating in the semiclassical limit,



Question: what kind of average is realized by the gravity path integral?

Averaging over N in holographic CFTs

In AdS/CFT, we have the parameter N in the boundary CFT. In the standard semiclassical limit, $N \sim \frac{1}{G_N}$.
What if gravity is taking an average over N at late times of the SFF?*
Proposals:

- Hong Liu '25: naive averaging $\overline{F} = \lim_{N\to\infty} \frac{1}{W_N} \sum_{K=N-W_N}^{N} F_K$
- Kudler-Flam, Witten '26: Mellin averaging $\overline{F_N} = \sum_{k=0}^{\infty} \frac{f_k}{N^K}$, where $f_k = -\mathrm{Res}_{s=k} \int_1^{+\infty} dN\ N^{s-1} F_N$.
- This talk: submatrix averaging in RMT: $\overline{F(H)} = \lim_{L\to\infty} \frac{1}{W_L} \sum_{K=L-W_L}^{L} F(H_K)$, where $H \in GUE(L)$ and H_K is the principal $K \times K$ submatrix.

Spectral form factor in GUE: generalities

The spectral form factor is defined as

$$\begin{aligned} S_L(\beta, T) &= \frac{1}{L} Z_L(\beta + iT) Z_L(\beta - iT) = \frac{1}{L} \sum_{i,j=1}^{L} e^{-\beta(E_i+E_j)+iT(E_i-E_j)} \\ &= \frac{1}{L} \int dE_1 dE_2\ \rho(E_1)\rho(E_2) e^{-\beta(E_1+E_1)+iT(E_1-E_2)}\,. \end{aligned}$$

where $\rho(E) = \sum_{i=1}^{L} \delta(E - \lambda_i)$ is the spectral measure. The normalization is chosen so that $SFF \sim O(1)$ as $L \to \infty$ in the plateau regime.

The ensemble averaged SFF is then given by

$$\begin{aligned} \langle S_L(\beta, T)\rangle &= \int \mathcal{D}H \exp\left(-\frac{L}{2} \operatorname{Tr} H^2\right) \operatorname{Tr}\ \mathrm{e}^{-\beta H - iHT} \operatorname{Tr}\ \mathrm{e}^{-\beta H + iHT} \\ &= \frac{1}{L} \int dE_1 dE_2\ \langle \rho(E_1)\rho(E_2)\rangle e^{-\beta(E_1+E_1)+iT(E_1-E_2)} \end{aligned}$$

Submatrix averaging in GUE SFF

Let $H \in GUE(L)$, and H_K is its principal minor (submatrix) of size K. The spectral form factor of H_K is given by

$$\begin{aligned} S_K(\beta, T) &= \frac{1}{K} \sum_{i,j=1}^{K} e^{-\beta(E_i^{(K)}+E_j^{(K)})+iT(E_i^{(K)}-E_j^{(K)})} = \\ &= \frac{1}{K} \int dE_1 dE_2 \ \rho_K(E_1)\rho_K(E_2) e^{-\beta(E_1+E_1)+iT(E_1-E_2)} \,. \end{aligned}$$

We are interested in considering a set of submatrices for sizes $K \in [L-W, L]$. Let us define the submatrix averaged SFF

$$\overline{S(\beta, T)} = \frac{1}{W} \sum_{K=L-W}^{L} S_K(\beta, T)\,.$$

Also define the submatrix averaged spectral measure product $\overline{\rho(\lambda_1)\rho(\lambda_2)} = \frac{1}{W} \sum_{K=L-W}^{L} \frac{1}{K} \rho_K(\lambda_1)\rho_K(\lambda_2)$.

Submatrix self-averaging: main result

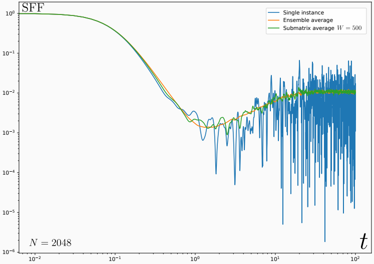
Submatrix self averaging statement, If $O(1) < W \leq O(L)$ for $L \to \infty$, then the spectral form factor averaged over the submatrices is self-averaging in the $L \to \infty$ limit, i.e.

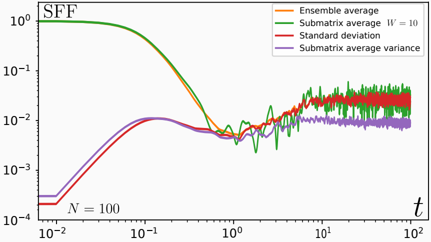
$$\lim_{L\to\infty} \overline{S(\beta, T)} = \langle S(\beta, T) \rangle \qquad \text{almost surely}\,.$$

Proof:

- Numerical evidence
- Field-theoretic analytic proof (using replica trick)

Numerics 1



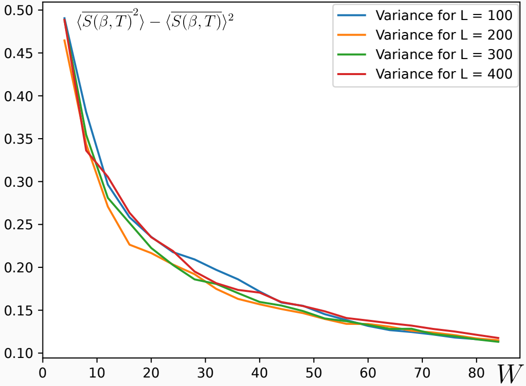


- **Left.** Submatrix average of the SFF in GUE compared to the single draw and ensemble average.
- **Right.** Variance of the submatrix average compared to variance of the ensemble average of the SFF and to the averaged SFF itself.

Ensemble variance $= \langle S(\beta, T)^2 \rangle - \langle S(\beta, T) \rangle^2 \sim O(1)$

Submatrix average variance: $D_L(\beta, T) = \langle \overline{S(\beta, T)}^2 \rangle - \langle \overline{S(\beta, T)} \rangle^2 \sim \frac{1}{W}$

Numerics 2



Variance of the submatrix average as function of L and window size

Idea of the analytic proof of self-averaging

Step 1. Show that the expectation value matches
(easy, follows from the definition):

$$\lim_{L\to\infty}\langle\overline{S(\beta,T)}\rangle = \langle S(\beta,T)\rangle\,.$$

Step 2. Consider the variance $D_L(\beta,T) = \left\langle\overline{S(\beta,T)}^2\right\rangle - \langle\overline{S(\beta,T)}\rangle^2\,.$
Show that

$$\lim_{L\to\infty} D_L(\beta,T) = 0\,.$$

(nontrivial)

Submatrix variance of the SFF

Rewriting the variance in terms of spectral correlators:

$$D_L(\beta, T) = \int dE_1 dE_2 dE_3 dE_4 \mathcal{D}(E_1, E_2, E_3, E_4) e^{-\beta(E_1+E_2+E_3+E_4)+iT(E_1-E_2+E_3-E_4)} \,.$$

where $\mathcal{D}$ is defined as follows:

$$\mathcal{D}(E_1, E_2, E_3, E_4) = \left\langle \overline{\rho(E_1)\rho(E_2)}\ \overline{\rho(E_3)\rho(E_4)} \right\rangle - \langle \overline{\rho(E_1)\rho(E_2)} \rangle \langle \overline{\rho(E_3)\rho(E_4)} \rangle \,.$$

Introducing the notations $X_K(E_1, E_2) = \rho_K(E_1)\rho_K(E_2)$;
$X_{IJ}(E_1, E_2) = \rho_I(E_1)\rho_J(E_2)$, we rewrite $\mathcal{D}$ as

$$\mathcal{D}(E_1, E_2, E_3, E_4) = \frac{1}{W^2} \sum_{I=L-W}^{L} \sum_{J=L-W}^{L} \frac{1}{IJ} \mathrm{Cov}(X_I(E_1, E_2), X_J(E_3, E_4)) \,.$$

Finally, the covariance is written as

$$\begin{aligned} \mathrm{Cov}(X_I(E_1, E_2), X_J(E_3, E_4)) &= \langle X_I(E_1, E_2) X_J(E_3, E_4) \rangle_{\mathrm{conn}} \\ &+ \langle X_{IJ}(E_1, E_3) \rangle \langle X_{IJ}(E_2, E_4) \rangle + \langle X_{IJ}(E_1, E_4) \rangle \langle X_{IJ}(E_2, E_3) \rangle \,. \end{aligned}$$

Resolvent approach

The resolvent is defined as $R(E) = \mathrm{Tr}\, \frac{1}{E-H}$.

The spectral density is recovered as the discontinuity

$$\rho(E) = \frac{1}{2\pi i} \lim_{\epsilon\to 0} \left(R(E-i\epsilon) - R(E+i\epsilon) \right) .$$

Key identity:

$$R(E) = \frac{\partial}{\partial E} \log \mathcal{Z}(E) , \quad \text{where } \mathcal{Z}(E) = \mathrm{Det}(E-H) .$$

This determinant can be written as the integral over fermions

(see Kamenev, Mezard '99 for the most complete replica treatment of standard GUE)

$$\mathcal{Z}(E) = \int \prod_{i=1}^{L} d\psi_i d\overline{\psi}_i e^{-\sum_{i,j=1}^{L} \overline{\psi}_i (E\delta_{ij} - H_{ij})\psi_j} .$$

Final ingredient is the replica trick

$$\log \mathcal{Z}(E) = \lim_{n\to 0} \frac{\mathcal{Z}(E)^n - 1}{n} .$$

Submatrix spectral correlators from the replica method

We need 2-pt and 4-pt spectral correlators for submatrices.

Example of 2-pt function:

$$\left\langle \operatorname{Tr} \frac{1}{E_1 - H_K} \operatorname{Tr} \frac{1}{E_2 - H_L} \right\rangle = \lim_{n,m \to 0} \frac{1}{nm} \frac{\partial}{\partial E_1} \frac{\partial}{\partial E_2} \langle \mathcal{Z}_{KL}^{n+m} \rangle \, ,$$

To derive $\langle \mathcal{Z}_{KL}^{n+m} \rangle$, we assume $K < L$ (H_K is a submatrix of GUE(L) matrix H) and proceed with the following steps:

1. Use the key identity above
2. Introduce fermionic variables ψ, ϕ arranged into $n + m$ replicas
3. Decompose $\operatorname{Tr} H^2 = \operatorname{Tr} H_K^2 + 2 \operatorname{Tr} V^\dagger V + \operatorname{Tr} h^2$, rearrange the fermions accordingly
4. Integrate out the random matrix
5. Introduce the auxiliary replica fields $G_{\alpha\beta}$, $F_{\alpha\beta}$ and Lagrange multiplier fields $\Sigma_{\alpha\beta}$ and $\Xi_{\alpha\beta}$.
6. Integrate out the fermions

Replica field theory for the submatrix 2-pt function

We arrive at the SYK-like path integral

(see Aref'eva, M. K., Tikhanovskaya, Volovich '18 for details on replicas in SYK)

$$\langle \mathcal{Z}_{KL}^{n+m} \rangle = \int \mathcal{D}G\mathcal{D}\Sigma\mathcal{D}F\mathcal{D}\Xi \exp\Big(K \,\mathrm{Tr}\log(\hat{E} - \Sigma) + (L-K)\,\mathrm{Tr}\log(\hat{E} - \Xi)_m + \\ + K\,\mathrm{Tr}\,\Sigma G + (L-K)\,\mathrm{Tr}(\Xi F)_m - \frac{K^2}{2L}\,\mathrm{Tr}\,G^2 - \frac{(L-K)^2}{2L}\,\mathrm{Tr}(F^2)_m \\ - \frac{K(L-K)}{L}\,\mathrm{Tr}(GF)_m \Big) \,.$$

Ingredients:

- $\hat{E} = \mathrm{diag}(E_1 \times \mathbf{1}_n, E_2 \times \mathbf{1}_m)$ is the matrix of energy arguments
- G and Σ are Hermitian $(n+m) \times (n+m)$ replica matrices
- F and Ξ are Hermitian $m \times m$ replica matrices

Symmetries of the action (unitary conjugation):

- $K < L$, general energies: $U(n) \times U(m)$
- $K = L$, $E_1 \neq E_2$: $U(n) \times U(m)$
- $K = L$, $E_1 = E_2$: $U(n+m)$

Saddle point structure

The symmetries determine the saddle point structure.

- $K < L$, general energies:

$$\Sigma = \begin{pmatrix} \bar{\Sigma} & 0_{n\times m} \\ 0_{m\times n} & \Xi \end{pmatrix} = \begin{pmatrix} U & 0_{n\times m} \\ 0_{m\times n} & V \end{pmatrix} \begin{pmatrix} \bar{S}_1 & 0_{n\times m} \\ 0_{m\times n} & S_2 \end{pmatrix} \begin{pmatrix} U^\dagger & 0_{n\times m} \\ 0_{m\times n} & V^\dagger \end{pmatrix}$$

  where $U \in U(n)$, $V \in U(m)$, $\bar{S}_1 = \mathrm{diag}(\bar{s}_{1,\alpha}\,,\ \alpha = \overline{1,\ n})$ and $S_2 = \mathrm{diag}(s_{2,\alpha}\,,\ \alpha = \overline{1,\ m})$
- $K = L$, $E_1 \neq E_2$:

$$\Sigma = \begin{pmatrix} U & 0_{n\times m} \\ 0_{m\times n} & V \end{pmatrix} \begin{pmatrix} S_1 & 0_{n\times m} \\ 0_{m\times n} & S_2 \end{pmatrix} \begin{pmatrix} U^\dagger & 0_{n\times m} \\ 0_{m\times n} & V^\dagger \end{pmatrix}$$

- $K = L$, $E_1 = E_2$: $\Sigma = U S U^\dagger$, where $U \in U(n+m)$ and $S = \mathrm{diag}(s_{1,\alpha}\,,\ \alpha = \overline{1,\ n+m})$

Here

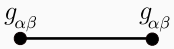
$$s_{1,2,\alpha} = \frac{E_{1,2} \pm \sqrt{E_{1,2} - 4}}{2}\,, \quad \bar{s}_{1,2,\alpha} = \frac{E_{1,2} \pm \sqrt{E_{1,2} - 4\frac{K}{L}}}{2\frac{K}{L}}\,.$$

Feynman rules for the replica submatrix theory

The replicated 2-pt function is

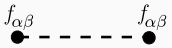
$$\frac{\partial}{\partial E_1}\frac{\partial}{\partial E_2}\langle \mathcal{Z}_{KL}^{n+m}\rangle = \langle\langle K\sum_{\alpha=1}^{n} G_{\alpha\alpha}\sum_{\beta=1}^{m}(KG_{\beta\beta}+(L-K)F_{\beta\beta})\rangle\rangle$$

We expand the action around the saddle point $G = G_0 + g$, $F = F_0 + f$, $\Sigma = \Sigma_0 + \sigma$, $\Xi = \Xi_0 + \xi$. Feynman rules:

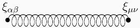


$= \dfrac{L}{K^2}\,,$

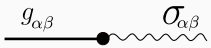
$= \dfrac{1}{K}A_{\alpha\beta\mu\nu}(\hat{E})\,,$



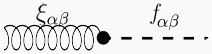
$= \dfrac{L}{(L-K)^2}\,,$



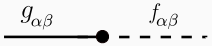
$= \dfrac{1}{L-K}B_{\alpha\beta\mu\nu}(\hat{E})\,,$



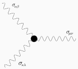
$= K\,,$



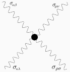
$= (L-K)\,,$



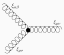
$= \dfrac{K(L-K)}{L}\,,$



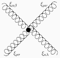
$= KC_{\dots}(\hat{E})\,,$



$= KD_{\dots}(\hat{E})\,,$



$= (L-K)P_{\dots}(\hat{E})\,,$



$= (L-K)Q_{\dots}(\hat{E})\,.$

Putting the covariances together

Recall that we need to compute

$$W^2\mathcal{D}(E_1, E_2, E_3, E_4) = \sum_{I=L-W}^{L} \sum_{J=L-W}^{L} \frac{1}{IJ} \mathrm{Cov}(X_I(E_1, E_2), X_J(E_3, E_4)) =$$

$$\sum_{K=L-W}^{L} \frac{1}{K^2} \mathrm{Cov}(X_K(E_1, E_2), X_K(E_3, E_4)) + \sum_{I \neq J} \frac{1}{IJ} \mathrm{Cov}(X_I(E_1, E_2), X_J(E_3, E_4)) \, .$$

Diagonal term: dominated by the disconnected contribution

$$\mathrm{Cov}(X_K(E_1, E_2), X_K(E_3, E_4)) = \langle X_K(E_1, E_2) X_K(E_3, E_4) \rangle_{\mathrm{conn}}$$
$$+ \langle X_K(E_1, E_3) \rangle \langle X_K(E_2, E_4) \rangle + \langle X_K(E_1, E_4) \rangle \langle X_K(E_2, E_3) \rangle \, .$$

Dominant diagram class:

$\langle X_K(E_1, E_3) \rangle \simeq$ 

$+ \cdots = K \; M_{\mathrm{diag}}(E_1) \delta(E_1 - E_3) \, .$

$\Rightarrow \mathrm{Cov}(X_K(E_1, E_2), X_K(E_3, E_4)) \sim K^2$

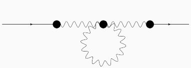
Putting the covariances together

Offdiagonal term: dominated by contributions from all terms!
(assume here $I < J$)

$$\mathrm{Cov}(X_I(E_1,E_2),X_J(E_3,E_4)) = \langle X_I(E_1,E_2)X_J(E_3,E_4)\rangle_{\mathrm{conn}}$$
$$+\langle X_{IJ}(E_1,E_3)\rangle\langle X_{IJ}(E_2,E_4)\rangle + \langle X_{IJ}(E_1,E_4)\rangle\langle X_{IJ}(E_2,E_3)\rangle\,.$$

Dominant diagrams:

$\langle X_{KL}(E_1,E_3)\rangle \simeq$



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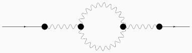
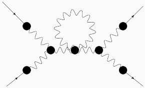


$= \frac{L^2}{K^2} M_{\mathrm{disc}}(E_1,E_3)\,.$

$\langle X_K(E_1,E_2)X_L(E_3,E_4)\rangle_{\mathrm{conn}} \simeq$



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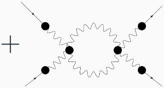

$= \frac{L^4}{K^4} M_{\mathrm{conn}}(\hat{E})\,.$

$\Rightarrow \mathrm{Cov}(X_I(E_1,E_2),X_J(E_3,E_4)) \sim \frac{J^4}{I^4}$ for $I < J$

Submatrix variance estimate

After taking the sum over the submatrix sizes I and J, we find

$$\mathcal{D}(E_1, E_2, E_3, E_4) = \frac{1}{W} M_1(E_1, E_2, E_3, E_4) + O\left(\frac{1}{W^2}\right).$$

This goes to zero if $W(L) \to \infty$ with $L \to \infty$, thus completing the proof of submatrix self-averaging.

Conclusions & outlook

- We have proven the self-averaging property of the submatrix average in the GUE.
- The argument should be easily generalizable to other Gaussian matrix ensembles and to Mellin averaging a-la Kudler-Flam-Witten.
- RMT universality of chaotic systems implies that this self-averaging should hold for any chaotic theory with respect to averaging over the (microcanonical) Hilbert space subspaces.
- Same should hold for averaging over N in holographic theories such as $\mathcal{N} = 4$ SYM, although the precise estimates for the variance need to be re-derived.

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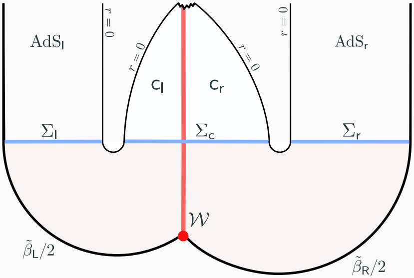
Thank you for attention!

Instead of outlook: baby universe puzzle



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Engelhardt, Gesteau 2504.14586
Balasubramanian, Yilidrim 2509.09763
Hong Liu 2509.14327

Kudler-Flam, Witten 2510.06376

This state is at odds with AdS/CFT-dictionary: it breaks ER = EPR. Yet it is a perfectly good state from the gravity point.
Proposal: baby universe states are not semiclassical, but are dual to a sequence of states in CFT averaged over N.