Black Hole formation by a scalar field

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$(\mathbb{M}, g)$ - space-time = four dimensional manifold with Lorentzian signature metric.

$x = (x^\alpha), \quad \alpha = 0,1,2,3$ - local coordinates

The Liouville metric (1849):

$$g_{\alpha\beta} := \Phi(x)\eta_{\alpha\beta}, \qquad \eta_{\alpha\beta} := \operatorname{diag}(+---)$$

$\Phi > 0$

$$\Phi := \phi_0(x^0) + \phi_1(x^1) + \phi_2(x^2) + \phi_3(x^3)$$

- the conformal factor

There is no Killing vector field for nontrivial functions $\phi_\alpha(x^\alpha)$

The advantage: the geodesic Hamilton-Jacobi equation admits complete separation of variables ⟹ Geodesic equations are integrable for arbitrary $\phi_\alpha(x^\alpha)$.

The Stackel problem (1893): which metric admits complete separation of variables in the geodesic Hamilton-Jacobi equation?
It was solved for metrics of arbitrary signature in any number of dimensions:
Katanaev arXiv2305.02222, PhysicaScripta (2023), TMΦ (2024)

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The action

$$S := \int dx \sqrt{|g|} \left(R - 2\Lambda + \frac{1}{2} g^{\alpha\beta} \partial_\alpha \varphi \partial_\beta \varphi - V(\varphi) \right), \qquad g := \det g_{\alpha\beta}$$

$\varphi(x)$ - scalar field
Λ - cosmological constant
$V(\varphi)$ - potential for a scalar field

The additional assumption $\varphi(x) := \Psi\big(\Phi(x)\big)$

A general solution to the field equations exists only for the potential

$$V + 2\Lambda = \pm 12 C^2 \mathrm{e}^{\mp k\varphi}$$

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$k := \dfrac{2}{\sqrt{3}}$ $\qquad C > 0$ is an integration constant

Black hole formation

$$V + 2\Lambda = -12 C^2 \mathrm{e}^{k\varphi}$$

A general solution

$$\Phi = c - s$$

$$\varphi = \pm\sqrt{3} \ln\big(C\Phi\big)$$

where $s := \eta_{\alpha\beta} x^\alpha x^\beta$ and $c \in \mathbb{R}$ is an integration constant

The solution

$$\boxed{g_{\alpha\beta} = (c-s)\eta_{\alpha\beta}, \qquad s := \eta_{\gamma\delta}x^\gamma x^\delta}$$

6 noncommuting Killing vector fields → Spontaneous symmetry emergence

Afanasev, Katanaev. Phys.Rev.D (2019), ibid. (2020)

Curvature tensor $R_{\alpha\beta\gamma}{}^{\delta} = \frac{1}{\Phi}\left(3-\frac{c}{\Phi}\right)\eta_{\alpha\gamma}\delta^\delta_\beta - \frac{3}{\Phi^2}(x_\alpha x_\gamma \delta^\delta_\beta - x_\alpha x^\delta \eta_{\beta\gamma}) - (\alpha \leftrightarrow \beta)$

Curvature invariants: $R^{\alpha\beta\gamma\delta}R_{\alpha\beta\gamma\delta} = \frac{12}{\Phi^4}\left(9-\frac{6c}{\Phi}+\frac{5c^2}{\Phi^2}\right)$

$$R^{\alpha\beta}R_{\alpha\beta} = \frac{36}{\Phi^4}\left(3+\frac{c^2}{\Phi^2}\right)$$

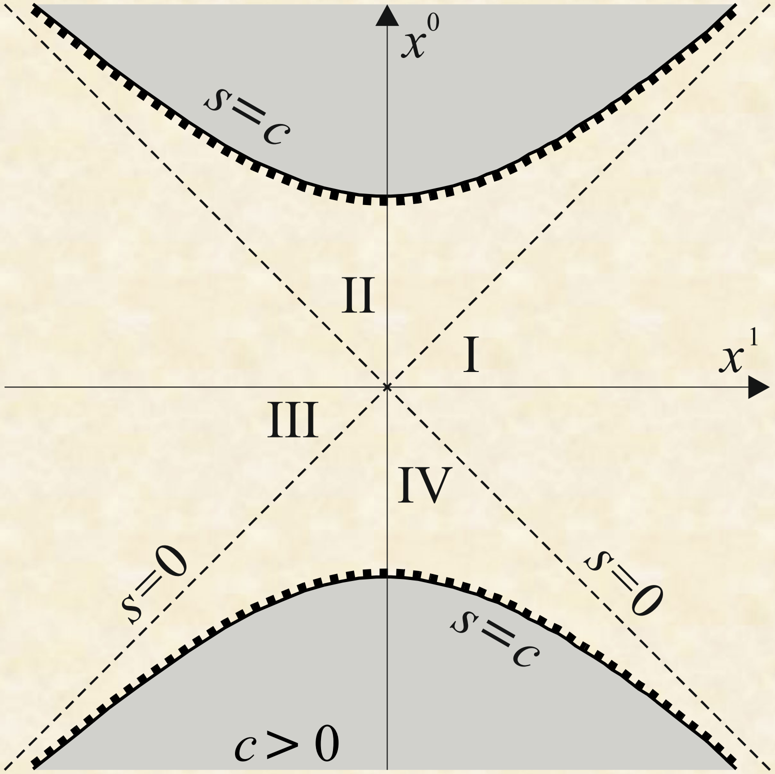
$$R = \frac{6}{\Phi^2}\left(3+\frac{c}{\Phi}\right)$$

The only singularity is located at $\Phi = 0$

The spacetime is asymptotically flat at space infinity $s \rightarrow -\infty$

Spherically symmetric black hole formation

$$\Phi > 0 \quad \Leftrightarrow \quad s < c, \qquad c > 0$$



The forbidden regions are shaded.

The picture must be rotated in two extra space dimensions around x^0 axis.

The solution is global: any geodesic can be either extended to infinite value of the canonical parameter in both directions or it ends up at the singular point.

No particle or light can escape the future light cone. Therefore it is the event horizon. At a fixed moment of time it is a sphere. The event horizon appears at $x^0 = 0$ with infinitesimal radius and afterwards expands with the velocity of light to infinity.
The singularity is geodesically incomplete.

Comparison with the Schwarzschild solution

Spherical coordinates: $x^0, x^1, x^2, x^3 \;\mapsto\; t := x^0, r, \vartheta, \varphi$

$$\boxed{ds^2 = \left(c - t^2 + r^2\right)\left[dt^2 - dr^2 - r^2\left(d\vartheta^2 + \sin^2 d\varphi^2\right)\right]}$$

In the limit $r \to \infty$, $\dfrac{t}{r} \to 0$ (spatial infinity) the metric is conformally Lorentzian

$$ds^2 = r^2\left[dt^2 - dr^2 - r^2\left(d\vartheta^2 + \sin^2 d\varphi^2\right)\right]$$

The event horizon: $t^2 - r^2 = 0, \quad t > 0$

The geometric defect

New coordinates: $R := \dfrac{1}{2} r^2, \quad T := t\,r$

$$ds^2 = \left(1 + \frac{c}{2R} - \frac{T^2}{4R^2}\right)\left[\left(dT - \frac{T}{2R}dR\right)^2 - dR^2 - 4R^2\left(d\vartheta^2 + \sin^2 d\varphi^2\right)\right]$$

At spatial infinity $R \to \infty$, $\dfrac{T}{R} \to 0$

$$ds^2 = dT^2 - dR^2 - 4R^2\left(d\vartheta^2 + \sin^2 d\varphi^2\right)$$

The Liouville solution to first order in $\dfrac{T}{R}$

$$ds^2 = dT^2 - \frac{T}{R}dTdR - dR^2 - 4R^2\left(d\vartheta^2 + \sin^2 d\varphi^2\right)$$

The Schwarzschild solution in the Painlevé-Gullstrand coordinates

$$ds^2 = dT^2 - \left(dR + \sqrt{\frac{2M}{R}}dT\right)^2 - R^2\left(d\vartheta^2 + \sin^2 d\varphi^2\right) \qquad M > 0$$

- mass of a black hole

At space infinity $R \to \infty$:

$$ds^2 = dT^2 - 2\sqrt{\frac{2M}{R}}\,dTdR - dR^2 - R^2\left(d\vartheta^2 + \sin^2 d\varphi^2\right)$$

$$\boxed{M(T,R) = \frac{T^2}{8R}}$$

- the mass function

$$R = 2M(T,R)$$

- horizon equation

In Schwarzschild coordinates:

$$\boxed{ds^2 = \left(1 - \frac{T^2}{4R^2}\right)dT^2 - \frac{dR^2}{1 - \dfrac{T^2}{4R^2}} - 4R^2\left(d\vartheta^2 + \sin^2 d\varphi^2\right)}$$

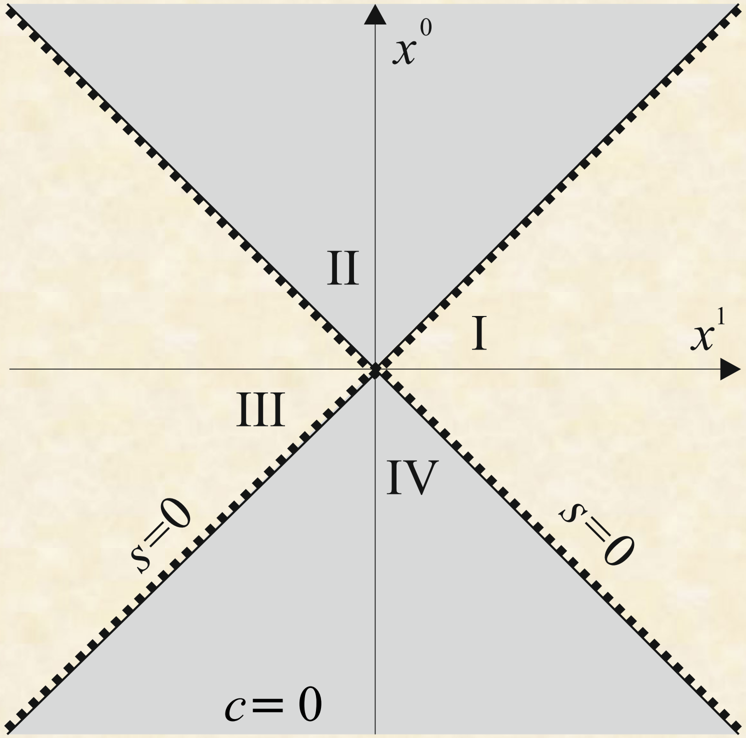
The potential unbounded from below

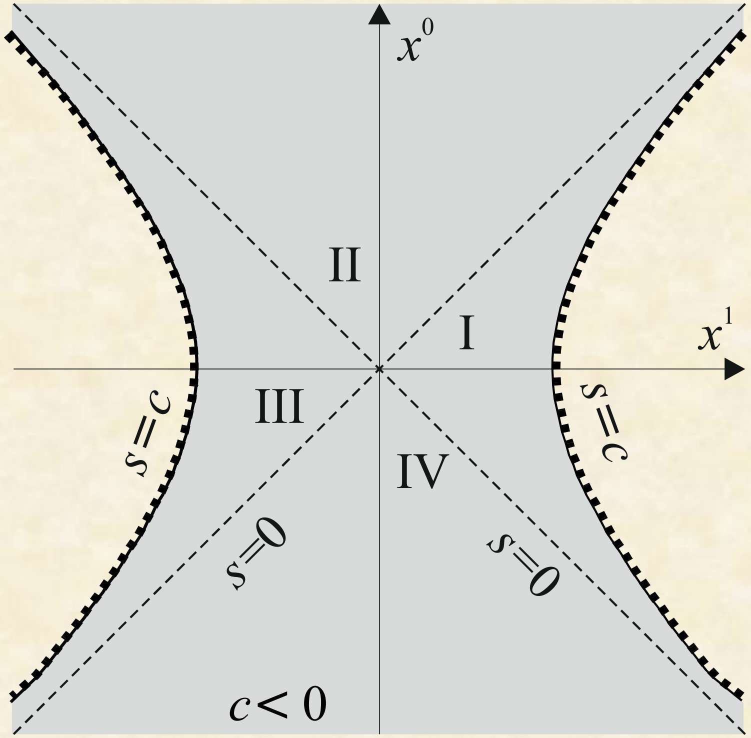
$$V + 2\Lambda = -12C^2 \mathrm{e}^{k\varphi}$$

A general solution

$$\Phi = c - s$$

$$\varphi = \pm\sqrt{3}\ln\left(C\Phi\right)$$





Solutions are defined for $\Phi > 0 \quad \Leftrightarrow \quad s < c$

The potential bounded from below

$$V+2\Lambda=12C^2\mathrm{e}^{k\varphi}$$

A general solution

$$\Phi=c+s$$

$$\varphi=\pm\sqrt{3}\ln\left(C\Phi\right)$$

Solutions are defined for $\Phi>0 \quad \Leftrightarrow \quad s>c$ Let $c<0$

The cosmological solution

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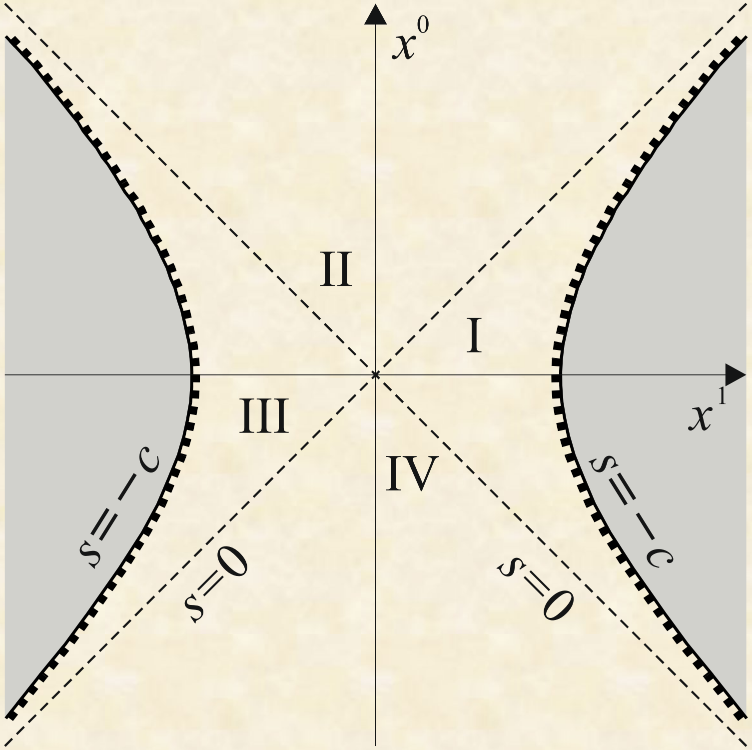

The naked singularity

The forbidden regions are shaded.

The picture must be rotated in two extra space dimensions around x^0 axis.

The solution is global: any geodesic can be either extended to infinite value of the canonical parameter in both directions or it ends up at the singular point.

Friedmann-like form of the Liouville metric

Pseudospherical coordinates in quadrant II (inside the future light cone):

$$x^0 := t\cosh\chi$$

$$x^1 := t\sinh\chi\sin\vartheta\cos\psi$$

$$x^2 := t\sinh\chi\sin\vartheta\sin\psi$$

$$x^3 := t\sinh\chi\cos\vartheta$$

$$\Rightarrow \quad s = t^2 > 0$$

$$ds^2 = (t^2 + c)\left(dt^2 - d\Omega\right)$$

$$d\Omega := d\chi^2 + \sinh^2\chi\,(d\vartheta^2 + \sin^2\vartheta\, d\psi^2)$$ - constant negative curvature metric

Next coordinate transformation $t \to \tau(t)$: $\dfrac{d\tau}{dt} = \sqrt{t^2 + c}$

$$\boxed{ds^2 = d\tau^2 - a^2(\tau)\, d\Omega}$$

$a(\tau) := t\sqrt{t^2 + c}$ - the scale factor

$$\frac{da}{d\tau} = \frac{2t^2 + c}{t^2 + c}$$

$$\frac{d^2 a}{d\tau^2} = \frac{2ct}{\left(t^2 + c\right)^{5/2}} > 0$$ - accelerated expansion

The expansion starts at the “horizon” $t = 0$ (future light cone), and there is no singularity in global coordinates. There are no Friedmann-like coordinates in quadrants I and III because lines $s = \text{const}$ are timelike.

Comparison with the perfect fluid stress-energy tensor

4-velocity: $u_\alpha = \frac{\partial_\alpha \varphi}{\sqrt{\partial\varphi^2}}$ $\qquad \partial\varphi^2 := g^{\alpha\beta}\partial_\alpha\varphi\,\partial_\beta\varphi > 0$

The 4-velocity can be defined only in quadrants II or IV.

$E = \frac{1}{2}\partial\varphi^2 + V = \frac{6(3\Phi - c)}{\Phi^3}$ - energy density

$P = \frac{1}{2}\partial\varphi^2 - V = -\frac{6(3\Phi + c)}{\Phi^3}$ - pressure

The weak energy condition holds: $E + P = \frac{12s}{\Phi^3} > 0$

The strong energy condition is violated: $E + 3P = -\frac{24c}{\Phi^3} < 0$

Conclusion

1) New global smooth solution in General Relativity with a scalar field is found. The scalar field has exponential potential unbounded from below.

2) The Liouville metric without a Killing vector is used as the initial ansatz. However the solution of Einstein's equations is invariant with respect to global Lorentz transformations and therefore has 6 noncommuting Killing vector fields (spontaneous symmetry emergence).

3) There are three global solutions depending on the integration constant. One solution describes the spherically symmetric black hole formation by the scalar field. The solution is global and infinitely smooth.

4) The horizon is dynamical: it is a sphere which arises with infinitesimal radius at a finite moment of time and afterwards expands with the velocity of light.

5) Metric for the naked singularity can be transformed into the Friedmann form only inside the light cone, where it describes accelerated expansion of the Universe. The corresponding global solution is infinitely smooth and geodesically complete except the naked singularity.