$\mathcal{R}$-rule and effective potential in subleading order

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based on works with D. Kazakov, A. Mukhaeva, D. Tolkachev and V. Filippov
arxiv:2602.11878, 2504.02418, 2209.08019

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JOINT INSTITUTE FOR NUCLEAR RESEARCH

Bogoliubov, Parasiuk'1957
Hepp'1966
Zimmerman'1967

$\mathcal{R}$'-operation and locality

$\mathcal{R}$-operation is a procedure of subtraction of all UV-divergences in the Feynman graph G

$$\mathcal{R} \circ G = (1 - \mathcal{K})\mathcal{R}' \circ G$$

$\mathcal{R}$'-operation is incomplete $\mathcal{R}$-operation, i.e. $\mathcal{R}$-operation without the last subtraction

Result of $\mathcal{R}$'-operation is always local (according to Bogoliubov-Parasiuk theorem)

$$\mathcal{R}'G_n = \frac{\mathcal{A}_n^{(n)}}{\epsilon^n}(\mu^2)^{n\epsilon} + \frac{\mathcal{A}_{n-1}^{(n)}}{\epsilon^n}(\mu^2)^{(n-1)\epsilon} + \dots \frac{\mathcal{A}_1^{(n)}}{\epsilon^n}\mu^{2\epsilon} +$$
$$+\frac{\mathcal{B}_n^{(n)}}{\epsilon^{n-1}}(\mu^2)^{n\epsilon} + \frac{\mathcal{B}_{n-1}^{(n)}}{\epsilon^{n-1}}(\mu^2)^{(n-1)\epsilon} + \dots \frac{\mathcal{B}_1^{(n)}}{\epsilon^{n-1}}\mu^{2\epsilon} +$$
$$+\frac{\mathcal{C}_n^{(n)}}{\epsilon^{n-2}}(\mu^2)^{n\epsilon} + \frac{\mathcal{C}_{n-1}^{(n)}}{\epsilon^{n-2}}(\mu^2)^{(n-1)\epsilon} + \dots \frac{\mathcal{C}_1^{(n)}}{\epsilon^{n-2}}\mu^{2\epsilon} +$$
$$+\text{lower pole terms like } \mathcal{D}_k^{(n)}, \mathcal{E}_k^{(n)} \text{ etc.}$$

Recurrence relations

Locality condition leads to relation between (sub...)leading poles of n-th and the first PT-order

$$\mathcal{A}_n^{(n)} = \frac{\mathcal{A}_1^{(n)}}{n};$$

$$\mathcal{B}_n^{(n)} = 2!\left(\frac{\mathcal{B}_1^{(n)}}{n} + \frac{\mathcal{B}_2^{(n)}}{n(n-1)}\right);$$

$$\mathcal{C}_n^{(n)} = 3!\left(\frac{\mathcal{C}_1^{(n)}}{2n} + \frac{\mathcal{C}_2^{(n)}}{2n(n-1)} + \frac{\mathcal{C}_3^{(n)}}{n(n-1)(n-2)}\right);$$

$$\dots$$

...and to relation between (sub...)leading poles and logarithms

$$\bar{\mathcal{A}}_n^{(n)} \equiv \mathcal{A}_n^{(n)};$$

$$\bar{\mathcal{B}}_n^{(n)} = (-1)^n\left(\mathcal{B}_1^{(n)} + \frac{2}{(n-1)}\mathcal{B}_2^{(n)}\right);$$

$$\bar{\mathcal{C}}_n^{(n)} = (-1)^{n-1}\left(\frac{n-1}{2}\mathcal{C}_1^{(n)} + 2\mathcal{C}_2^{(n)} + \frac{3}{n-2}\mathcal{C}_3^{(n)}\right);$$

$$\dots$$

These recurrence relations can be used in any ways to construct RG-like equations in any local quantum field theory whether renormalizable it is or not

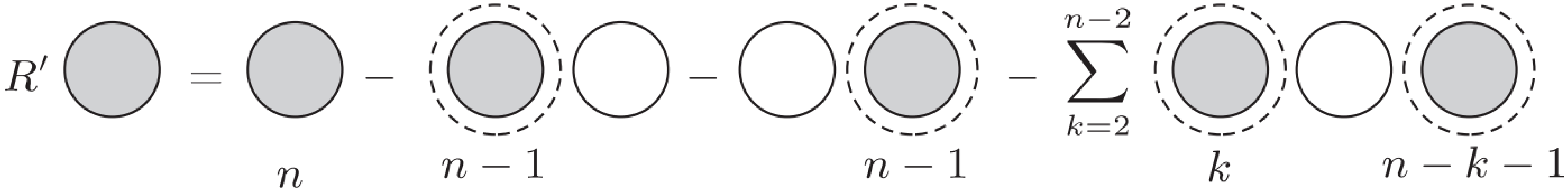
How to use?

- Write down UV-divergent Feynman diagrams for some quantity (amplitude, effective potential, effective action etc)
- Evaluate integrals
- Find relation between orders with respect to general recurrence relations

Our playground are effective potentials

Effective potential (EP) is a part of effective action which do not depend on momenta (derivatives)

Typical example of use $\mathcal{R}$'-operation to obtain leading order of PT series



For renormalizable models the task sometimes simple but one need to calculate at least up to third loops. For the non-renormalizable quantum field theory the same is possible one needs to reconstruct operators generating new PT counter-terms

For the subleading order one needs 5 loops to find specific recurrence relations!

$\mathcal{R}$-rule

- $\mathcal{R}$'-operation form in the given order repeats the structure of divergent diagrams in the same order (rhs of recurrence relations has to have the structure of loop graphs)

- Recurrence equations except ones for the highest divergences should be linear in their own order, i.e. all divergent and logarithmic contributions should be linear

Effective potential quantum field theory: setup

Generating functional

$$Z(J) = \int \mathcal{D}\phi \ \exp\left(i \int d^D x \ \mathcal{L}(\phi, d\phi) + J\phi \right)$$

1PI generating functional
and effective action

$$W(J) = -i \log Z(J), \ \Gamma(\phi) = W(J) - \int d^D x J(x)\phi(x)$$

Effective potential

$$\Gamma[\phi] \ = \ \int d^4 x \left\{ - \ V_{eff}(\phi) + \frac{1}{2} Z(\phi) \partial^\mu \phi \partial^\mu \phi + ... \right\}.$$

Shifted action

$$e^{i\Gamma[\hat{\phi}]} = \int D\phi e^{iS[\phi+\hat{\phi}]-i\hat{\phi}S'[\phi]} \qquad S[\phi+\hat{\phi}] = S[\phi] + \hat{\phi}^2 S''[\phi] + \dots$$

Formal expansion of
effective potential

$$V_{eff} = \sum_{k=0}^{\infty} (-\lambda)^k V_k$$

Effective potential in logarithmic expansion

2209.08019

2602.11878

$$V_{eff} = \begin{pmatrix} V_0 & 0 & \cdots \\ a_1\lambda^3 L & b_1\lambda^4 & \cdots \\ a_2\lambda^4 L^2 & b_2\lambda^5 L & \cdot\cdot \\ \cdots & \cdots & \cdots \\ a_n\lambda^{2n} L^n & b_n\lambda^{2n+1} L^{n-1} & \cdots \end{pmatrix}$$

Tree-level

One-loop level

Two-loop level

...

...

...

n-loop level

LLA

NNLA

...

Leading EP in arbitrary scalar field theory

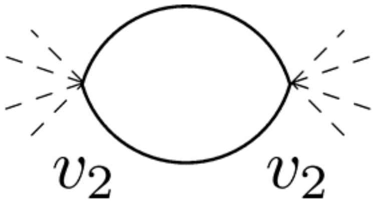
Let us consider the model in $d = 4 - 2\epsilon$

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)^2 - \lambda V_0(\phi)$$

Quantum correction to effective potential can be represented as formal sum

$$V_{eff} = \lambda \sum_{n=0}^{\infty} (-\lambda)^n V_n \qquad v_n = \frac{\partial^n V_0}{\partial \phi^n}$$

One-loop singular contribution to effective potential can be written formally:



$$\Delta V_1^A = -\frac{v_2^2}{4\epsilon}$$

$$D_k = \frac{\partial^k}{\partial \phi^k}$$

Recurrence relation

$$n\Delta V_n^A = -\frac{1}{2} v_2 D_2 V_{n-1}^A + \frac{1}{4} \sum_{k=2}^{n-1} D_2 \Delta V_k^A D_2 \Delta V_{n-k-1}^A$$

Can be rewritten

$$n\Delta V_n^A = -\Delta A_1 \sum_{k=1}^{n-1} D_2 \Delta V_k^A D_2 \Delta V_{n-k-1}^A$$

$$\Sigma_A(z, \phi) = \sum_{n=0}^{\infty} (-z)^n \Delta V_n^A$$

$$z = \lambda/\epsilon$$
$$z \to \lambda \log(\mu^2/m_1^2)$$
$$m_1 = \lambda v_2$$

RG-equation

$$\frac{\partial}{\partial z} \Sigma_A = -\frac{1}{4} \left(D_2 \Sigma_A\right)^2, \ \Sigma_A(0, \phi) = V_0$$

Leading EP in arbitrary scalar field theory

Recurrence relation

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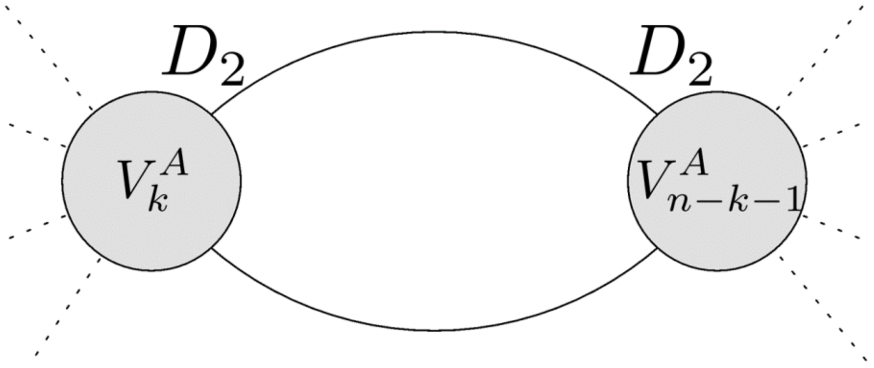
General form of recurrence relation for leading divergencies



Approved for the SO(N)-symmetric models, non-minimal interaction with curvature etc.
arxiv:2504.02418, 2512.16613

Renormalizable case:

$$\Sigma_A(z,\phi) = \frac{\phi^4}{4!} f_A(z) \qquad f_A' = -\frac{3}{2} f_A^2 \qquad f(z) = \frac{1}{1+\frac{3}{2}z}$$

Subleading EP in arbitrary scalar field theory

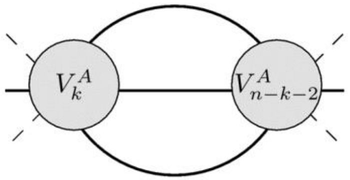
n-bubble topology | leading | subleading

$$\left(\frac{a}{\epsilon}+b+\ldots\right)^n = \frac{a^n}{\epsilon^n} + \frac{n\ a^{n-1}\boxed{b}}{\epsilon^{n-1}} + ..$$

$$D_k = \frac{\partial^k}{\partial \phi^k}$$

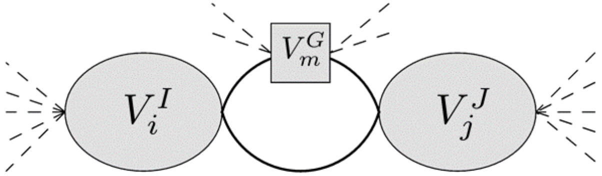
correction to propagator

V_n^G

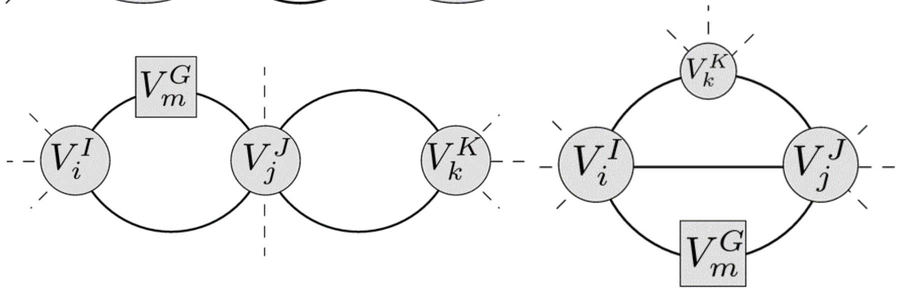


1-loop and 2-loop NNLA corrections

$V_n^{B_1}$

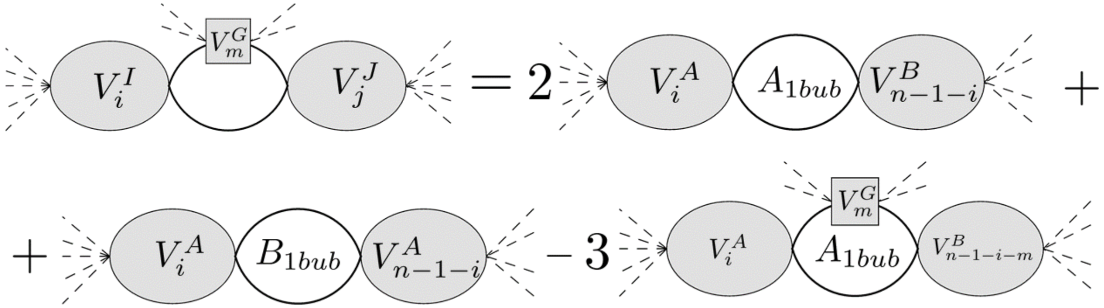


$V_n^{B_2}$



$$\{I,J,K\} = \{A,B\}$$

One-loop contribution



$$\sum_{k=0}^{n-1}\left[(B_{bub})D_2V_k^A D_2V_{n-k-1}^A + 2(A_{1bub})D_2V_k^B D_2V_{n-k-1}^A\right]$$

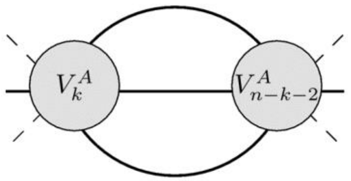
$$-3(A_{1bub})\sum_{l,k=0}^{l+k<n-2} D_2V_l^A D_2V_k^A D_2V_{2,n-l-k-1}^G$$

Subleading EP in arbitrary scalar field theory

$$D_k = \frac{\partial^k}{\partial \phi^k}$$

correction to propagator

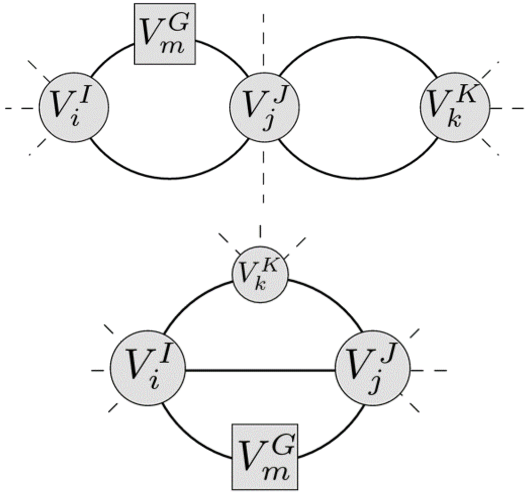
V_n^G



$$(G_{2sun}) \sum_{k=0}^{n-2} D_3 V_k^A D_3 V_{n-k-2}^A$$

1-loop and 2-loop NNLA corrections

$V_n^{B_2}$



$$\{I, J, K\} = \{A, B\}$$

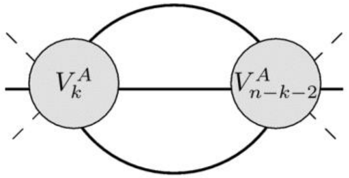
$$- \sum_{l,k=0}^{l+k<n-1} [(B_{2bub}) D_2 V_l^A D_2 V_k^A D_4 V_{n-l-k-2}^A$$
$$+ (A_{2bub}) D_2 V_l^A D_2 V_k^A D_4 V_{n-l-k-2}^B + 2(A_{2bub}) D_2 V_l^A D_2 V_k^B D_4 V_{n-l-k-2}^A]$$
$$+ \sum_{l,k,m=0}^{l+k+m<n-3} 3(A_{2bub}) D_2 V_l^A D_2 V_k^A D_4 V_m^A D_2 V_{2,n-l-k-m-2}^G$$
$$- \sum_{l,k=0}^{l+k<n-1} [(A_{2sun}) D_3 V_l^A D_3 V_k^A D_2 V_{n-l-k-2}^B + 2(A_{2sun}) D_3 V_l^A D_3 V_k^B D_2 V_{n-l-k-2}^A$$
$$+ (B_{2sun}) D_3 V_l^A D_3 V_k^A D_2 V_{n-l-k-2}^A$$
$$+ \sum_{l,k,m=0}^{l+k+m<n-3} 2(A_{2sun}) D_3 V_l^A D_3 V_k^A D_2 V_m^A D_2 V_{2,n-l-k-m-2}^G$$

Subleading EP in arbitrary scalar field theory

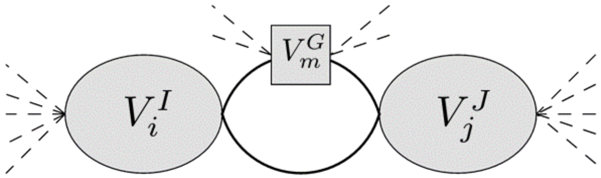
$$\Sigma_G = \sum_n V_n^G (-z)^n, \Sigma_B = \sum_n V_n^B (-z)^n$$

$$\mathcal{B}_n^{(n)} = 2! \left(\frac{\mathcal{B}_1^{(n)}}{n} + \frac{\mathcal{B}_2^{(n)}}{n(n-1)} \right); \qquad D_k = \frac{\partial^k}{\partial \phi^k}$$

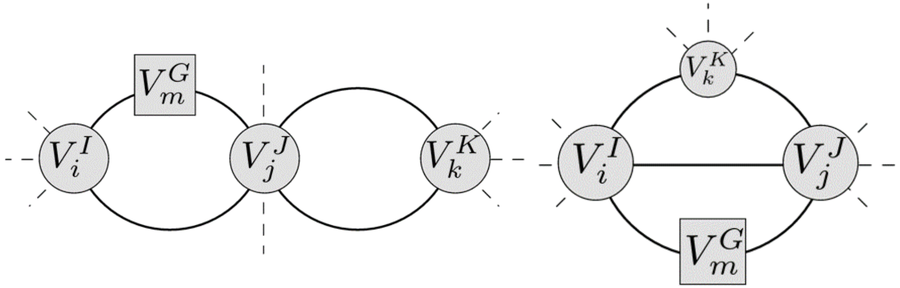
V_n^G



$V_n^{B_1}$



$V_n^{B_2}$



$\{I, J, K\} = \{A, B\}$

Equations summing all subleading contributions in effective potential

$$\frac{\partial^2 \Sigma_B}{\partial z^2} = -2 \frac{1}{4} \frac{\partial}{\partial z} \left\{ 2 D_2 \Sigma_A D_2 \Sigma_B - (D_2 \Sigma_A)^2 D_2 \Sigma_G \right\} -$$
$$-2 \left\{ \frac{1}{8} (D_3 \Sigma_A)^2 D_2 \Sigma_B - 2 \frac{1}{8} (D_3 \Sigma_A)^2 D_2 \Sigma_A D_2 \Sigma_B + 2 \frac{1}{8} D_3 \Sigma_A D_3 \Sigma_B D_2 \Sigma_A + \right.$$
$$+ 2 \frac{1}{8} D_3 \Sigma_A D_3 \Sigma_B D_2 \Sigma_A - \frac{1}{8} (D_3 \Sigma_A)^2 D_2 \Sigma_A + \frac{1}{8} (D_2 \Sigma_A)^2 D_4 \Sigma_B -$$
$$\left. -2 \frac{1}{8} (D_2 \Sigma_A)^2 D_4 \Sigma_A D_2 \Sigma_G + 2 \frac{1}{8} D_2 \Sigma_A D_2 \Sigma_B D_4 \Sigma_A \right\}$$

contributions with the propagators in effective potential

$$\frac{\partial^2 \Sigma_G}{\partial z^2} = \frac{1}{12} (D_3 \Sigma_A)^2$$

Scheme dependence

Transformation to nonminimal subtraction scheme

$$z \to z(1 + \epsilon\ c_1)$$

... leads to extra term in solution

$$\Sigma_B^{c_1}(z,\phi) = \Sigma_B^{\mathrm{MS}} + c_1 \Delta\Sigma_B$$

Equation for the additional part

$$\frac{\partial^2}{\partial z^2}\Delta\Sigma_B = (D_2\Sigma_A)^2\left(-D_4\Delta\Sigma_B\right) - 2D_3\Sigma_A D_2\Sigma_A D_3\Delta\Sigma_B - 2D_4\Sigma_A D_2\Sigma_A - \\ - 4D_2\Sigma_A D_2\frac{\partial}{\partial z}\Delta\Sigma_B - \left((D_3\Sigma_A)^2 + 4D_2\frac{\partial}{\partial z}\Sigma_A\right)D_2\Delta\Sigma_B$$

can be simplified to

$$\Delta\Sigma_B(z,\phi) = \epsilon z \frac{\partial}{\partial z}\Sigma_A(z,\phi)$$

The whole result can be written

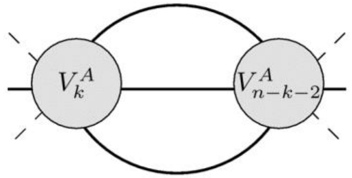
$$\Sigma(z,\phi) = \Sigma_A(z,\phi) + \epsilon z \Sigma_B(z,\phi) + \epsilon z c_1 \frac{\partial}{\partial z}\Sigma_A(z,\phi)$$

without any particular integrals of complicated PDE we can track the one-loop subtraction scheme coefficient!

Subleading EP in renormalizable scalar theory

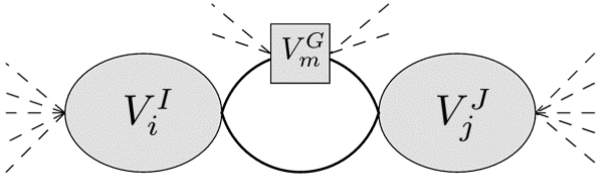
correction to propagator

V_n^G

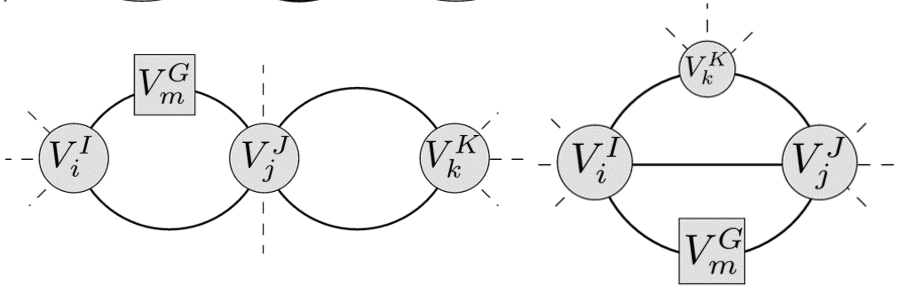


1-loop and 2-loop NNLA corrections

$V_n^{B_1}$



$V_n^{B_2}$



$$\{I, J, K\} = \{A, B\}$$

$$\Sigma_A = \frac{\phi^4}{4!} f_A(y), \Sigma_B = \frac{\phi^4}{4!} f_B(y), \Sigma_G = \frac{\phi^2}{2} f_G(y)$$

Equations summing all subleading contributions in effective potential

$$f_B'' + 6 f_A f_B' + \frac{3}{2} f_B \left(4 f_A' + 9 f_A^2\right) = 3 f_A^3 \left(1 + 4 \frac{f_A'}{f_A^2} f_G + 2 \frac{f_G'}{f_A} + 6 f_G\right)$$

$$f_G'' = \frac{1}{12} f_A^2$$

Equation linking subleading logarithms with subleading poles

$$\frac{d}{dl} \bar{f}_B(l) = \frac{1}{8} \frac{d}{dl} \left(f_A \left(f_G - f_B\right)\right) + 2 f_A^2 \left(-\frac{1}{16} f_A + \frac{9}{32} f_B - \frac{3}{8} f_G\right)$$

$$l = \lambda \log(\mu^2 / m_1^2)$$

These equations can be easily integrated!
Let us check solutions!

Subleading EP in renormalizable scalar theory

Solution for the subleading poles contribution to effective potential

$$f_B(y) = \frac{2\left(3y(3y-32)+32(2+3y)\log\left((1+\frac{3}{2}y)\right)\right)}{27(2+3y)^2}$$

Solution for the poles contribution to propagator

$$f_G(y) = \frac{1}{54}\left(3y - 2\log\left(1+\frac{3y}{2}\right)\right)$$

Kazakov,Vladimirov,Tarasov'1979
Kastening'1992 etc

Solution for the subleading logarithms in
minimal subtraction scheme

$$\bar{f}_B(l) = \frac{6l + 68\log\left(1+\frac{3l}{2}\right)}{9(1+\frac{3}{2}l)^2}$$

We can reproduce beta-function!

$$\beta(\lambda) = \frac{3}{2}\lambda^2 - \frac{17}{6}\lambda^3$$

Even here results are consistent with results of Ovsyannikov-Callan-Symanzik-equations!

Conclusions and prospectives

- There is hidden structure in quantum field theory that exactly can be recovered with the help of locality directly from Feynman diagrams
- Useful tool for that is the $\mathcal{R}$-rule which allow to build analogous of Ovsyannikov-Callan-Symanzik equation without using multiplicativity of renormalization
- Equations for EP in any local quantum field theory can be obtained using $\mathcal{R}$-rule easily. They reproduce known perturbative results for simple models.
- Scheme dependence of one-loop subtraction can be extracted and controlled by the LLA equation

- Consequences for usual RG-equations in renormalizable models (QED,QCD: 2604.04766)
- Cosmological applications? (in work)
- Anything on amplitudes or EA? Nonlinear sigma-models?
- Can the R-rule help to analyze scheme-dependence in non-renormalizable models?

Thank you for your attention!