

Field-Theoretic Renormalization Group in Models of Self-Organized Criticality, Growth Processes and Surface Roughening

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Isotropic stochastic models of non-equilibrium critical behavior

Kardar–Parisi–Zhang model

$$\partial_t h = \varkappa_0 \partial^2 h + (\lambda_0/2)(\partial h)^2 + \eta,$$

Pavlik model

$$\partial_t h = \varkappa_0 \partial^2 h + (\lambda_0/2) \partial^2 h^2 + \eta.$$

Here η is random noise,

$$h = h(x), \quad x = \{t, \mathbf{x}\}, \quad \mathbf{x} = \{x_i\}, \quad i = 1, \dots, d,$$

$$\partial_t = \partial/\partial t, \quad \partial_i = \partial/\partial x_i, \quad \partial^2 = \partial_i \partial_i.$$

Anisotropic stochastic models of non-equilibrium critical behavior

Hwa–Kardar model

$$\partial_t h = \varkappa_{\perp 0} \partial_{\perp}^2 h + \varkappa_{\parallel 0} \partial_{\parallel}^2 h - (\lambda_0/2) \partial_{\parallel} h^2 + \eta,$$

Pastor-Satorras–Rothman model

$$\partial_t h = \varkappa_{\perp 0} \partial_{\perp}^2 h + \varkappa_{\parallel 0} \partial_{\parallel}^2 h + (\lambda_0/3!) \partial_{\parallel}^2 h^3 + \eta.$$

Here \mathbf{n} defines a certain distinguished direction:

$$\mathbf{x} = \mathbf{x}_{\perp} + \mathbf{n}x_{\parallel}, \quad x_{\parallel} = \mathbf{n}\mathbf{x}, \quad \mathbf{n}\mathbf{x}_{\perp} = 0.$$

Random Gaussian noise with pair correlator

Ordinary white noise:

$$\langle \eta(\mathbf{x})\eta(\mathbf{x}') \rangle = C_0 \delta(t - t') \delta^{(d)}(\mathbf{x} - \mathbf{x}'), \quad C_0 > 0.$$

Conservation case:

$$\langle \eta(\mathbf{x})\eta(\mathbf{x}') \rangle = -C_0 \delta(t - t') \partial^2 \delta^{(d)}(\mathbf{x} - \mathbf{x}'),$$

Static noise:

$$\langle \eta(\mathbf{x})\eta(\mathbf{x}') \rangle = C_0 \delta^{(d)}(\mathbf{x} - \mathbf{x}'),$$

Quenched noise:

$$\langle \eta(\mathbf{x})\eta(\mathbf{x}') \rangle = \Delta(h - h') \delta^{(d)}(\mathbf{x} - \mathbf{x}'), \quad \text{e.g.} \quad \Delta(h - h') = \delta(h - h').$$

Moving medium or interaction with environment

Lagrangean (Galilean covariant) derivative:

$$\partial_t h \rightarrow \nabla_t h = \partial_t + v_i \partial_i h$$

with a certain velocity field $v_i(x) = v_i(t, \mathbf{x})$.

Kazantsev–Kraichnan Gaussian ensemble:

$$\langle v_i(x) v_j(x') \rangle = D_0 \delta(t - t') \int \frac{d\mathbf{k}}{(2\pi)^d} \frac{1}{k^{d+\xi}} \left\{ P_{ij}^\perp(\mathbf{k}) + \alpha P_{ij}^\parallel(\mathbf{k}) \right\} \exp\{i\mathbf{k}(x - x')\}.$$

Kolmogorov scaling: $\xi = 4/3$, Batchelor's limit: $\xi \rightarrow 2$.

Stirred Navier–Stokes equation for incompressible (e.g.) fluid:

$$\nabla_t v_i = \nu_0 \partial^2 v_i - \partial_i \wp + f_i,$$

pressure $\wp(v)$ makes the dynamics consistent with incompressibility condition $\partial_i v_i = 0$.

The correlator of the Gaussian stirring force:

$$\langle f_i(\mathbf{x}) f_j(\mathbf{x}') \rangle = \delta(t - t') \int \frac{d\mathbf{k}}{(2\pi)^d} D(k) P_{ij}^\perp(\mathbf{k}) \exp\{i\mathbf{k}(\mathbf{x} - \mathbf{x}')\},$$

where

$$D(k) = D_0 k^2 \quad \text{or} \quad D(k) = D_0 \quad \text{or} \quad D(k) = D_0 k^{4-d-y}.$$

Note that

$$\lim_{y \rightarrow 4} D_0 k^{4-d-y} \sim W \delta^{(d)}(\mathbf{k}) : \quad \text{fully developed turbulence.}$$

Quantities of interest and expected IR behaviour

Structure functions, isotropic cases:

$$\mathcal{S}_n(t, r) = \langle [h(t, \mathbf{x}) - h(0, 0)]^n \rangle \simeq r^{-n\Delta_h} F\left(t r^{-\Delta_\omega}\right), \quad r = |\mathbf{x}|,$$

argument of F is dimensionless.

Anisotropic case:

$$\mathcal{S}_n(t, r_\perp, r_\parallel) \simeq r_\perp^{-n\Delta_h} F\left(t r_\perp^{-\Delta_\omega}, r_\parallel r_\perp^{-\Delta_\parallel}\right), \quad r_\perp = |\mathbf{x}_\perp|, \quad r_\parallel = |x_\parallel|,$$

with $\Delta_\parallel \neq 1$ in general case.

Action functional: General rules

Theorem: any stochastic equation of the type

$$\partial_t \phi(x) = U(x, \phi) + f(x), \quad \langle f(x)f(x') \rangle = D(x, x'),$$

where $\phi(x) = \phi(t, \mathbf{x})$ is a random field, $U(x, \phi)$ is a t -local functional depending on the fields and their derivatives, $f(x)$ is a random force, **is equivalent to quantum field model** of the double set of fields $\tilde{\phi} = \{\phi, \phi'\}$ and action functional

$$S[\varphi] = \underbrace{\frac{1}{2} \varphi' D \varphi'}_{\text{noise term}} + \varphi' \underbrace{[-\partial_t \varphi + U]}_{\text{dynamics}},$$

integration over t and \mathbf{x} implied.

Action functional: General rules

What does it mean:

- ▶ statistical average is equivalent to functional integration with weight $\exp S[\phi]$;
- ▶ classical random field \rightarrow quantum field;
- ▶ we may use all techniques from quantum field theory: Feynman graphs, renormalization group, operator product expansion, *etc.*

RG: General scheme

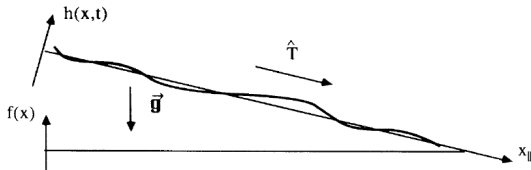
The main steps (general scheme) are following:

- ▶ stochastic formulation of the model;
- ▶ quantum field action and Feynman diagrams;
- ▶ divergences of the diagrams;
- ▶ renormalization, RG, RG flow and fixed points;
- ▶ critical dimensions at different fixed points.

Different features may include:

- ▶ appearing of counterterms;
- ▶ logarithmic (not power) corrections to ordinary scaling;
- ▶ line (or surface) of fixed points;
- ▶ node and spiral types of attractors;
- ▶ node and spiral types of attractors;
- ▶ different numbers of scaling equations.

Hwa-Kardar model



T. Hwa and M. Kardar, Phys. Rev. Let., Vol.62, 16 (1989).

Stochastic equation (f is a random force)

$$\partial_t h(t, x) = \nu_{||} \partial_{||}^2 h(t, x) + \nu_{\perp} \partial_{\perp}^2 h(t, x) - \frac{1}{2} \partial_{||} h^2(t, x) + f(t, x).$$

Anisotropy of the system

$$\mathbf{x} = \mathbf{x}_{\perp} + \mathbf{n}x_{||}, \quad \mathbf{n} \perp \mathbf{x}_{\perp} \quad (\hat{T} \equiv \mathbf{n}).$$

Hwa-Kardar model: type of noise and turbulent mixing

Thermal noise

$$\langle f(x)f(x') \rangle = D_0 \delta(t - t') \delta^{(d)}(\mathbf{x} - \mathbf{x}'), \quad D_0 > 0;$$

rapid correlations in both space and time.

Introducing of velocity field (isotropic one):

$$\langle v_i(t, \mathbf{x}) v_j(t', \mathbf{x}') \rangle = \delta(t - t') D_{ij}(\mathbf{x} - \mathbf{x}'),$$

$$D_{ij}(\mathbf{r}) = B_0 \int_{k>m} \frac{d\mathbf{k}}{(2\pi)^d} \frac{1}{k^{d+\xi}} P_{ij}(\mathbf{k}) \exp\{i\mathbf{k} \cdot (\mathbf{x} - \mathbf{x}')\}.$$

Actions functional

Quantum field action:

$$S(\Phi) = \frac{1}{2} h' D_0 h' + h' (-\nabla_t h + \nu_{\parallel 0} \partial_{\parallel}^2 h + \nu_{\perp 0} \partial_{\perp}^2 h - \frac{1}{2} \partial_{\parallel} h^2) + S_v.$$

All integrations are implied:

$$h' D_0 h' \equiv D_0 \int dt \int d^d x h'(t, x) h'(t, x).$$

Feynman rules and divergent functions

According to general rules both models contain three propagators

$$\langle hh' \rangle_0 = \longrightarrow, \quad \langle hh \rangle_0 = \text{---}, \quad \langle vv \rangle_0 = \text{~~~~}$$

and two vertices

$$-\frac{1}{2}h'\partial_{\parallel}h^2 = h^2\partial_{\parallel}h' = \text{---}\bullet\text{---}; \quad -h'(v\partial_{\parallel})h = h(v\partial_{\parallel})h' = \text{---}\bullet\text{~~~~}$$

The propagators are

$$\langle hh' \rangle_0 = \frac{1}{-i\omega + \epsilon(k)}, \quad \langle hh \rangle_0 = \frac{D_0}{\omega^2 + \epsilon^2(k)}; \quad \epsilon(k) \equiv \nu_{\parallel_0}k_{\parallel}^2 + \nu_{\perp_0}k_{\perp}^2.$$

Logarithmic dimension is $d = 4$ and the only divergent Green function is $\langle h'h \rangle$.

Dimensions and scales

The key difference between anisotropic velocity ensemble

$$\mathbf{v}(t, \mathbf{x}) = \mathbf{n}v(t, x_{\perp})$$

and isotropic one (under consideration)

$$\langle v_i(t, \mathbf{x}) v_j(t', \mathbf{x}') \rangle = \delta(t - t') D_{ij}(\mathbf{x} - \mathbf{x}')$$

is possibility/impossibility to introduce two independent momentum scales.

In first case any quantity F is described by three canonical dimensions

$$[F] \sim [T]^{-d_F^{\omega}} [L_{\parallel}]^{-d_F^{\parallel}} [L_{\perp}]^{-d_F^{\perp}},$$

in second case only by two:

$$[F] \sim [T]^{-d_F^{\omega}} [L]^{-d_F^k}.$$

Renormalization constants Z and β -functions

The renormalization constants Z are

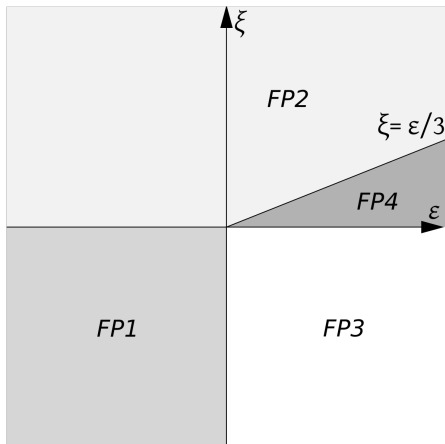
$$Z_{\nu_{\parallel}} = 1 - \frac{3}{8u} \frac{x}{\xi} - \frac{3}{16} \frac{g}{\varepsilon},$$
$$Z_{\nu_{\perp}} = 1 - \frac{3x}{8\xi}, \quad \text{where } x = uw.$$

The β -functions are

$$\beta_g = g \left(-\varepsilon + \frac{9}{32}g + \frac{9}{16} \frac{x}{u} + \frac{9}{16}x \right),$$
$$\beta_x = x \left(-\xi + \frac{3}{8}x \right),$$
$$\beta_u = u \left(-\frac{3}{16}g - \frac{3x}{8u} + \frac{3}{8}x \right).$$

Fixed points and scaling regimes

Four fixed points depending on the parameters ξ and η



IR scaling behaviour

Canonical scale invariance:

$$\left(\sum_i d_i^\omega \mathcal{D}_i - d_G^\omega \right) G = 0,$$
$$\left(\sum_i d_i^k \mathcal{D}_i - d_G^k \right) G = 0.$$

The differential RG equation:

$$(\mathcal{D}_\mu + \beta_g \partial_g + \beta_x \partial_x + \beta_u \partial_u - \gamma_{\nu_\perp} \mathcal{D}_{\nu_\perp} - \gamma_G) G = 0.$$

At the fixed points $\beta_i = 0$ for all $i = g, x, u$.

Critical dimensions: Fixed point FP2

The equation of critical scaling is a combination of equations of canonical scaling with RG equation taken at fixed point

$$\left(\mathcal{D}_{k_{\parallel}} + \mathcal{D}_{k_{\perp}} + \Delta_{\omega} \mathcal{D}_{\omega} - d_G^k - \Delta_{\omega} d_G^{\omega} - \gamma_G^* \right) G^R = 0,$$

where $\Delta_{\omega} = 2 - \gamma_{\nu_{\perp}}^*$. The critical dimension Δ_F of a quantity F reads

$$\Delta_F = d_F^k + \Delta_{\omega} d_F^{\omega} + \gamma_F^*.$$

Usually this step is very straightforward, and for the fixed point FP2 they are

$$\Delta_h = 1 - \xi, \quad \Delta_v = 1 - \xi, \quad \Delta_{h'} = 3 - \varepsilon + \xi, \quad \Delta_{\omega} = 2 - \xi.$$

Restoration/Emergence of anisotropic scaling: point FP3

The points FP3 and FP4 has the coordinate $\alpha^* = 1/u^* = 0$. Direct substitution of the coordinates g^* , x^* , α^* into $\gamma_{\nu\perp}$ gives us trivial answer: $\gamma_{\nu\perp}^* = 0$ and, consequently, $\Delta_F = d_F^k + 2d_F^\omega$ are simply canonical dimensions.

To obtain nontrivial corections we should expand β_α !

$$\left(\mathcal{D}_{k_{\parallel}} + \mathcal{D}_{k_{\perp}} + \Delta_\omega \mathcal{D}_\omega - \lambda^* \mathcal{D}_\alpha - d_G^k - \Delta_\omega d_G^\omega - \gamma_G^* \right) G^R = 0,$$

where $\lambda = \partial\beta_\alpha/\partial\alpha$ at $\alpha = 0$ and λ^* denotes $\lambda(g^*, x^*)$.

This trick allows us to determine nontrivial one-loop corrections to canonical dimensions at fixed points FP3 and FP4 and, moreover, reproduce well-known one-loop answers for pure Hwa-kardar model [Phys. Rev. Lett. 62, 1813 (1989); Phys. Rev. A 45, 7002 (1992)], which corresponds in our terminology to fixed point FP3.

The IR scaling equation then becomes

$$\left(-\mathcal{D}_{r_{\parallel}} - \mathcal{D}_{r_{\perp}} - \Delta_{\omega} \mathcal{D}_t - \lambda_{\alpha} \mathcal{D}_{\alpha} - d_G^k - \Delta_{\omega} d_G^{\omega} - \gamma_G^*\right) G = 0,$$

and for FP3 this gives

$$\left(-\mathcal{D}_{r_{\parallel}} - \mathcal{D}_{r_{\perp}} - 2\mathcal{D}_t - \frac{2\varepsilon}{3} \mathcal{D}_{\alpha} - d_G^k - 2d_G^{\omega} - \gamma_G^*\right) G = 0.$$

We recall two canonical scale equations:

$$\left(\sum_i d_i^{\omega} \mathcal{D}_i - d_G^{\omega}\right) G = 0,$$

$$\left(\sum_i d_i^k \mathcal{D}_i - d_G^k\right) G = 0.$$

Solutions to set of these three equations are

$$z_1 = \frac{r_\perp^2}{t\nu_\perp}, \quad z_2 = \frac{r_\perp}{r_\parallel} \quad \text{and} \quad z_3 = \alpha (\mu r_\perp)^{-2\varepsilon/3}.$$

To obey canonical symmetries of pure HK, choose two of them: z_1 and $z_0 = z_2 z_3^{-1/2}$, which are dimensionless for both \parallel and \perp dimensions:

$$z_0 = \frac{r_\perp^{\Delta_\parallel}}{r_\parallel} \mu^{\varepsilon/3} \alpha^{-1/2} \quad \text{with} \quad \Delta_\parallel = 1 + \varepsilon/3.$$

For fixed IR irrelevant parameters μ and α we arrive at strongly anisotropic scaling:

$$G(t, r_\perp, r_\parallel) \simeq r_\perp^{-\Delta_G} F\left(t r_\perp^{-\Delta_\omega}, r_\parallel r_\perp^{-\Delta_\parallel}\right), \quad r_\perp = |\mathbf{x}_\perp|, \quad r_\parallel = |x_\parallel|,$$

with $\Delta_\omega = 2$ for FP3.

Conclusion

We considered some model of non-equilibrium critical behavior, applied methods of quantum field theory to it.

- ▶ The key point is the possibility to reformulate initial stochastic problem into some quantum field theory.
- ▶ Feynman graphs are divergent. Renormalization group allows us to work with these objects.
- ▶ Critical dimensions of measurable quantities are connected with fixed points of RG functions. This means, that RG allows us to calculate critical dimensions.

Some review paper: N. V. Antonov *et al.*, *Symmetry* 15, 1556 (2023).

Thank you for your attention!