# Generalized Models for Spinning Field Lumps on Plane

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# Shrödinger group

We recall free Schrödinger equation in arbitrary space dimensions

$$i\frac{\partial}{\partial t}\psi = -\frac{\nabla^2}{2m}\psi. \tag{1}$$

It is well known that this equation is invariant under space-time transformations of Schrödinger group. In 2 space dimensions, it takes the form

| Schrödinger group | | |
|---|---|---|
| Subgroup | Transformations | Infinitesimal generators $G$ |
| Time Translation | $t^{'} = t + \beta$ | $\frac{\partial}{\partial t}$ |
| Space Translation | $x^{i'} = x^i + a^i$ | $\frac{\partial}{\partial x^i}$ |
| Rotation | $x^{i'} = L^{ij}x^j, \quad L \in SO(2)$ | $x^i\frac{\partial}{\partial x^j} - x^j\frac{\partial}{\partial x^i}$ |
| Galilean boost | $x^{i'} = x^i + v^i \cdot t$ | $t\frac{\partial}{\partial x^i} - imx^i$ |
| Dilatation | $t^{'} = e^{2\sigma}t,\ x^{i'} = e^{\sigma}x^i$ | $2t\frac{\partial}{\partial t} + x^i\frac{\partial}{\partial x^i} + \frac{1}{2}$ |
| Special conformal symmetry | $t^{'} = \frac{t}{1+\eta t},\ x^{i'} = \frac{x^i}{1+\eta t}$ | $\frac{i(x^i)^2}{2} - \frac{t}{2} - x^i t\frac{\partial}{\partial x^i} - t^2\frac{\partial}{\partial t}$ |

Unbroken dilatation and conformal symmetry are present in a model with potential term $|\psi|^{2n}$ that satisfies relation[1]

$$nd = d + 2, \tag{2}$$

where d is the number of space dimensions.

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[1] M. O. deKok and J.W. van Holten, Nucl. Phys. B 803 (2008), arXiv: 0712.3686 [hep-th].

# $(2+1)$-dimensional theory

In case of $d = 2$ space dimensions, non-relativistic theory with Shrödinger invariance is written as

$$\mathcal{L}_{NR} = i\psi^* \dot{\psi} - \frac{1}{2m} \nabla \psi^* \nabla \psi + \frac{\lambda}{8m^2} \left(\psi^* \psi\right)^2 . \tag{3}$$

The corresponding equation of motion supports conformal Q-tube solutions of the form[2] $\psi(t, r) = e^{i\mu t} e^{in\theta} h(r)$. This ansatz leads to the equation

$$h^{''}(r) + \frac{h^{'}(r)}{r} - \frac{n^2}{r^2} h(r) = 2m\mu h(r) - \frac{\lambda}{2m} h^3(r). \tag{4}$$

This equation allows for scaling

$$\bar{r} = r\sqrt{2m\mu} \; , \quad \bar{h} = h\sqrt{\frac{\lambda}{2m\mu}} \tag{5}$$

and can be rewritten into

$$\bar{h}^{''}(\bar{r}) + \frac{\bar{h}^{'}(\bar{r})}{\bar{r}} - \frac{n^2}{\bar{r}^2} \bar{h}(r) = \bar{h}(\bar{r}) - \frac{1}{2m} \bar{h}^3(\bar{r}). \tag{6}$$

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[2] M.Volkov and E.Wohnert 2002; P.Brax and P. Valageas, 2025

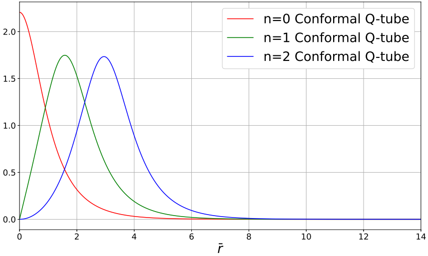

Рис.: Conformal Q-tubes with winding number $n = 0, 1, 2$.

The integral characteristics of Q-tubes

$$\begin{aligned} H &= 2\pi \int_0^\infty \left[ \frac{|\nabla\psi|^2}{2m} - \frac{\lambda}{8m^2}|\psi|^4 \right] rdr = 0, \\ N &= 2\pi \int_0^\infty |\psi|^2 rdr = const. \end{aligned} \tag{7}$$

We support this result by considering scale invariance of theory (3) and the influence of unbroken conformal symmetry.

Dilatations: $e^{\sigma} = \sqrt{2m\mu}$, so that $t^{'} = 2m\mu t$ and $x^{i'} = \sqrt{2m\mu} \cdot x^{i}$. The complex field $\psi^{'} = (2m\mu)^{-\frac{1}{2}}\psi$.

$$\nabla^2\psi^{'} = \psi^{'} - \frac{\lambda}{4m^2}\left|\psi^{'}\right|^2\psi^{'}. \tag{8}$$

$$N = \frac{\sqrt{2m\mu}}{\sqrt{2m\mu}}\int_{-\infty}^{\infty} d^2x^{'}\left|\psi^{'}\left(t^{'}, x^{i'}\right)\right|^2, \quad H = 0. \tag{9}$$

The latter is a result of an unbroken scale invariance and conformal symmetry, the corresponding symmetry generators $D$ and $K$

$$D = 2tH + \frac{i}{2}\int \vec{x}\left(\psi^{*}\vec{\nabla}\psi - \psi\vec{\nabla}\psi^{*}\right)d^2x,$$
$$K = t^2H - tD - \frac{m}{2}\int \vec{x}^2(\psi^{*}\psi)d^2x.$$

are evolve as [3]

$$\frac{dK}{dt} = -t\frac{dD}{dt}, \qquad \frac{dD}{dt} = 2H. \tag{10}$$

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[3] M. O. deKok and J.W. van Holten, Nucl. Phys. B 803 (2008), arXiv: 0712.3686 [hep-th].

# Relativistic $-\lambda\phi^4$

Consideration of the relativistic corrections leads to violation of conformal symmetry. The corresponding relativistic model is

$$\mathcal{L} = \partial^\mu \phi^* \partial_\mu \phi - m^2 \phi^* \phi + \frac{\lambda}{2} (\phi^* \phi)^2 \tag{11}$$

The ansatz for Q-tubes is $\phi = e^{-i\omega t} e^{in\theta} f(r)$, where $\omega - m = \mu$.
The non-relativistic regime of the solitons corresponds to $|\omega - m| \ll m$.

# E(Q) for solitons in relativistic theory

Let us consider the integral characteristics of solitons in the relativistic theory. The $U(1)$ Noether charge is

$$Q = \int \mathrm{i}\left[\phi^*\partial_0\phi - \phi\partial_0\phi^*\right] d^2x = 2\pi\int_0^\infty 2\omega f^2(r) r dr. \tag{12}$$

The energy of soliton $E$ is provided by the integral

$$\begin{aligned} E = \int \left[\partial_0\phi^*\partial_0\phi + \partial_i\phi^*\partial_i\phi + m^2\phi^*\phi - \frac{\lambda}{2}(\phi^*\phi)^2\right] d^2x = \\ 2\pi\int_0^\infty \left[(\omega^2 + m^2 + \frac{n^2}{r^2})f(r)^2 + (f'(r))^2 - \frac{\lambda}{2}f^4(r)\right] r dr. \end{aligned} \tag{13}$$

$H = E - mQ$, $Q = N$ at $\omega = m$ (which is equivalent to $\mu = 0$).

# Results for E(Q)

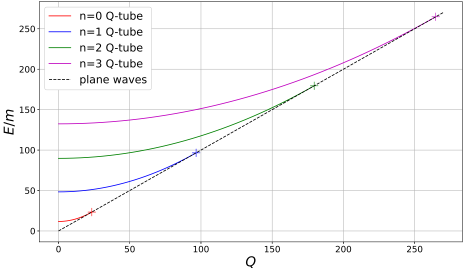

Рис.: Integral characteristics of Q-tubes with different values of the parameter n. Calculations were performed by setting $\frac{\lambda}{m} = 1$.

Note that $\omega = m$ we obtain $E - mQ = H = 0$, as in the theory with Shrödinger invariance, and $Q = mN = const$.

# Model with additional $\phi^6$ interaction

We can also add higher-order terms of ineraction to the theory, e. g. consider the lagrangian

$$\mathcal{L} = \partial^\mu \phi^* \partial_\mu \phi - m^2 \phi^* \phi + \frac{\lambda}{2} (\phi^* \phi)^2 - \frac{\sigma}{3} (\phi^* \phi)^3, \tag{14}$$

where $[m] = M$, $[\lambda] = M$, $[\sigma] = 1$; $\lambda, \sigma > 0$.
The theory is U(1)-invariant. At the regime $|\phi|^2 \ll \lambda/\sigma$ the model approximately coincides with (11). In this model, there is also the conformal limit with an isolated point. However, the kinematic and linear stability near this point depends on the parameter $\sigma$.

# Results for E(Q)

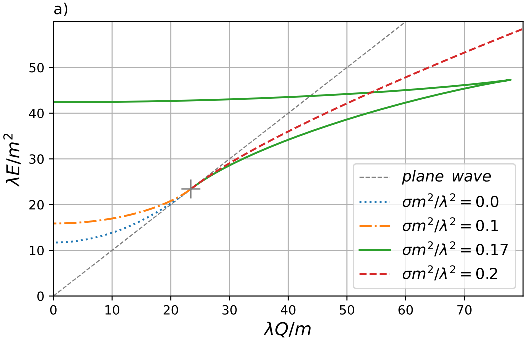

Рис.: Integral characteristics of Q-tubes in theory with additional sextic term. Calculations were performed by setting $\lambda = m = 1$, $n = 0$.

# Results for E(Q)

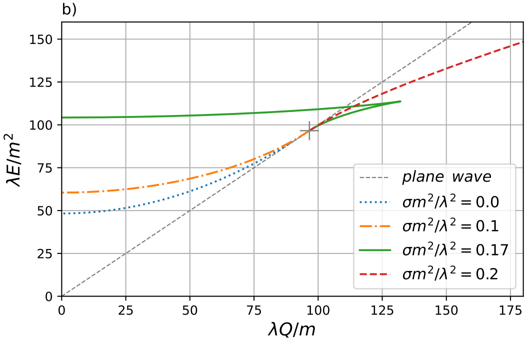

Рис.: Integral characteristics of Q-tubes in theory with additional sextic term. Calculations were performed by setting $\lambda = m = 1$, $n = 1$.

# Q-tubes in (2+1) Friedeberg-Lee-Sirlin theory

Let us consider the theory with the lagrangian

$$\mathcal{L} = \partial_\mu \phi^* \partial^\mu \phi + \frac{1}{2} \partial_\mu \chi \partial^\mu \chi - V(\chi, |\phi|), \tag{15}$$

where

$$V(\chi, |\phi|) = h^2 |\phi|^2 \chi^2 + \frac{\varkappa^2}{2} \left(\chi^2 - v^2\right)^2, \tag{16}$$

and $[v] = [h] = [\varkappa] = M^{1/2}$.
The equations of motion (15) are

$$\begin{aligned} &\partial_\mu \partial^\mu \phi + h^2 \chi^2 \phi = 0, \\ &\partial_\mu \partial^\mu \chi + 2h^2 |\phi|^2 \chi + 2\varkappa^2 (\chi^2 - v^2) \chi = 0. \end{aligned} \tag{17}$$

# Effective potential

In the limit $m_\chi = \varkappa v \gg m_\phi$ the kinetic term for the field $\chi$ can be neglected, and we obtain that $\chi^2 = v^2 - \frac{h^2}{\varkappa^2}|\phi|^2$ or $\chi = 0$. In this case, we can consider an effective theory of a massive self-interacting field $\phi$ with a piece-wise potential

$$V_{eff}(|\phi|^2) = \begin{cases} m_\phi^2|\phi|^2 - \frac{h^4}{2\varkappa^2}|\phi|^4, & \text{if } |\phi| < \frac{\varkappa v}{h}, \\ \frac{\varkappa^2 v^4}{2}, & \text{if } |\phi| > \frac{\varkappa v}{h}. \end{cases} \tag{18}$$

As one can see, for small $\phi$ this theory coincides with the theory (11).

# Ansatz and equations

The ansatz for Q-tubes is

$$\begin{cases} \phi(t,\vec{x}) = e^{-\mathrm{i}\omega t} e^{\mathrm{i}n\theta} f(r), \\ \chi(t,\vec{x}) = g(r). \end{cases} \tag{19}$$

Equations on these functions are

$$\begin{aligned} & f^{''} + \frac{f^{'}}{r} - \frac{n^2}{r^2} f - (h^2 g^2 - \omega^2) f = 0, \\ & g^{''} + \frac{g^{'}}{r} - 2h^2 f^2 g - 2\varkappa^2 (g^2 - v^2) g = 0. \end{aligned} \tag{20}$$

# E(Q) for Q-tubes in FLS

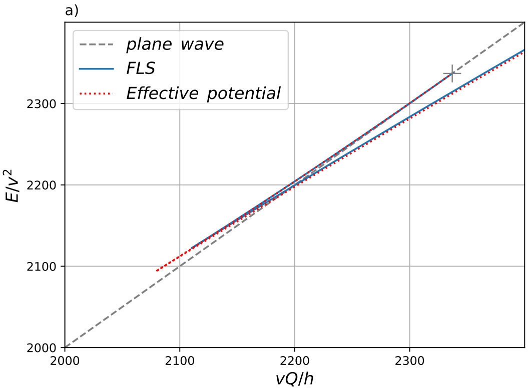

Рис.: Integral characteristics of Q-tubes with n = 0, $\varkappa/h = 10$

# E(Q) for Q-tubes in FLS

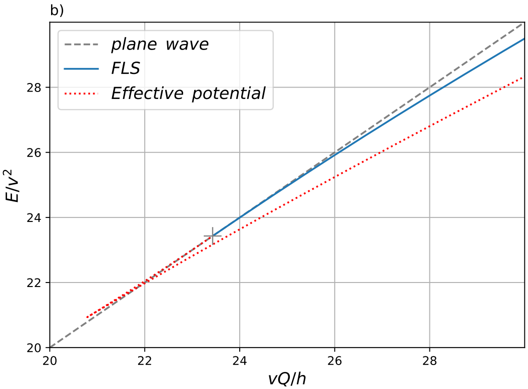

Рис.: Integral characteristics of Q-tubes with n = 0, $\varkappa/h = 1$

# Conclusion

- We have found and examined Q-tube solutions in the scalar field theories with the conformal symmetry restoration.
- In the relativistic $-\lambda\phi^4$ theory, the Q-tubes are kinematicaly unsable, but the conformall symmetry restoration in the non-relativistic limit leads to the enhanced stability of solitons.
- It the theory with additional $\phi^6$ self-interaction, the conformal symmetry is restored at $\omega = m$. The kinematic stability of Q-tubes in the vicinity of this point depends on the coupling constant of the sextic term. These results are easily generalized on the complex scalar theory with an arbitrary polynomial potential.
- We have shown that in FLS theory, the conformal symmetry is also restored at $\omega = m$. The kinematic stability of Q-tubes in the vicinity of this point depends on the parameters of the theory.

Thanks for your attention!

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# Appendix 1: Analytical approximation

It can be noted that equation of motion (6) allows for following integral relation

$$\int_0^\infty d\bar{r}\frac{\left(\bar{h}'(\bar{r})\right)^2}{\bar{r}} = \int_0^\infty d\bar{r}\frac{n^2\bar{h}^2(\bar{r})}{\bar{r}^3}. \tag{21}$$

A known asymptotic behavior of the Q-tube $\bar{h}(\bar{r}) \sim \bar{r}^n$ at $\bar{r} \to 0$ leads to approximation

$$\bar{h}''(\bar{r}) = \bar{h}(\bar{r})\left(1+\frac{n^2}{R^2}\right) - \frac{1}{2m}\bar{h}^3(\bar{r}), \text{ for } n \to \infty, \tag{22}$$

where $R$ is the radius of the solution's peak. Soliton solution of Eq.(22) has an exact analytical form

$$\bar{h}(\bar{r}) = \sqrt{4m\left(1+\frac{n^2}{R^2}\right)}\operatorname{sech}\left(\sqrt{1+\frac{n^2}{R^2}}(\bar{r}-R)\right), \tag{23}$$

where $R = \sqrt{2}n$ in the limit of large $n$.

$$N \approx \frac{8\pi m}{\lambda}\ln\left(1+e^{2\sqrt{3}n}\right) n \to \infty \frac{16\sqrt{3}\pi m}{\lambda} n. \tag{24}$$

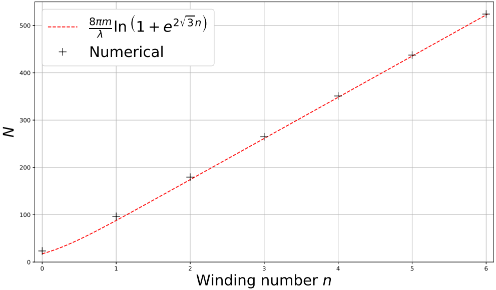

Рис.: The value of global $U(1)$ charge $N$ plotted versus winding number $n$ at $\lambda/m = 1$. Cross markers indicate results of numerical integration, while analytical estimations are represented by a dashed line.

# Appendix 2: The mechanical potential

$$\mathrm{w} = \frac{\omega}{m}, \quad \tilde{r} = mr, \quad \tilde{f} = f\frac{\sqrt{\lambda}}{m}, \quad g = \sigma\frac{m^2}{\lambda^2} \tag{25}$$

$$U(\tilde{f}) = -\frac{n^2}{2\tilde{r}^2}\tilde{f}^2 - \frac{(1-\mathrm{w}^2)}{2}\tilde{f}^2 + \frac{\tilde{f}^4}{4} - g\frac{\tilde{f}^6}{6} \tag{26}$$

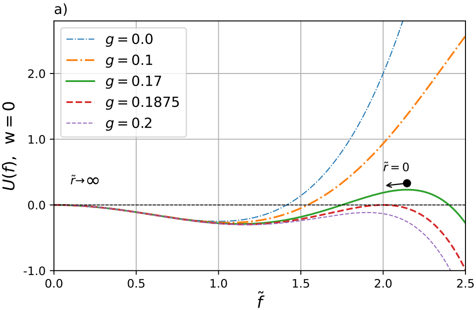

Рис.: The mechanical potential $U(\tilde{f})$ for different values of $g$.

# Appendix 3: Q-tube profiles in the theory with an additional sextic term

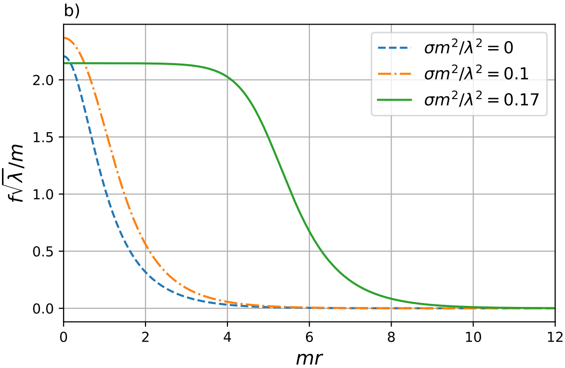

Рис.: Profiles of solitons for different values of $\sigma m^2/\lambda^2$; $n = 0$, $\mathrm{w} = 0$.

# Appendix 4: E and Q for the theory with an additional sextic term

The expression for U(1) Noether charge is

$$Q = 2\pi \int_0^\infty 2\omega f^2(r) r dr. \tag{27}$$

The energy of soliton is

$$E = 2\pi \int_0^\infty \left[ (\omega^2 + m^2 + \frac{n^2}{r^2}) f(r)^2 + (f'(r))^2 - \frac{\lambda}{2} f^4(r) + \frac{\sigma}{3} f^6(r) \right] r dr. \tag{28}$$

# Appendix 5: E and Q for FLS theory

The expression for U(1) Noether charge is

$$Q = 2\pi \int_0^\infty 2\omega f^2(r) r dr. \tag{29}$$

The energy is written as

$$\begin{aligned} E = \int \Big[ & \partial_0\phi^* \partial_0\phi + \partial_i\phi^*\partial_i\phi + \frac{1}{2}\partial_0\chi^*\partial_0\chi + \\ & + \frac{1}{2}\partial_i\chi^*\partial_i\chi + h^2|\phi|^2\chi^2 + \frac{\varkappa^2}{2}\left(\chi^2 - v^2\right)^2 d^2x = \\ & 2\pi \int_0^\infty \Big[ (\omega^2 + \frac{n^2}{r^2}) f(r)^2 + (f'(r))^2 + \frac{1}{2}(g'(r))^2 + \\ & + h^2 f(r)^2 g(r)^2 + \frac{\varkappa^2}{2}\left(g(r)^2 - v^2\right)^2 r dr. \end{aligned} \tag{30}$$

# Appendix 6: E(Q) for Q-tubes in FLS, n=1

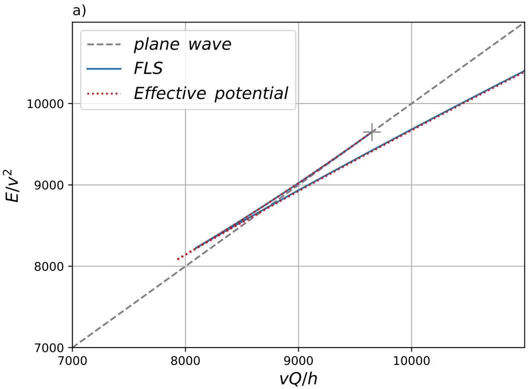

Рис.: Integral characteristics of Q-tubes with n = 1, $\varkappa/h = 10$

# Appendix 6: E(Q) for Q-tubes in FLS, n=1

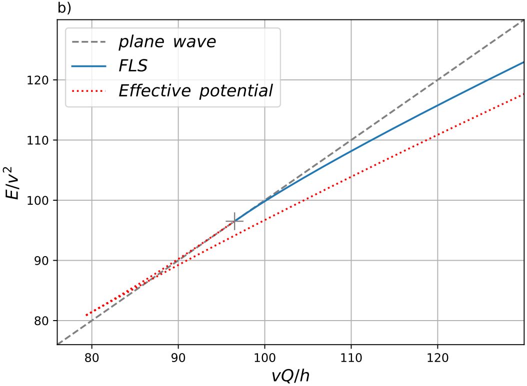

Рис.: Integral characteristics of Q-tubes with n = 1, $\varkappa/h = 1$