Gravitational S-duality

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Electric-magnrtic duality

- Electromagnetic duality is a fundamental concept in theoretical physics, representing the symmetry in which electric and magnetic fields can be interchanged, along with charges and monopoles, without changing Maxwell's equations.
- It readily generalizes to curved space electrodynamics and Yang-Mills theory (Deser 1976) and is crucial for understanding dualities in quantum field theories and string theory.
- This duality implies that in many scenarios, a strongly coupled electric theory can be mapped to a weakly coupled magnetic theory (S-duality), offering powerful tools in calculations.

Gravitational duality

- It extends to higher linear spins and most importantly, to linearized gravity (Szekeres 1971, Maartens 1997, Henneaux 2004, Barnich 2008, Boos 2021), but the extension to full non-linear gravity was questioned (Ellwanger 2001, Deser 2005,Monteiro 2023).
- Recently, it was shown that using the frame definition of duality transformation one can express the self-dual part of the curvature tensor (Kol: 2022) in the form bearing the elements of exact S-duality, but this does not solve the full problem.
- Gravitational S-duality was also extensively discussed in supergravity (Argurio 2008, Argurio 2009, Moutsopoulos 2009, Dehouck 2011) and string theory (Hull 2000, 2001, 2023, Kol 2023).

Non linear S-duality in gravity

- The cornerstone of non-linear gravitational S-duality is the Taub-NUT solution of the vacuum Einstein equations This solution, meanwhile, has severe problems associated with Misner strings.
- Dropping the Misner time identification which eliminates Misner strings at the price of introducing closed timelike curve everywhere, one can adopt Bonnor's view (Bonnor 1969), that Misner strings are true line singularities. Developing this idea, it was shown (Clement and DG 2015) that one of the long-standing problems — closed timelike curves in the neighborhood of MS — can be partially solved, since these lines are not geodesics, while MS itself are geodesically transparent.
- Mass formulas and the thermodynamics of nutty black holes were developed (Clement and DG 2017, 2019, Henigar et al 2019, Wu 2019, Chen 2019, Awad)

Negative mass problem

- Meanwhile, Bonnor's interpretation was criticized (Manko and Ruiz-Ruiz 2005) arguing that singular MS have unphysical matter sources.
- Based on linearized theory, Bonnor proposed that Misner strings are massless sources of rotation. However, in nonlinear theory, the Komar mass integral over the sidewall of Misner strings is negative, which was interpreted in as an unphysical negative mass of the string.
- Here, we show that this interpretation can be challenged. Our argument is based on the approach of localizing Komar integrals using the concept of Komar-Killing local fields.

Non-linear gravielectic and gravimagnetic fields

- In linearized general relativity the curvature tensor can be split into electric and magnetic sectors leading to Maxwell-like equations for GE and GM fields.
- Here we suggest the definition applicable in the non-linear regime for stationary spacetime, which is directly related to the Komar two-form associated with stationarity. .
  Consider the Komar formula for the mass of the stationary gravitational field in the volume V surrounded by the closed two-surface ∂V given by the integral

$$M = -\frac{1}{8\pi}\oint_{\partial V} \star dk,$$

where $k = g_{t\nu}dx^\nu$ is the Komar one-form corresponding to the timelike Killing vector $k^\mu = \delta^\mu_t$, and the star denotes the Hodge dual.

- The Komar integral for the dual gravimagnetic charge is

$$N = \frac{1}{8\pi} \oint_{\partial V} dk.$$

- Commuting the covariant derivatives and using Killing equations one can derive Maxwell equations and Bianchi identities for Riemann tensor in terms of Komar two-form and its dual

$$d \star K = 2 \star dx^{\mu} R_{\mu\nu} k^{\nu},$$

$$dK = -\frac{1}{2} k^{\sigma} R_{[\rho\mu\nu]\sigma} dx^{\rho} \wedge dx^{\mu} \wedge dx^{\nu},$$

These equations are analogous to the Maxwel equations, which now hold in the full non-linear general relativity. Our proposal is to consider Killing two forms as non-linear generalization of gravimaxwell tensor containing gravielectric and gravimagnetic fields as usual.

Taub-NUT as gravitational dyon

- Taub-NUT metric is an exact solution of the vacuum Einstein equations with two parameters: mass m, and NUT parameter n:

$$ds^2 = -\frac{\Delta}{\Sigma}(dt + 2n\cos\theta d\varphi)^2 + \frac{\Sigma}{\Delta}dr^2 + \Sigma d\Omega^2,$$
$$\Delta = r^2 - 2mr - n^2, \qquad \Sigma = r^2 + n^2.$$

This metric has the timelike Killing vector $k^\mu = \delta^\mu_t$, which becomes null on the horizon $\Delta(r_H) = 0$: $\quad r_H = m + \sqrt{m^2 + n^2}$. In contrast to the Schwarzschild case, the metric has no singularity at $r = 0$. For definiteness, we assume $m > 0$, $n > 0$.
This metric also has a rotational Killing vector $m^\mu = \delta^\mu_\varphi$ and admits a canonical representation in Weyl-Papapetrou coordinates

$$ds^2 = -F(dt - \omega d\varphi)^2 + F^{-1}[e^{2k}(d\rho^2 + dz^2) + \rho^2 d\varphi^2], \qquad (1)$$

related as $\rho = \sqrt{\Delta}\sin\theta,\ z = (r - m)\cos\theta$. The polar axis $\rho = 0$ is divided into three rods (Harmark 2004, Clement and DG 2019): the horizon rod $|z| < r_H$ and two Misner string rods $z > r_H,\ z < -r_H$ (north and south strings).

Misner strings are tubes

- Each rod is a Killing horizon of a certain linear combination of k^μ and m^μ and is characterized by a constant two-dimensional directional vector whose components are related to surface gravity κ, and angular velocity Ω of the rod. For the Taub-NUT metric the horizon is non-rotating, $\Omega_H = 0$, while Misner strings are rotating with velocities $\Omega_\pm = \mp 1/2n$.
- Although Misner strings appear as one-dimensional defects, they are actually two-dimensional cylinders with finite lateral surface area per unit length (Clement and DG 2019):

$$\mathcal{A} = \oint d\varphi \sqrt{|g_{zz} g_{\varphi\varphi}|} = 2\pi |e^k \omega|.$$

Thus, the image of infinitely thin hollow tube is a more accurate geometric representation of Misner strings than a line.

GE and GM fluxes in Misner strings

- Let us show that the MS cylinders carry distribution-valued fluxes of gravielectric and gravimagnetic fields defined above. To discover them we have to consider the potentials for K and $\star K$. For the first it is simply the Komar one-form

$$k = -\frac{\Delta}{\Sigma}(dt + 2n\cos\theta d\varphi),$$

and it leads to the bulk Komar two-form

$$K = 2m\frac{\tilde{\Delta}}{\Sigma^2}(dt + 2n\cos\theta d\varphi) \wedge dr + 2n\frac{\Delta}{\Sigma}\sin\theta d\theta \wedge d\varphi,$$

where we have introduced $\tilde{\Delta} = r^2 + 2\frac{n^2}{m}r - n^2$. Using the fact that $d\varphi$ is not closed on the polar axis (DG and Karsanov 2026)

$$dd\varphi = 2\pi\,\delta^2(\mathbf{x})\,dx \wedge dy,$$

where x, y are the local Cartesian coordinates in the plane orthogonal to z axis, one finds that the exterior derivative dk has in addition the singular part

Distributional fluxes

$$dk = K + \mathcal{K},$$

proportional to the NUT charge:

$$\mathcal{K}^{\pm} = \mp 4\pi n \frac{\Delta}{\Sigma} \delta(x)\delta(y) dx \wedge dy.$$

The dual quantity $\star K$ whose reverse flux determines the Komar mass is given by

$$\star K = 2n \frac{\Delta}{\Sigma^2}(dt + 2n \cos\theta d\varphi) \wedge dr - 2m \frac{\tilde{\Delta}}{\Sigma} \sin\theta d\theta \wedge d\varphi.$$

The corresponding potential one-form $\tilde{k}$ such that $\star K = d\tilde{k}$ is equal to

$$\tilde{k} = \frac{m}{n} \frac{\tilde{\Delta}}{\Sigma} (dt + 2n \cos\theta d\varphi).$$

Also one finds that $d\tilde{k}$ apart of the bulk field contains a singular term

$$d\tilde{k} = \star K + \tilde{\mathcal{K}},$$

where the second term is localized on the Misner strings

Asymptotics

- These singular fluxes have the following features. GE fluxes are constant at infinities $z \to \pm\infty$ and are incoming there. They increase approaching the horizon. GM fluxes are also constant at infinity, but vanish at the horizon
- In contrast with the electromagnetic case (DG and Karsanov 2026) where the corresponding electric and magnetic fluxes are self-dual, here S-duality holds only as $r \to \infty$.
- So far it was assumed that to get regular spacetime it is enough to cut the Misner string from the lateral side by cylindrical surfaces. Then calculating the flux of $\star K$ through the lateral surface led to a negative mass of the MS strings. But presence of singular fluxes of GE and GM fields inside the strings and finiteness of their transverse sections demands to include these sections too in the definition of the closed surface surrounding the strings.

Balance of fluxes

The total boundaries $\partial V_{\pm}$ consist of the sum of the surfaces orthogonal to the string at infinity $S_{\perp}^{\infty}$, at the horizon $S_{\perp}^{H}$ and the lateral surface S_L:

$$\partial V_{MS} = S_{\perp}^{\infty} \cup S_{\perp}^{H} \cup S_L.$$

Motivated by the form of the Komar integrals we define the fluxes of the GE and GM fields in the bulk as

$$\Phi = -\oint_S \star K, \qquad \tilde{\Phi} = \oint_S K.$$

Here S is either a lateral surface of cylinders S_L either sphere of some radius S_r, and we will assume that the normals to the sphere and cylinders around the strings are all directed outward of these surfaces. For the fluxes through the cross sections of the strings then we have analogously a definition

$$\Phi_{\perp} = -\oint_S \tilde{\mathcal{K}}, \qquad \tilde{\Phi}_{\perp} = \oint_S \mathcal{K}.$$

Balance of fluxes

- Total flux through the surface $S_\perp$ orthogonally intersecting the strings with the normal in outer direction (i.e. towards increasing $|z|$) at the point r on the strings will read

$$\Phi_\perp^\pm(r) = -4\pi m \frac{\tilde{\Delta}(r)}{\Sigma(r)}$$

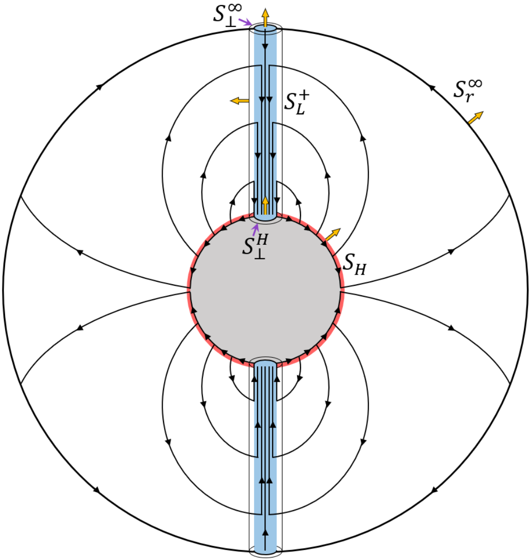
- The fluxes through the transverse sections will then be given by $\Phi_\perp^\pm(\infty)$ and $\Phi_\perp^\pm(r_H)$ respectively, while the flux over the lateral surface Φ_L must be computed using the expression for the bulk integral (13). Then the results of calculation will read

$$\begin{aligned} \Phi_\perp^\pm(\infty) &= -4\pi m, \\ \Phi_\perp^\pm(r_H) &= -4\pi\sqrt{m^2+n^2}, \\ \Phi_L^\pm &= -4\pi\frac{n^2}{r_H} = 4\pi(m - \sqrt{m^2+n^2}). \end{aligned}$$

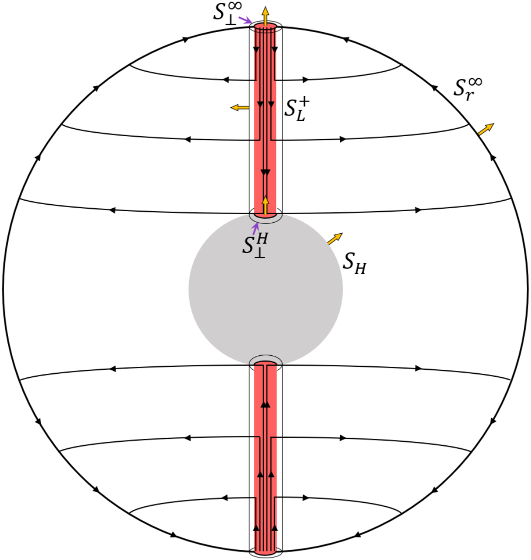
Misner string masses

- The masses of Misner strings according to the Komar definition (6) will read
$$M_{\pm} = \frac{1}{8\pi}[\Phi^{\pm}_{\perp}(\infty) - \Phi^{\pm}_{\perp}(r_H) + \Phi^{\pm}_{L}] = 0.$$
- So the Misner strings are massless indeed, as was originally assumed by Bonnor!
- Thus Taub-NUT solution has no "material" sources of gravielectric and gravimagnetic masses. Only closed GE and GM loops

Shematic GE lines picture



Shematic GM lines picture



Conclusions

- Non-linear generalization of GE and GM fields can be given in stationary spacetime in terms of the Komar-Killing two-form field
- GE and GM fields introduced in such a way exhibit asymptotic S-duality (linearized level)
- Misner strings are not rigid rods but hollow tubes of non-zero transverse area, carrying GE and GM fluxes
- Balance of fluxes shows that lateral Komar integrals are compensated but longitudinal integrals through the MS strings sections, do Misner strings are massless as Bonnor suggested on the linearized level. No more problem of negative mass for Taub-NUT gravitational dyon