Quantum black holes and unitarity test

Amplitudes and semiclassical optical theorem for fields and gravity

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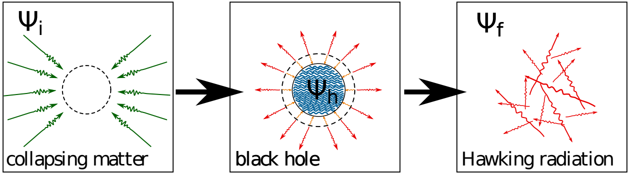
Quarks-2026

2026 May 18, Petrozavodsk

Information loss in a nutshell

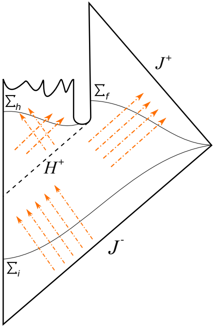


$$\hat{\rho}_i = |\Psi_i\rangle\langle\Psi_i| \;\mapsto\; \hat{\rho}_f = \operatorname*{Tr}_{\Sigma_h}\left(|\Psi_f\rangle|\Psi_h\rangle\langle\Psi_h|\langle\Psi_f|\right)$$

Apparent unitarity violation: $\boxed{\mathrm{Tr}(\hat{\rho}_f^2) < 1}$.

This is paradoxical!

S.W. Hawking (1976)



Gravitational S–matrix:

$$\boxed{\langle\Psi_f|\hat{S}|\Psi_i\rangle = \int_{\Phi:=\{g_{\mu\nu},\,\phi,\,\chi\}} \mathcal{D}\Phi\, e^{iS[\Phi]}\;\Psi_f^*\Psi_i\,, \qquad (PI)}$$

does not seem to exist...

Goal: check the identities $\hat{\mathcal{S}}^\dagger\hat{\mathcal{S}} = \hat{\mathcal{S}}\hat{\mathcal{S}}^\dagger = \hat{1}$ directly as formally defined path integral.

Evaporation as tunnelling

Tunneling of particles through dynamical black hole spacetime:

ArXiv: 9907001 [hep-th] Parikh, Wilczek (1999)

- Semiclassical amplitude:

$$\langle \Psi_f | \hat{S} | \Psi_i \rangle \simeq F[\Phi_s] e^{\frac{i}{\hbar} S[\Phi_s]} (1 + O(\hbar)) \,, \qquad \frac{\delta}{\delta \Phi} S[\Phi_s] = 0 \,.$$

- Scattering: find (analytic) solution Φ_s with asymptotics corresponding to boundary problem $\Psi_{in} \mapsto \Psi_{out}$. Real (non singular) solutions trivially exist at energy below black hole formation.
- Analytic continuation: deform saddles to avoid singularities to describe $\Psi_{in} \mapsto \Psi_{out}$ for energy above black hole formation. Trickiest part!

ε–regularization method

- Faddeev–Popov–like trick: insert into path integral $\langle\Psi_f|\hat{S}|\Psi_i\rangle$ a unity

$$1 \equiv \int_0^{+\infty} dT \int_{-i\infty}^{+i\infty} \frac{d\varepsilon}{2\pi i} e^{\varepsilon(T - T_{int}[\Phi])} .$$

Find deformed saddle points from complexified action,

$$S_\varepsilon[\Phi] = S[\Phi] + i\varepsilon T_{int}[\Phi] - i\varepsilon T \;,$$

and then the amplitude $\mathcal{A} = \lim_{\varepsilon\to+0} \mathcal{A}_\varepsilon$.

ArXiv: 0707.0433, Levkov, Panin, Sibiryakov (2007)

- Applied to shells models: "interaction time" $T_{int}[\Phi]$ follows:
  - diffeomorphism invariant
  - positive–definite for real solutions
  - diverges for eternal black holes (to ensure asymptotic flatness)

Limit $\varepsilon \to 0 \;\Rightarrow \mathcal{P}_{fi} = |\mathcal{A}_{fi}|^2 \approx \exp\left(-2\Im m\, S_{tot}\right) = \exp\left(-S_{BH}\right)$, where $S_{BH} = \frac{1}{4}\mathrm{Area}$ - black hole entropy.

ArXiv: 1503.07181, Bezrukov, Levkov, Sibiryakov (2015)

Dynamical boundary model

CGHS model with boundary of spacetime $\phi = \phi_0$

$$S = \int d^2x\sqrt{-g}e^{-2\phi}\left(R+4(\nabla\phi)^2+4\lambda^2\right) + \int d\tau e^{-2\phi}(2K+4\lambda) - m\int ds\ .$$

Minimal BH mass $M_{cr} = 2\lambda e^{-2\phi_0}$ chosen so than coupling is weak.

ArXiv: 2006.03606, MF, Levkov, Sibiryakov (2020)

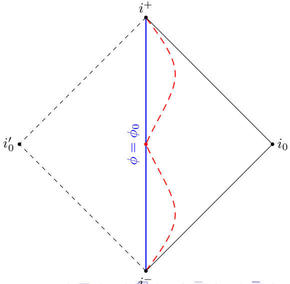
Metric inside (T, r) and outside (t, r):

$$ds_-^2 = -dT^2 + dr^2\ , \quad ds_+^2 = -f(r)dt^2 + \frac{dr^2}{f(r)}\ ,$$

$$f(r) = 1 - \frac{M}{2\lambda}e^{-2\lambda r}\ , \quad \phi = -\lambda r\ .$$

Israel condition $\Rightarrow$ $\left(\frac{dr}{d\tau}\right)^2 + V_{eff}(r) = 0,$

where $V_{eff}(r) = 1 - \left(\frac{M}{m} + \frac{m}{8\lambda}e^{-2\lambda r}\right)^2.$

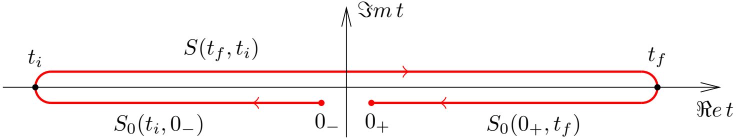


Calculation of total action S_{tot}

Semiclassical scattering amplitude $\mathcal{A}_{fi} = \langle \Psi_f | \hat{S} | \Psi_i \rangle \simeq e^{iS_{\text{tot}}}$,

$$S^C_{tot} = S(t_f, t_i) + S_0(t_i, 0_-) + S_0(0_+, t_f) - i \ln \left(\Psi_i \, \Psi_f^* \right) ,$$

where S and S_0 are interacting and free actions (from $\hat{S} = \hat{U}_0 \hat{U} \hat{U}_0$), and $\Psi_{f,\,i} = e^{ip_{f,\,i}x}$ are semiclassical wave functions of particle.

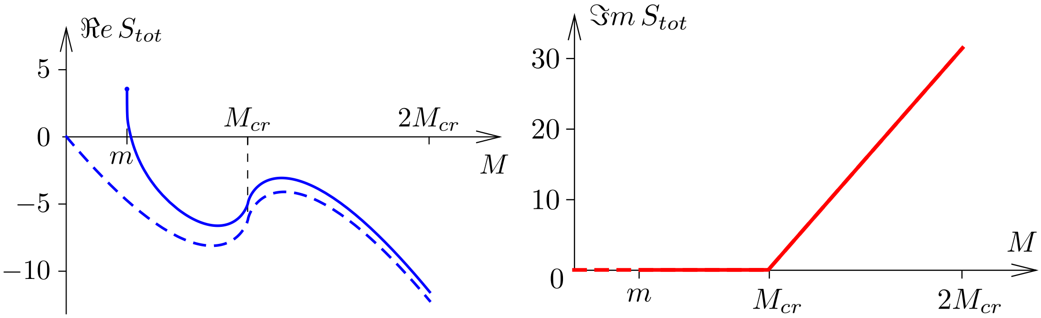


For massless particle

$$S_{\text{tot}} = -\frac{M - M_{\text{cr}}}{\lambda} \log \left(1 - \frac{M + i\varepsilon}{M_{\text{cr}}} \right) + \frac{M}{\lambda} \left(1 - \log \frac{M_{\text{cr}}}{2\lambda} \right) ,$$

where $+i\varepsilon$ follows from specific T–functional.

Point-particle scattering amplitude



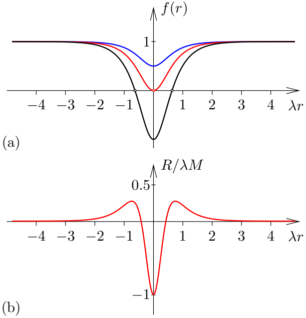
- Imaginary part of action $\Im m\, S_{\rm tot} = \frac{\pi}{\lambda}(M - M_{\rm cr})\theta(M - M_{\rm cr})$ contributes to suppression exponent for transition probability $\mathcal{P}_{fi} = \exp\left(-S_{BH}\right)$, where $S_{BH} = \frac{2\pi}{\lambda}(M - M_{cr})$ is corrected Bekenstein entropy.
- Apparent non–unitarity, but consistent with interpretation of point–like particle as exclusive QFT state. Analogy: Klein paradox in QM. Second quantization of the boundary?

Regular model

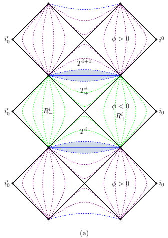
Consider a deformed CGHS action

$$S_{\text{sinh}} = -2\int d^2x\,\sqrt{-g}\,\sinh(2\phi)\,\left(R + 4(\nabla\phi)^2 + 4\lambda^2\right)$$

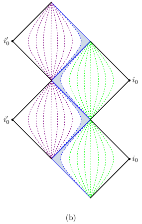
Vacuum solutions: $ds^2 = -fdt^2 + dr^2/f$, $\boxed{f(r) = 1 - \frac{M}{4\lambda\cosh(2\lambda r)}}$



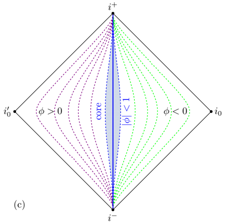
Non-extremal $M > M_{ext}$



Extremal $M = M_{ext}$



Horizonless $M < M_{ext}$



ArXiv: 2507.17933, MF (2026)

Transition amplitude

- Shock wave: $t = \int^r dr'/f(r')$, energy $E = M_i - M_f$.
- Amplitude:

$$\mathcal{A}_{fi} := \langle \Psi_f | \hat{S} | \Psi_i \rangle = \lim \int \mathcal{D}\Phi \, e^{iS_0(0,t_f)+iS(t_f,t_i)+iS_0(t_i,0)} \Psi_f^*[\Phi]\Psi_i[\Phi] \, .$$

- Total action

$$S_{tot} = S_{sw} + S_{gr} + S_{GH} + S_{free} - i\log(\Psi_f^*\Psi_i)$$

includes

  - shock wave action $S_{sw} = 0$
  - pure gravitational term

$$S_{LDV} \simeq \int_{-\infty}^{+\infty} dr \left(\frac{M_f}{f_L(r)\cosh^2(2\lambda r)} - \frac{M_i}{f_R(r)\cosh^2(2\lambda r)} \right)$$

  - Gibbons–Hawking term $S_{GH} = (M_i - M_f)/\lambda$
  - free evolution action $S_{free} = \int p dr - \int E dt$
  - wave functions $\Psi_{f,i} = \exp(iEr_{f,i})$

$$S_{free} - i\log(\Psi_f^*\Psi_i) = (M_i - M_f)(t_f - r_f - t_i + r_i) + iM_f\tau_{delay}$$

Transition amplitude

- The answer: $\mathcal{A}_{fi} = \lim_{\varepsilon,\,\varepsilon'\to+0} \exp(i\varphi(M_i + i\varepsilon) - i\varphi(M_f + i\varepsilon'))$,

$$\varphi(M) = \frac{M}{\lambda} - \frac{2M_{ext}}{\lambda}\sqrt{1 - \frac{M^2}{M_{ext}^2}} \arctan\sqrt{\frac{M_{ext} + M}{M_{ext} - M}}$$

- Transition probability: $\boxed{\mathcal{P}_{fi} = |\mathcal{A}_{fi}|^2 \simeq \exp(-\Delta S_{BH})}$
- Statistical ensemble of interior situated shock waves:

$$\langle M_f \rangle = M_i - T_H(M_i)$$

for large M_i.

Numerically: if M_i becomes comparable with M_{ext}

$$\langle M_f \rangle < M_{ext} \ ,$$

black hole decays completely.

ArXiv: 2507.17933, MF (2026)

Coherent states formalism

Choose basis $|a\rangle$: $\hat{a}_k|a\rangle = a_k|a\rangle\ \forall k$, where $\hat{a}_k$ is free field annihilation operator,

$$\hat{\chi}(t,x) = \int \frac{d^d k}{(2\pi)^{d/2}\sqrt{2\omega_k}} \left(\hat{a}_k e^{-i\omega_k t + ikx} + \hat{a}_k^\dagger e^{i\omega_k t - ikx}\right) \ , \quad \omega_k = \sqrt{k^2+m^2} \ .$$

Crucially, the wave functional is exponent of gaussian integral,

$$\log\langle\chi|a\rangle = \int d^d k \left(-\frac{1}{2}\omega_k \chi(k)\chi(-k) + \sqrt{2\omega_k}\chi(k)a_k - \frac{1}{2}\,a_k a_{-k}\right) \ ,$$

where $\chi(t,k) = \int \frac{d^d x}{(2\pi)^{d/2}} e^{ikx}\chi(t,x)$ is spatial Fourier transform.

Original application: multiparticle scattering $A_{2\to \mathrm{many}}$.

Rubakov, Son, Tinyakov et al. ($\sim$ 1990)

Matrix elements to calculate:

$$\boxed{\langle b|\hat{\mathcal{S}}^\dagger\hat{\mathcal{S}}|a\rangle = \langle b|\hat{\mathcal{S}}\hat{\mathcal{S}}^\dagger|a\rangle = \langle b|a\rangle} \equiv e^{\int d^d k\, b_k^* a_k} \ .$$

In–in formalism for unitarity test

Rewrite coherent states matrix element as path integral:

$$\langle b|\hat{S}^\dagger \hat{S}|a\rangle = \mathcal{N}\int \mathcal{D}\chi_a \mathcal{D}\chi_b \mathcal{D}\chi_f \; \langle b|\chi_b\rangle\langle\chi_b|\hat{S}^\dagger|\chi_f\rangle\langle\chi_f|\hat{S}|\chi_a\rangle\langle\chi_a|a\rangle \; ,$$

where propagators $\langle\Phi''|\hat{S}|\Phi'\rangle = \int_{\Phi(t')=\Phi'}^{\Phi(t'')=\Phi''} \mathcal{D}\Phi\; e^{iS[\Phi]}$. Integrating out gaussian integrals we get

$$\langle b|\hat{S}^\dagger \hat{S}|a\rangle = \boxed{\mathcal{N}\int \mathcal{D}\chi_+ \mathcal{D}\chi_- \; e^{iS_{BP}[\chi_+]-iS_{BP}[\chi_-]}}\langle b|a\rangle$$

with action $S_{BP} = -\int d^d x\,\left(\frac{1}{2}\chi\Box\chi + V(\chi)\right)$.

Boundary value problem:

$$\omega_k\chi_+(t_i,k) + i\dot{\chi}_+(t_i,k) = \sqrt{2\omega_k}e^{-i\omega_k t_i}a_{-k}\;, \quad t_i \to -\infty\;,$$

$$\omega_k\chi_-(t_i',k) - i\dot{\chi}_-(t_i',k) = \sqrt{2\omega_k}e^{i\omega_k t_i'}b_k^*\;, \quad t_i' \to -\infty\;,$$

$$\chi_+(t_f,k) = \chi_-(t_f',k)\;, \qquad \dot{\chi}_+(t_f,k) = \dot{\chi}_-(t_f,k)\;, \qquad t_f,\; t_f' \to +\infty\;.$$

Quartic scalar field

Extract self–interaction coupling λ in front of the action,

$$S_{BP} = \frac{\tilde{S}_{BP}}{\lambda} \,, \qquad a_k,\, b_k \;\mapsto\; a_k,\, b_k/\sqrt{\lambda}$$

$$\tilde{S}_{BP} = -\frac{1}{2}\int d^4x \left(\chi\Box\chi + \chi^4\right) \,, \qquad \Rightarrow \qquad \Box\chi^{cl}_{\pm} + (\chi^{cl}_{\pm})^3 = 0 \,.$$

Note the BVP solution $\chi^{cl}_{\pm}(t,x) = \chi^{cl}(t,x)$ because of final condition at $t_f \to +\infty$. We linearize fluctuations of the field around classical solution,

$$\chi_+ = \chi^{cl}_+ + \delta\chi_+ \,, \qquad \chi_- = \chi^{cl}_- + \delta\chi_- \,,$$

to satisfy vacuum (Feynman) boundary condition,

$$\omega_k \delta\chi_+ + i\dot{\delta\chi}_+ = 0 \,, t_i \to -\infty \,,$$

$$\omega_k \delta\chi_- - i\dot{\delta\chi}_- = 0 \,, t_i \to -\infty \,,$$

$$\delta\chi_+ = \delta\chi_+ \,, \quad \dot{\delta\chi}_+ = \dot{\delta\chi}_- \,, \qquad t_f \to +\infty \,.$$

Sadlle point calculation

From free scalar theory

$$\langle b|\hat{S}^\dagger \hat{S}|a\rangle = \mathcal{N}\int \mathcal{D}\delta\chi_+\mathcal{D}\delta\chi_-\ e^{-\frac{1}{2}\int d^dx\,\delta\chi_+\Box\delta\chi_+-\frac{1}{2}\int d^dx\,\delta\chi_-\Box\delta\chi_-}$$

we define the normalization factor:

$$\mathcal{N} = \sqrt{\frac{2\pi}{det\ G_F^{-1} det\ G_{\bar{F}}^{-1}}} = \langle 0|\hat{S}^\dagger \hat{S}|0\rangle^{-1}\ ,$$

where $G_F(p) = \frac{i}{p^2+i\epsilon}$ ($G_{\bar{F}}(p)$) is Feynman (anti–Feynman) propagator.

For quartic scalar field theory

$$\frac{i}{\lambda}\tilde{S}^C_{BP}[\chi_{cl}] - \frac{1}{2}Tr\ \log[G_F\cdot G_F^{-1}[\chi_{cl}]] - \frac{1}{2}Tr\ \log[G_{\bar{F}}\cdot G_{\bar{F}}^{-1}[\chi_{cl}]] = 2\pi i\, n + O(\lambda)\ ,$$

where $\tilde{S}^C_{BP} = -\frac{1}{2}\int_C dt\int d^3x(\chi_{cl}\Box\chi_{cl} + (\chi_{cl})^4/2)$, $n\in\mathbb{Z}$.
Contour C is closed and given no singularities of χ_{cl} one has $\tilde{S}^C_{BP} = 0$.

Perturbation series

Trace of log can be expanded into

$$Tr\,\log[G_F G_F^{-1}[\chi_{cl}]] = Tr[-\sum_{n=1}^{\infty}\frac{(3i)^n}{n}(G_F\chi_{cl}^2)^n]\,.$$

$$(G_F\chi_{cl}^2) = \int\frac{d^4k}{(2\pi)^4}\frac{i}{k^2+i\epsilon}\int d^4x\,\chi_{cl}^2(x)\,,$$

$$(G_F\chi_{cl}^2)^2 = \int\frac{d^4p}{(2\pi)^4}[\chi^2]_{cl}(p)[\chi^2]_{cl}(-p)\int\frac{d^4k}{(2\pi)^4}\frac{-1}{(k^2+i\epsilon)((k+p)^2+i\epsilon)}\,.$$

$$Tr\,\log[...] = -3i(G_F(0)-G_{\bar{F}}(0))\int d^4x\,\chi_{cl}^2(x)+\frac{9}{2}\int d^4xd^4y\chi_{cl}^2(x)[G_F(x-y)^2+G_{\bar{F}}(x-y)^2]\chi_{cl}^2(y)$$

First term is zero, the second is non trivial.

“Quantization”

Second term:

$$\frac{9}{2}\int\frac{d^4q}{(2\pi)^4}\chi_{cl}^2(q)\chi_{cl}^2(-q)\int\frac{d^4p}{(2\pi)^4}\left(G_F(p)G_F(p+q)+G_{\bar F}(p)G_{\bar F}(p+q)\right)$$

$$=-\frac{1}{4}\int\frac{d^4q}{(2\pi)^4}\chi_{cl}^2(q)\chi_{cl}^2(-q)\left(\mathcal{M}_{s,1-loop}(q)-\mathcal{M}^*_{s,1-loop}(q)\right)\ .$$

We find

$$\frac{i}{\lambda}\tilde{S}^C_{BP}[\chi_{cl}]-\frac{1}{4}\int\frac{d^4q}{(2\pi)^4}\chi_{cl}^2(q)\chi_{cl}^2(-q)\cdot\Im\mathrm{m}\,\mathcal{M}_{1-loop}(q)+\ldots=2\pi i\,n\ ,\quad n\in\mathbb{Z}\ .$$

For massless quartic field ($\Im\mathrm{m}\,\mathcal{M}_{1-loop}(k)=9/4\pi$, $\lambda=1$) we obtain a weird “quantization” rule:

$$\frac{i}{\lambda}\tilde{S}^C_{BP}[\chi_{cl}]-\frac{9}{4\pi}\tilde{S}_{BP}[\chi_{cl}]+\ldots=2\pi i\,n,\quad n\in\mathbb{Z}\ ,$$

which is necessary for QFT unitarity around classical scattering solution. Let us check this out!

Lipaton

Solution in massless quartic scalar theory (Euclid: $ds^2 = d\tau^2 + dx^2$):

$$\chi(\tau, x) = \frac{\sqrt{8}\rho}{\tau^2 + x^2 - \rho^2} \,,$$

Fourier:

$$\chi(\tau, k) = \frac{2\sqrt{\pi}\rho}{k}\, e^{-\mathrm{sign}(\Re\mathrm{e}\,(\sqrt{\tau^2-\rho^2}k))\sqrt{\tau^2-\rho^2}k} \,,$$

falls off in the limit $\tau \to \pm\infty$.
Linearization: vacuum coherent state limit is $\rho \to 0$.
Classical action

$$\tilde{S}_{BP} = 64\pi\rho^4 \int_{-\infty}^{+\infty} dt \int_0^{+\infty} dr \frac{r^2}{(r^2 - t^2 - \rho^2)^4} = -\frac{8\pi^2}{3} i$$

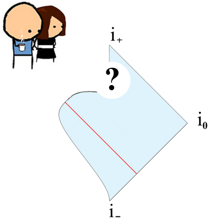
$$-\frac{9}{4\pi}\tilde{S}_{BP}[\chi_{cl}] = 2\pi i\, n \,, \quad \Rightarrow \quad \boxed{n = 3} \,.$$

Unitarity test: passed.

Final remarks

- We presented a semiclassical method to calculate scattering amplitudes in gravity with examples:
    - Model with regular core and shock wave tunneling from the black hole.
    - Model with dynamical boundary and massive particle consistent with thermodynamics.
- The test for unitarity of the model in terms of the path integral is proposed:
    - We derive a "quantization" condition for classical action in quartic scalar theory.

Thank you!



ε-regularization for dilaton gravity

We choose explicitly

$$T_{\rm int} = \int d^2x\sqrt{-g}\,\frac{f(\phi)}{\lambda^2}(\lambda^2 - (\partial_\mu\phi)^2)^2$$

where $f(\phi(r))$ has support on $r \gg r_0$

The metric has form $ds^2 = -e^{\nu(r)}dt^2 + e^{\zeta(r)}dr^2$ and $\phi = -\lambda r$, complexified field eqiations, e.g.

$$\partial_r\left(1 - e^{-\zeta}\right) + 2\lambda\left(1 - e^{-\zeta}\right) + \frac{i\varepsilon\lambda}{2}f(-\lambda r)e^{-2\lambda r}\left(1 - e^{-\zeta}\right)^2 = 0\,,$$

have solution

$$1 - e^{-\zeta(r)} = \frac{M}{2\lambda}e^{-2\lambda r}\left(1 - \frac{i\varepsilon M}{4\lambda}\int_{-\infty}^{\phi(r)} d\phi\, f(\phi)\right)^{-1}$$

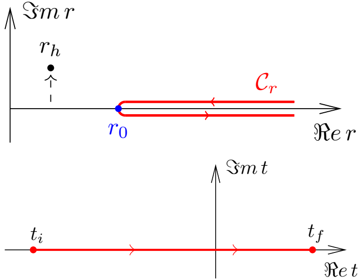
We see that inserting $iT_{\rm int}$ is equivalent to imaginary shift $M \mapsto M + i\varepsilon$.

How to deform time contour C

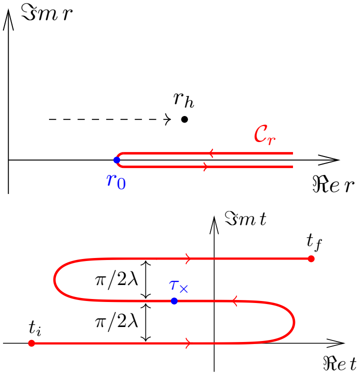
$$t(r) = \int dr \frac{\sqrt{f(r) - V_{eff}(r)}}{f(r)\dot{r}_*(r)}\,,$$

$$\dot{r}_*(r) = \mp\sqrt{-V_{eff}(r)}$$

$M < M_{\mathrm{cr}}$



$M > M_{\mathrm{cr}}$



$$\Im m(t_f - t_i) = 2\pi \mathrm{Res}\frac{1}{f(r{=}r_h)} = \frac{\pi}{\lambda}$$

Calculation S_{tot} in boundary model

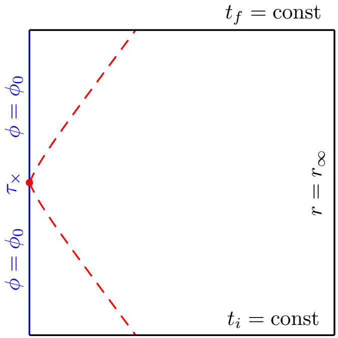
Gravitational part

- CGHS action

$$S_{\mathrm{CGHS}} = 2\int d^2x\sqrt{-g}\,\Box e^{-2\phi}$$

- Gibbons-Hawking action

$$S_{GH} = 2\kappa\int d\sigma\, e^{-2\phi}(K - K_0)$$



- $K_0 = 2\lambda$, $\kappa = 1$ at $r \to +\infty$
- $K_0 = 0$, $\kappa = -1$ at $t \to \pm\infty$

Field equations of motion $\Rightarrow$ $S_{gr} = 2\kappa\oint d\sigma\, e^{-2\phi_0} K$

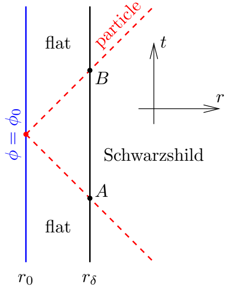
Calculation S_{tot} in boundary model

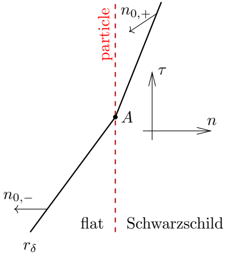
- Boundary $\phi = \phi_0$

$$S_{\phi_0} = 2e^{-2\phi_0} \int\limits_{\phi=\phi_0} d\tau\, K$$

$$(n^\tau, n^n) = (-\mathrm{sh}\psi(\tau), -\mathrm{ch}\psi(\tau))$$

$$\int\limits_{\phi=\phi_0} d\tau\, K = \psi_+ - \psi_-$$

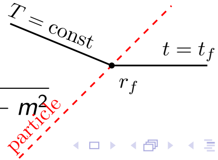




$$K = 2\delta(\tau - \tau_0)\left(\mathrm{arsh}\sqrt{-V_{\mathrm{eff}}(r_0)} - \mathrm{arsh}\sqrt{-V_{\mathrm{eff}}(r_0)/f(r_0)}\right)$$

- Cauchy surfaces $t = t_{f,i}$

$$S_{t_f} = -2\int d\sigma\, e^{-2\phi} K\,, \qquad S_{t_f} = S_{t_i} \simeq \frac{p}{2\lambda}\,, \quad p = \sqrt{M^2 - m^2}$$



Calculation S_{tot} in boundary model

- Point particle action $S_m = -m \left[\int_{r_0}^{r_i} + \int_{r_0}^{r_f}\right] \frac{dr}{\sqrt{-V_{\mathrm{eff}}(r)}}$

$$S_m = \frac{m^2}{\lambda p} \ln \left[\frac{1}{2} + \frac{Mm^2}{8M_{\mathrm{cr}}p^2} + \frac{p_0}{2p}\right] - \frac{m^2(r_i + r_f - 2r_0)}{p}$$

- Contributions from in- and out- states

$$\Psi_{f,i} = \exp(\mp ipr_\mp)$$

- Free point particle action $S_{m,0}$

$$S_0(t_i, 0_-) = p(r_- - r_i) - Mt_i \;, \qquad S_0(0_+, t_f) = p(r_+ - r_f) + Mt_f$$

$$t_f - t_i = \frac{M(r_i + r_f - 2r_0)}{p} + \ldots$$

Calculation S_{tot} in boundary model

Nasty crocodile

$$S_{\rm tot} = -\frac{M-M_{\rm cr}}{\lambda}\ln\left(1-\frac{M+i\varepsilon}{M_{\rm cr}}\right)+\frac{p}{\lambda}\left(1-\ln\frac{M_{\rm cr}}{2\lambda}\right)+$$
$$-\frac{p}{\lambda}\ln\left(\frac{1}{2}+\frac{Mm^2}{8M_{\rm cr}p^2}+\frac{p_0}{2p}\right)+\frac{2M_{\rm cr}}{\lambda}\ln\left(\frac{4M_{\rm cr}(p_0+M)+m^2}{4M_{\rm cr}(p_0+M)-m^2}\right)+$$
$$+\frac{M}{\lambda}\ln\left(\frac{4M^3-3m^2M+(4M^2-m^2)p_0}{(p+M)^3}+\frac{m^2(4M^2+m^2)}{4M_{\rm cr}(p+M)^3}\right),$$

where $p_0 = \sqrt{(M+m^2/4M_{\rm cr})^2-m^2}$.

Part which survives in the limit $m \to 0$.