

Renormalized multiparticle amplitudes from exact Landau method

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Threshold multiparticle amplitudes

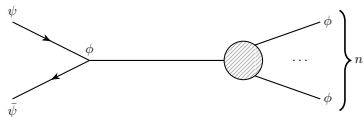
In 3 + 1 dimensions consider theory

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\phi)^2 - \frac{1}{2}\phi^2 - \frac{\lambda}{4}\phi^4, \quad m = 1.$$

Definition

$$\mathcal{A}_{1 \rightarrow n} = \langle n, E = n | \hat{\phi}(0) | 0 \rangle$$

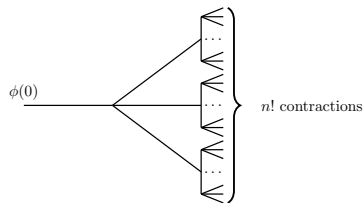
Appear in



$$\langle 0 | \hat{\phi}(x) \hat{\phi}(0) | 0 \rangle = \sum_n \langle 0 | \hat{\phi}(x) | n \rangle \langle n | \hat{\phi}(0) | 0 \rangle$$

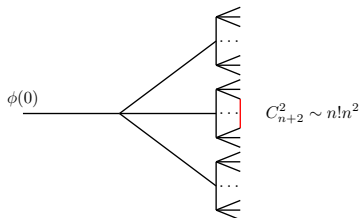
Growth of amplitudes at large n

Amplitudes at tree level (*Brown, 1992*)



$$\mathcal{A}_{1 \rightarrow n}^{\text{tree}} = n! \left(\frac{\lambda}{8} \right)^{\frac{n-1}{2}}$$

Loop corrections (*Argyres, 1993*)



$$\mathcal{A}_{1 \rightarrow n}^{l\text{-loop}} = \mathcal{A}_{1 \rightarrow n}^{\text{tree}} \cdot \lambda^l (B_{l,2l} \cdot n^{2l} + \dots + B_{l,0})$$

$\lambda n \gtrsim 1$ is non-perturbative even at $\lambda \ll 1$!

Double scaling semiclassical limit

Leading n contributions in **all loops** sum into an exponent
(Libanov et al., 1994)

$$\mathcal{A}_{1 \rightarrow n} = \mathcal{A}_{1 \rightarrow n}^{\text{tree}} \cdot e^{\frac{B}{\lambda}(\lambda n)^2} \cdot (1 + \dots), \quad B = \frac{\sqrt{27}}{64\pi^2} \left[i\pi - \ln(7 + 4\sqrt{3}) \right].$$

Conjecture

$$\mathcal{A}_{1 \rightarrow n} = \mathcal{A}_{1 \rightarrow n}^{\text{tree}} \cdot \exp \left(\frac{F_{-1}(\lambda n)}{\lambda} + F_0(\lambda n) + \lambda F_1(\lambda n) + \dots \right),$$

λn - fixed, $\lambda \ll 1$ — semiclassical limit

Perturbative result is restored in **double limit**

$$\lambda n \ll 1, \quad \lambda \ll 1.$$

Expression for $\mathcal{A}_{1 \rightarrow n}$

$$\mathcal{A}_{1 \rightarrow n} = \frac{\mathcal{A}_{1 \rightarrow n}^{\text{tree}}}{\sqrt{2}} \lim_{\tilde{\phi}_0 \rightarrow +\infty} \frac{\tilde{\phi}_0^2}{\lambda} \int_{-i\infty}^{+i\infty} dj e^{\frac{j\tilde{\phi}_0}{\lambda}} \int_0^{2\pi i} \frac{d\tau_0}{2\pi i} e^{\frac{\lambda n \tau_0}{\lambda}} \frac{\mathcal{A}_j}{\mathcal{A}_{j=0}}$$

- Scalar field is redefined $\phi(x) = \lambda^{-1/2} \tilde{\phi}(x)$;
- Action is rewritten in terms of $\tilde{\phi}$ and classical source is added

$$S[\phi] = \lambda^{-1} \int d^4x \left[(\partial_\mu \tilde{\phi}(x))^2 / 2 - \tilde{\phi}^2(x) / 2 - \tilde{\phi}^4(x) / 4 \right] + \lambda^{-1} i j \tilde{\phi}(0);$$

- \mathcal{A}_j is a sum of vacuum diagrams in presence of j on a classical background $\tilde{\phi}_B(t)$;
- Integrals in j and τ_0 are taken via saddle-point approximation.

After field rescale $\lambda \rightarrow 0$ become semiclassical limit
and λ now counts loops.

$$\mathcal{A}_{1 \rightarrow n} = \frac{\mathcal{A}_{1 \rightarrow n}^{\text{tree}}}{\sqrt{2}} \lim_{\tilde{\phi}_0 \rightarrow +\infty} \frac{\tilde{\phi}_0^2}{\lambda} \int_{-i\infty}^{+i\infty} dj e^{\frac{j\tilde{\phi}_0}{\lambda}} \int_0^{2\pi i} \frac{d\tau_0}{2\pi i} e^{\frac{\lambda n \tau_0}{\lambda}} \frac{\mathcal{A}_j}{\mathcal{A}_{j=0}}$$

Introduce coherent state

$$|\xi\rangle = \exp\left(4\pi^{3/2}\xi\hat{a}_{\vec{p}=\vec{0}}^\dagger\right)|0\rangle.$$

Amplitude from generating function

$$\langle n|\hat{\phi}(0)|0\rangle = \frac{n!}{2\pi i} \oint \frac{d\xi}{\xi^{n+1}} \langle \xi|\hat{\phi}(0)|0\rangle.$$

After redefinition $\xi = (8/\lambda)^{1/2}e^{-\tau_0}$

$$\mathcal{A}_{1 \rightarrow n} = -\frac{\mathcal{A}_{1 \rightarrow n}^{\text{tree}}\lambda^{1/2}}{2\sqrt{2}} \int_0^{2\pi i} \frac{d\tau_0}{2\pi i} e^{\frac{\lambda n \tau_0}{\lambda}} \langle \xi|\hat{\phi}(0)|0\rangle.$$

$$\mathcal{A}_{1 \rightarrow n} = \frac{\mathcal{A}_{1 \rightarrow n}^{\text{tree}}}{\sqrt{2}} \lim_{\tilde{\phi}_0 \rightarrow +\infty} \frac{\tilde{\phi}_0^2}{\lambda} \int_{-i\infty}^{+i\infty} dj e^{\frac{j\tilde{\phi}_0}{\lambda}} \int_0^{2\pi i} \frac{d\tau_0}{2\pi i} e^{\frac{\lambda n \tau_0}{\lambda}} \frac{\mathcal{A}_j}{\mathcal{A}_{j=0}}$$

Insert identity

$$1 = \int_{-\infty}^{+\infty} d\tilde{\phi}_0 \delta(\tilde{\phi}_0 - \tilde{\phi}(0)) = \int_{-\infty}^{+\infty} d\tilde{\phi}_0 \int_{-i\infty}^{+i\infty} \frac{dj}{2\pi i \lambda} e^{\frac{j[\tilde{\phi}_0 - \tilde{\phi}(0)]}{\lambda}}.$$

Now we can remove $\tilde{\phi}(0)$ from path integral for $\langle \xi | \hat{\phi}(0) | 0 \rangle$

$$\mathcal{A}_{1 \rightarrow n} = -\frac{\mathcal{A}_{1 \rightarrow n}^{\text{tree}}}{\sqrt{2}} \int_{-\infty}^{+\infty} \frac{d\tilde{\phi}_0 \tilde{\phi}_0}{4\pi i \lambda} \int_{-i\infty}^{+i\infty} dj e^{\frac{j\tilde{\phi}_0}{\lambda}} \int_0^{2\pi i} \frac{d\tau_0}{2\pi i} e^{\frac{\lambda n \tau_0}{\lambda}} \frac{\mathcal{A}_j}{\mathcal{A}_{j=0}}.$$

Exponent of $j\tilde{\phi}(0)$ is absorbed in \mathcal{A}_j .

Classical background

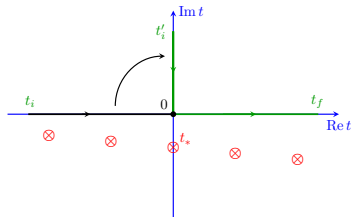
Path integral representation

$$\mathcal{A}_{1 \rightarrow n} = \mathcal{N} \int \mathcal{D}\tilde{\phi} \mathcal{D}\tilde{\phi}_{\text{in}} \mathcal{D}\tilde{\phi}_{\text{out}} \langle \xi | \tilde{\phi}_{\text{out}} \rangle \langle \tilde{\phi}_{\text{in}} | 0 \rangle e^{\frac{iS[\tilde{\phi}]}{\lambda} \Big|_{\tilde{\phi}_{\text{in}}}}^{\tilde{\phi}_{\text{out}}}$$

Integration of $\tilde{\phi}_{\text{in, out}}$ imposes BC-s ($\omega_k = \sqrt{\vec{k}^2 + 1}$)

$$\tilde{\phi}(t \rightarrow -\infty, \vec{x}) = b_{\vec{k}}^* e^{i\omega_k t} - \text{Wick rotation}$$

$$\tilde{\phi}(t \rightarrow +\infty, \vec{x}) = 2^{3/2} e^{-\tau_0 + it} + a_{-\vec{k}} e^{-i\omega_k t}$$



Shift $\tilde{\phi}(x) = \tilde{\phi}_B(t) + \tilde{\varphi}(x)$ and adiabatic switch-off $\lambda \rightarrow \lambda \cdot g(t) = \lambda e^{-2\epsilon t}$

$$\tilde{\phi}_B(t) = \sqrt{\frac{2}{g(t)}} \frac{i}{\sin[(1 + i\epsilon)t + i\tau_0]}, \quad \epsilon \ll 1.$$

Evaluation of \mathcal{A}_j



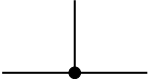

\mathcal{A}_j is sum of vacuum diagrams, thus

$$\mathcal{A}_j = e^{W[\lambda, j, \tau_0]}$$

We use BPHZ in initial theory

$$S[\tilde{\phi}] = -\frac{1}{2\lambda} \int d^4x \left\{ \tilde{\phi}(x)(\square + 1)\tilde{\phi}(x) + g(t) \frac{\tilde{\phi}^4(x)}{2} + \delta Z_\phi \tilde{\phi}(x)(\square + 1)\tilde{\phi}(x) - Z_\phi \delta m^2 \tilde{\phi}^2(x) + \frac{\delta Z_\lambda}{2} g(t) \tilde{\phi}^4(x) \right\}.$$

Then add term $ij\tilde{\phi}(0)$, shift $\tilde{\phi}(x) = \tilde{\phi}_B(t) + \tilde{\varphi}(x)$
and $G(x, x') \rightarrow G_B(x, x')$

			
$-j\lambda^{-1}\tilde{\phi}_B(0)$	$-j\lambda^{-1}\delta^{(4)}(x)$	$-6i\lambda^{-1} \int d^4x \tilde{\phi}_B(t) \cdot g(t)$	$-6i\lambda^{-1} \int d^4x g(t)$

Evaluation of \mathcal{A}_j

Loop expansion

$$W[\lambda, j, \tau_0] = \frac{W_{-1}[j, \tau_0]}{\lambda} + W_0[j, \tau_0] + \lambda \cdot W_1[j, \tau_0] + \dots;$$

Expansion in j

$$W_{-1}[j, \tau_0] = \text{⊗} + \text{⊗} \text{---} \text{⊗} + \mathcal{O}(j^3)$$

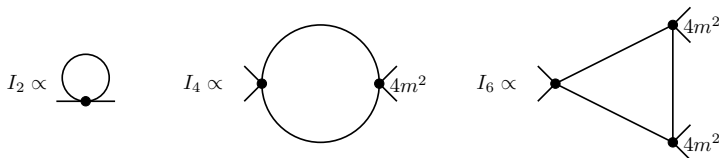
$$W_0[j, \tau_0] = \text{⊙} \text{---} \text{⊗} + \text{⊗} \text{---} \text{⊗} + \mathcal{O}(j^2)$$

$$W_0[j, \tau_0] = 3ij \int_{-\infty}^{+\infty} dt g(t) \tilde{\phi}_B(t) G_B(0, t, \vec{0}) \left[\left(I_2 - \frac{[\delta m^2]_1}{3} \right) - \tilde{\phi}_B^2(t) g(t) \left(\frac{3}{2} I_4 - \frac{[\delta Z_\lambda]_1}{3} - \frac{[\delta Z_\phi]_1}{3} \right) + \frac{9}{4} \tilde{\phi}_B^4(t) g^2(t) I_6 \right].$$

Renormalization prescriptions

$$W_0[j, \tau_0] = 3ij \int_{-\infty}^{+\infty} dt g(t) \tilde{\phi}_B(t) G_B(0, t, \vec{0}) \left[\left(I_2 - \frac{[\delta m^2]_1}{3} \right) - \tilde{\phi}_B^2(t) g(t) \left(\frac{3}{2} I_4 - \frac{[\delta Z_\lambda]_1}{3} - \frac{[\delta Z_\phi]_1}{3} \right) + \frac{9}{4} \tilde{\phi}_B^4(t) g^2(t) I_6 \right].$$

One-loop amplitude depend on integrals



On-shell renormalization: $[\delta Z_\phi]_1 = 0$, $[\delta m^2]_1 = 3I_2$.

Coupling renormalization: $I_4^{\text{REG}} = I_4 - \frac{2}{9}[\delta Z_\lambda]_1$.

After renormalization we can set $g(t) \rightarrow 1$.

Saddle-point approximation

$$\mathcal{A}_{1 \rightarrow n} = \frac{\mathcal{A}_{1 \rightarrow n}^{\text{tree}}}{\sqrt{2}} \lim_{\tilde{\phi}_0 \rightarrow +\infty} \frac{\tilde{\phi}_0^2}{\lambda} \int_{-i\infty}^{+i\infty} dj e^{\frac{j\tilde{\phi}_0}{\lambda}} \int_0^{2\pi i} \frac{d\tau_0}{2\pi i} e^{\frac{\lambda n \tau_0 + W_{-1}[j, \tau_0]}{\lambda} + W_0[j, \tau_0] + \dots}$$

Saddle-point equations

$$\begin{cases} \partial_j W_{-1} + \tilde{\phi}_0 = 0; \\ \partial_{\tau_0} W_{-1} + \lambda n = 0. \end{cases}$$

Solutions

$$\begin{cases} \tau_0[\tilde{\phi}_0, \lambda n] = \frac{\sqrt{2}}{\tilde{\phi}_0} + \mathcal{O}(\lambda n); \\ j[\tilde{\phi}_0, \lambda n] = -\frac{\sqrt{2}\lambda n}{\tilde{\phi}_0^2} + \mathcal{O}(\lambda n)^2. \end{cases}$$

Thus

- Limit $\tilde{\phi}_0 \rightarrow \infty$ is $\tau_0 \rightarrow 0$;
- Expansion in j is expansion in λn .

Limit from Exact Landau method

$$\mathcal{A}_{1 \rightarrow n} = \frac{\mathcal{A}_{1 \rightarrow n}^{\text{tree}}}{\sqrt{2}} \lim_{\tilde{\phi}_0 \rightarrow +\infty} \frac{\tilde{\phi}_0^2}{\lambda} \int_{-i\infty}^{+i\infty} dj e^{\frac{j\tilde{\phi}_0}{\lambda}} \int_0^{2\pi i} \frac{d\tau_0}{2\pi i} e^{\frac{\lambda n \tau_0 + W_{-1}[j, \tau_0]}{\lambda} + W_0[j, \tau_0] + \dots}$$

If one considers Quantum Mechanics (amplitudes in $0 + 1$ dimensions)

$$\mathcal{A}_{1 \rightarrow n} \xrightarrow{\text{QFT to QM}} \langle n | \hat{x}(0) | 0 \rangle \xrightarrow{\text{Adiabatics}} e^{\frac{i\alpha}{\epsilon}} \langle n | \hat{x} | 0 \rangle \Big|_{\lambda \cdot g(0) = \lambda}.$$

Highlighted matrix element is evaluated via Exact Landau method

$$\langle n | \hat{x} | 0 \rangle = -4\pi i \lim_{x \rightarrow +\infty} x^2 \psi_{n,I}(x) \psi_0(x) + \text{exp. small.}$$

Limit can be proved in QM and successfully used as a conjecture in QFT.

Exact Landau method: main idea

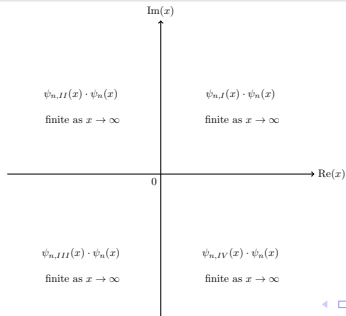
Wavefunction decomposition

- Consider 1d Schrödinger equation

$$\lambda^2 \frac{d^2 \psi_n}{dx^2} + (2\lambda E_n - x^2 - x^4/2) \psi_n = 0;$$

- There are four linearly independent solutions

$$\psi_n(x) = C_\alpha \psi_{n,\alpha}(x) + C_\beta \psi_{n,\beta}(x), \quad \alpha \neq \beta.$$

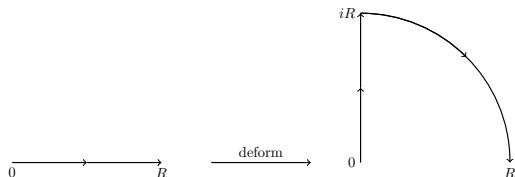


Contour deformation

Initial integral

$$\langle n|\hat{x}|0\rangle = \lim_{R \rightarrow +\infty} \int_{-R}^R dx \psi_n(x) \cdot x \cdot \psi_0(x).$$

is considered at finite R . After decompositions and contour deformations



it can be written as a residue at infinity:

$$\langle n|\hat{x}|0\rangle = -2\text{Res}_{x \rightarrow +\infty} \psi_{n,I}(x) \cdot x \cdot \psi_0(x) + \text{exp. small.}$$

$$\mathcal{A}_{1 \rightarrow n} = \mathcal{A}_{1 \rightarrow n}^{\text{tree}} \cdot \exp \left(\frac{F_{-1}(\lambda n)}{\lambda} + F_0(\lambda n) + \lambda F_1(\lambda n) + \dots \right).$$

In QFT

$$\mathcal{A}_{1 \rightarrow n} = \mathcal{A}_{1 \rightarrow n}^{\text{tree}} e^{\frac{9\lambda}{4} [(n-3)(n-1)I_6 - (n-1)I_4^{\text{REG}}]} (1 + \dots),$$

Coincides with *Voloshin, 1992* at $I_4^{\text{REG}} = 0$.

In QM (adiabatic phase included) :

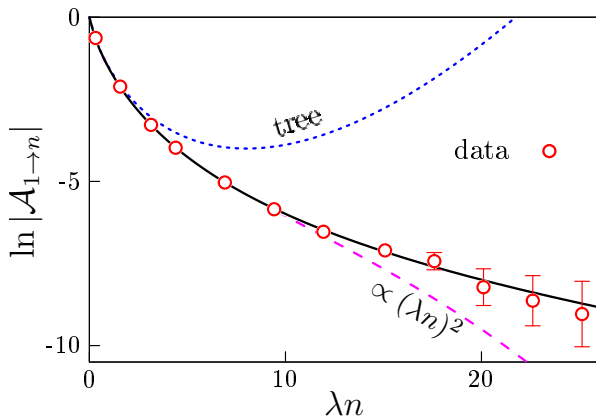
$$F_{-1} = \lambda^2 n^2 \left(-\frac{3i}{16\epsilon} - \frac{17}{32} \right) + \lambda^3 n^3 \left(\frac{17i}{256\epsilon} + \frac{125}{256} \right) + \mathcal{O}(\lambda n)^4;$$

$$F_0 = \lambda n \left(-\frac{3i}{16\epsilon} - \frac{5}{32} \right) + \mathcal{O}(\lambda n)^2.$$

Coincides with brute force resummation *Jaeckel, Schenk, 2018*.

Non-perturbative regime

One can **numerically solve** full set of saddle-point equations w.r.t $j, \tau_0, \tilde{\phi}(x)$ and then **take limit** $j \rightarrow 0$.
(*Son, '95; Demidov, BF, Levkov, 2023*)



- Exponentiation conjecture is successfully proved and exponent is calculated;
- Our formula works both in QFT and QM and can be used to obtain higher order corrections;
- Expression can be used in fully non-perturbative regime $\lambda n \gtrsim 1$.

Thank you for your attention!

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Backup slides

Classical momentum

$$p(x) = \sqrt{2\lambda E_n - x^2 - x^4/2}.$$

Solution of Schrödinger equation is given by asymptotic series normalized at x_0 ($p(x_0) \neq 0$)

$$\psi(x) \sim a_+(\lambda)\xi_{x_0}^+(\lambda, x) + a_-(\lambda)\xi_{x_0}^-(\lambda, x),$$

$$\xi_{x_0}^\pm(\lambda, x) = \frac{\exp\left(\pm i \cdot \lambda^{-1} \int_{x_0}^x \mathcal{P}(\lambda, x') dx'\right)}{\sqrt{\mathcal{P}(\lambda, x)}}, \quad \mathcal{P}^2 = p^2 + \lambda^2 \sqrt{\mathcal{P}} \left(\frac{1}{\sqrt{\mathcal{P}}}\right)'.$$

$\xi_{x_0}^\pm(\lambda, x)$ are multivalued analytic.

Stokes line

A curve L in \mathbb{C} starting from turning point $x_* : p(x_*) = 0$

$$\operatorname{Im} i \int_{x_*}^x p(x') dx' = 0, \quad x \in L.$$

Consider L connecting x_* with ∞ . For chosen determination of $p(x)$ one of ξ^\pm will decay at ∞ (**recessive**), another will grow (**dominant**).

Condition $\psi \xrightarrow[x \rightarrow \infty \text{ on } L]{} 0$ **unambiguously fixes** asymptotic expansion of a solution.

$$\psi(x) \sim a_+(\lambda) \xi_{x_0}^+(\lambda, x), \quad x \in L \quad \text{if e.g. } \xi^+ \text{ is recessive}$$

Recessive solution is Borel summable.

Relation between exponent and loop corrections

Exponential form

$$\frac{\mathcal{A}_{1 \rightarrow n}}{\mathcal{A}_{1 \rightarrow n}^{\text{tree}}} = \exp \left(F_{-1}(\lambda n)/\lambda + F_0(\lambda n) + \lambda F_1(\lambda n) + \lambda^2 F_2(\lambda n) + \dots \right),$$

Expansion in λn

$$F(\lambda n) = f_{-1,1}\lambda n + f_{-1,2}(\lambda n)^2 + f_{-1,3}(\lambda n)^3 + \dots,$$

$$F_0(\lambda n) = f_{0,1}\lambda n + f_{0,2}(\lambda n)^2 + \dots,$$

$$F_1(\lambda n) = f_{1,0} + \dots$$

Expansion of the exponent

$$e^{F_{-1}(\lambda n)/\lambda + F_0(\lambda n) + \lambda F_1(\lambda n) + \mathcal{O}(\lambda^2)} = 1 + \lambda(n^2 f_{-1,2} + n f_{0,1} + f_{1,0}) + \mathcal{O}(\lambda^2).$$