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Yaroslav Drachov

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Matrix integrals and their expression in radial-angular coordinates today is one of the most thoroughly studied subjects in the modern mathematical physics. Radial-angular decomposition of matrix derivatives remains quite obscure topic, though. The goal of the presented work is to clarify this matter. This is the key object to construct β-deformed $\widetilde{W}$-algebras.

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- symmetric matrices

$$dX = \left(\prod_i dX_{ii}\right) \prod_{\substack{i,j \\ i<j}} dX_{ij}, \qquad \beta = 1 \tag{1}$$

- Hermitian matrices

$$dX = \left(\prod_i dX_{ii}\right) \prod_{\substack{i,j \\ i<j}} d\Re X_{ij} d\Im X_{ij}, \qquad \beta = 2 \tag{2}$$

- quaternionic Hermitian matrices

$$dX = \left(\prod_i dX_{ii}\right) \prod_{i,j} dX_{ij}^{(0)} dX_{ij}^{(1)} dX_{ij}^{(2)} dX_{ij}^{(3)}, \qquad \beta = 4. \tag{3}$$

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X can be diagonalized

$$X = U \operatorname{diag}(\lambda_1, \ldots, \lambda_N) U^\dagger \tag{4}$$

where U is orthogonal ($\beta = 1$), unitary ($\beta = 2$), or quaternionic unitary ($\beta = 4$) and $\dagger$ means transposition, Hermitian conjugation and quaternionic Hermitian conjugation, correspondingly. Diagonalization provides one a set of radial-angular coordinates on the matrix spaces.

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Matrix integrals in radial-angular coordinates

Matrix measure can be easily rewritten in these coordiantes given the Jacobian of coordinate change (Dyson 1962)

$$J = \prod_{\substack{i,j \\ i<j}} |\lambda_i - \lambda_j|^\beta \tag{5}$$

and reads as

$$dX = \left(\prod_{\substack{i,j \\ i<j}} |\lambda_i - \lambda_j|^\beta \right) \left(\prod_i d\lambda_i \right) dU_{\text{Haar}}. \tag{6}$$

This fact is used extensively while evaluating matrix integrals.

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One can also consider matrix derivatives, whose matrix elements are

$$\left(\frac{\partial}{\partial X}\right)_{ji} = \frac{\partial}{\partial X_{ij}}, \tag{7}$$

higher matrix derivatives

$$\left(\frac{\partial}{\partial X}\right)^{n}, \tag{8}$$

where contraction of matrix indices is assumed, and matrix operators

$$\mathcal{O}\left(X, \frac{\partial}{\partial X}\right) = \sum_{\alpha} C_{\alpha} X^{\alpha_{\ell(\alpha)}} \frac{\partial}{\partial X} \cdots X^{\alpha_2} \frac{\partial}{\partial X} X^{\alpha_1}, \tag{9}$$

where sum is taken over weak compositions α.

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Matrix derivatives in radial-angular coordinates

- Little to no work is done about symmetric and quaternionic Hermitian matrices. Contemporary literature is still full of misconceptions on this topic (Srinivasan and Panda 2022).
- For Hermitian matrices the best we have is the expression for

$$\left(\frac{\partial^n}{\partial X^n}\right)_{ii}, \tag{10}$$

  on the space of functions of radial coordinates, see (Marshakov, Mironov, and Morozov 1992b).
- A lot is done about higher Calogero-Moser Hamiltonians, their expressions in terms of Dunkl operators with no direct relation to matrix derivatives, see e.g. (Mironov and Morozov 2023) and references therein. Serves as an inspiration for this work.

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$$F(X) = \sum_{k=0}^{\infty} \sum_{\lambda} C_{k,\lambda} X^k \prod_{i=1}^{\ell(\lambda)} \operatorname{tr} X^{\lambda_i}. \tag{11}$$

Their angular-radial decomposition reads as

$$F = U \operatorname{diag}(f_1, \ldots, f_N) U^{\dagger}. \tag{12}$$

We've rewritten its derivative in angular-radial coordinates

$$\frac{\partial}{\partial X} F = U \operatorname*{diag}_{1 \le i \le N} \left(\frac{\partial f_i}{\partial \lambda_i} + \beta \sum_{j \ne i} \frac{f_i - f_j}{\lambda_i - \lambda_j} \right) U^{\dagger}. \tag{13}$$

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Matrix derivatives in radial-angular coordinates

Repeated application of the proposed rule allows one to prove on the space of symmetric (in λ_i, $i = 1, \ldots, N$) functions

$$\frac{\partial^n}{\partial X^n} = U \operatorname*{diag}_{1 \le i \le N} \left(\frac{\mathfrak{d}^n}{\mathfrak{d}\lambda_i^n} \right) U^\dagger, \tag{14}$$

where $\mathfrak{d}/\mathfrak{d}\lambda_i$ is the Dunkl operator

$$\frac{\mathfrak{d}}{\mathfrak{d}\lambda_i} = \frac{\partial}{\partial \lambda_i} + \frac{\beta}{2} \sum_{j \ne i} \frac{1}{\lambda_i - \lambda_j} (1 - (ij)) \tag{15}$$

and (ij) is the operator of transposition of λ_i with λ_j. Even more general statement is shown to be true

$$\mathcal{O}\left(X, \frac{\partial}{\partial X} \right) = U \operatorname*{diag}_{1 \le i \le N} \left(\mathcal{O}\left(\lambda_i, \frac{\mathfrak{d}}{\mathfrak{d}\lambda_i} \right) \right) U^\dagger. \tag{16}$$

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$$H_n(y) = \int_{n\times n} e^{-\operatorname{tr} X^2/2} \det(X+y)dX. \tag{17}$$

They satisfy stationary Shrödinger equation

$$H_n''(x) - xH_n'(x) + nH_n(x) = 0. \tag{18}$$

This is matrix differential equation, though matrix is 1×1 now. The generalization to the $N\times N$ case are Gaussian Hermitian matrix models

$$Z(Y) = \int_{n\times n} e^{-\operatorname{tr} X^2/2} \det(X\otimes 1 + Y\otimes 1)dX. \tag{19}$$

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$$Z''(X) - XZ'(X) + nZ(X) = 0. \tag{20}$$

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$\widetilde{W}$-operators are defined as

$$\left(\frac{\partial}{\partial X} - \frac{1}{X}\frac{\partial}{\partial t_0}\right)^s Z(\mathbf{t}) = \sum_{k\geq 1-s} X^{-k-s}\widetilde{W}_s^{(k)}(\mathbf{t})Z(\mathbf{t}), \qquad t_k = -\frac{\operatorname{tr} X^k}{k} \quad (23)$$

They close in algebra (Marshakov, Mironov, and Morozov 1992a) in the same fashion as W-operators do. Moreover,

$$W^{(1)} = \widetilde{W}^{(1)}, \qquad W^{(2)} = \widetilde{W}^{(2)}, \qquad W^{(3)} \neq \widetilde{W}^{(3)}, \ldots \quad (24)$$

Hence the name. Any $\widetilde{W}^{(n)}$ operator can be obtained by a recursion relation

$$\widetilde{W}_k^{(n+1)} = \sum_{m=0}^{k-1}\frac{\partial}{\partial t_m}\widetilde{W}_{k-m}^{(n)} + \sum_{m\geq 0} m t_m \widetilde{W}_{k+m}^{(n)}. \quad (25)$$

For $W^{(n)}$ operators no such relation is known yet. It turns out that $\widetilde{W}^{(s)}(z) = \ddagger(i\partial\varphi)^s\ddagger/s$, where the new normal ordering $\ddagger\ldots\ddagger$ implies that the terms with "wrong" order of current modes should be just thrown out.

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$$\left(\frac{\partial}{\partial X} - \frac{1}{X}\frac{\partial}{\partial t_0}\right)^s Z(\mathbf{t}) = \sum_{k \geq 1-s} X^{-k-s} \widetilde{W}_s^{(k)}(\mathbf{t}) Z(\mathbf{t}), \qquad t_k = -\frac{\operatorname{tr} X^k}{k} \quad (23)$$

They close in algebra (Marshakov, Mironov, and Morozov 1992a) in the same fashion as W-operators do. Moreover,

$$W^{(1)} = \widetilde{W}^{(1)}, \qquad W^{(2)} = \widetilde{W}^{(2)}, \qquad W^{(3)} \neq \widetilde{W}^{(3)}, \ldots \quad (24)$$

Hence the name. Any $\widetilde{W}^{(n)}$ operator can be obtained by a recursion relation

$$\widetilde{W}_k^{(n+1)} = \sum_{m=0}^{k-1} \frac{\partial}{\partial t_m} \widetilde{W}_{k-m}^{(n)} + \sum_{m \geq 0} m t_m \widetilde{W}_{k+m}^{(n)}. \quad (25)$$

For $W^{(n)}$ operators no such relation is known yet. It turns out that $\widetilde{W}^{(s)}(z) = \ddagger (i\partial\varphi)^s \ddagger / s$, where the new normal ordering $\ddagger \ldots \ddagger$ implies that the terms with "wrong" order of current modes should be just thrown out.

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β-deformed $\widetilde{W}$-algebras

The following definition was shown to produce $\widetilde{W}$-operators for Ward identities of β-deformed two-matrix models

$$\left(\frac{\mathfrak{d}}{\mathfrak{d}\lambda_i} - \frac{1}{\lambda_i}\frac{\partial}{\partial t_0}\right)^s Z(\mathbf{t}) = \sum_{k\geq 1-s} \lambda_i^{-k-s}\widetilde{W}_s^{(k)}(\mathbf{t})Z(\mathbf{t}), \tag{26}$$

where $t_k = \sum_{i\geq 1}\lambda_i^{-k}/k$, $k \geq 1$, $\widetilde{W}$ recursion reads

$$\widetilde{W}_k^{(n+1)} = \beta\sum_{m\geq 0} mt_m\widetilde{W}_{k+m}^{(n)} + \sum_{m=1}^{k+n}\frac{\partial}{\partial t_m}\widetilde{W}_{k-m}^{(n)} + (1-\beta)(k+n+1)\widetilde{W}_k^{(n)}. \tag{27}$$

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The powerful technique of (β-deformed) matrix derivatives diagonalization was developed and applied to several problems, including description of β-deformed $\widetilde{W}$-algebras. Now we have cute and handy formulae like

$$\frac{\partial^n}{\partial X^n} = U \,\mathrm{diag}\left(\frac{\mathfrak{d}^n}{\mathfrak{d}\lambda_1^n}, \ldots, \frac{\mathfrak{d}^n}{\mathfrak{d}\lambda_N^n}\right) U^\dagger. \tag{28}$$

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