Path Integrals in the Future Light Cone

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Path integral measure

$$\int D\phi \exp(iS[\phi])F(\phi)\,, \tag{1}$$

1d QM: $\xi(\tau) \in C([0,T],\mathbb{R})$

$$\int d\xi \exp\left(\frac{i}{\hbar}\int_0^T \frac{1}{2}\dot{\xi}^2 d\tau\right) F(\xi)\,, \tag{2}$$

$\hbar \to -i\sigma^2$,

Wiener measure on the space $C([0,T],\mathbb{R})$

$$w_\sigma(d\xi) = d\xi \exp\left(-\frac{1}{2\sigma^2}\int_0^T d\tau\,\dot{\xi}^2\right)\,. \tag{3}$$

Quasi-invariance

Let $x \in \mathbb{R}$. Consider the shift transformation

$$T_a x \equiv x + a \,. \tag{4}$$

Gaussian measure

$$\gamma(dx) \equiv \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) dx \tag{5}$$

is quasi-invariant

$$\gamma(d(T_a x)) = \exp\left(-ax - \frac{a^2}{2}\right) \gamma(dx)\,. \tag{6}$$

Delta measure

$$\delta_{x_0}(dx) \equiv \delta(x - x_0) dx \tag{7}$$

is not quasi-invariant

$$\delta_{x_0}(d(T_a x)) = \delta_{x_0 - a}(dx) \neq \mathcal{P}(x)\delta_{x_0}(dx) \tag{8}$$

Quasi-invariance under the $Diff_+^3([0,T])$

The group $Diff_+^1([0,T])$ of the one-time differentiable diffeomorphisms $\varphi(\tau)$ of the segment $[0,T]$ with positive derivative $\dot{\varphi}(\tau) > 0$ acts on $C([0,T])$

$$(\varphi\xi)(\tau) = \frac{\xi(\varphi^{-1}(\tau))}{\sqrt{\dot{\varphi}^{-1}(\tau)}}\,. \tag{9}$$

The Wiener measure is quasi-invariant under this action

$$w_\sigma(d(\varphi\xi)) = \mathcal{P}_\varphi(\xi) w_\sigma(d\xi)\,, \tag{10}$$

where

$$\mathcal{P}_\varphi(\xi) = \sqrt[4]{\dot{\varphi}(0)\dot{\varphi}(T)}\ \exp\left(\frac{1}{4}\left[(\xi(T))^2\frac{\ddot{\varphi}(T)}{\dot{\varphi}(T)} - (\xi(0))^2\frac{\ddot{\varphi}(0)}{\dot{\varphi}(0)}\right]\right)\ \exp\left(\frac{1}{4}\int_0^T d\tau\ (\xi(\tau))^2 \mathcal{S}ch\{\varphi,\tau\}\right)$$

is Radon-Nikodym derivative

Quasi-invariant measure on $Diff_+^1([0,T])$

On compact Lie groups there exists Haar measure that is left- and right-invariant $h(d(g_0 g)) = h(d(g g_0)) = h(dg)$.

On the group $Diff_+^1([0,T])$ such measure does not exist. There exists measure $\mu_\sigma(d\varphi)$ that is quasi-invariant under left action of the subgroup $Diff_+^3([0,T])$

$$\mu_\sigma(d(f \circ \varphi)) = \tilde{\mathcal{P}}_f(\varphi)\mu_\sigma(d\varphi) \tag{11}$$

Quasi-invariant measure on $Diff_+^1([0, T])$

Consider the map

$$A : C([0, T]) \to Diff_+^1([0, T]) \,, \tag{12}$$

$$\varphi(\tau) = (A\xi)(\tau) = T \frac{\int_0^{\tau} \exp\left(\xi(\tau_1)\right) d\tau_1}{\int_0^{T} \exp\left(\xi(\tau_1)\right) d\tau_1} \,. \tag{13}$$

It transforms the Wiener measure $w_\sigma(d\xi)$ on $C([0, T])$ to the measure $\mu_\sigma(d\varphi)$ on $Diff_+^1([0, T])$.

E. T. Shavgulidze Russ. J. Math. Phys. (2000)

Orbit decomposition

Let $C_+([0, T]) \subset C([0, T])$ be the space of continuous positive functions on the interval $[0, T]$. $C_+([0, T])$ is decomposed into the orbits of the action

$$(\varphi\xi)(\tau) = \frac{\xi(\varphi^{-1}(\tau))}{\sqrt{\dot{\varphi}^{-1}(\tau)}}\,. \tag{14}$$

of the group $Diff_+^1([0, T])$ with definite value of the group invariant

$$\frac{1}{\rho^2} \equiv \frac{1}{T}\int_0^T \frac{d\tau}{\xi^2(\tau)}\,. \tag{15}$$

Orbit decomposition

Thus, we have one-to-one correspondence

$$\xi \longleftrightarrow (\rho, \varphi)\,, \tag{16}$$

$$\xi(\tau) = \frac{\rho}{\sqrt{\dot{\varphi}^{-1}(\tau)}}\,, \tag{17}$$

$$\frac{1}{\rho^2} = \frac{1}{T}\int_0^T \frac{d\tau_1}{\xi^2(\tau_1)}\,, \qquad \varphi^{-1}(\tau) = \frac{\int_0^\tau \frac{d\tau_1}{\xi^2(\tau_1)}}{\frac{1}{T}\int_0^T \frac{d\tau_1}{\xi^2(\tau_1)}}\,. \tag{18}$$

Decomposition of the Wiener measure

The Wiener measure on the space $C_+([0, T])$ can be represented as the decomposition over the orbits of the group $Diff_+^1([0, T])$

$$w_\sigma(d\xi) = \exp\left(-\frac{\sigma^2}{4\rho^2}\right) (\dot{\varphi}(0)\dot{\varphi}(T))^{3/4} \, \mu_{\frac{2\sigma}{\rho}}(d\varphi)\, d\rho\,. \qquad (19)$$

Belokurov and Shavgulidze
Mod. Phys. Lett. A 2018, arXiv:1806.05605.
Theor. Math. Phys. 2019, arXiv:1812.04039.

Two-dimensional case

Now consider the space $C([0,T],\mathbb{R}^2)$ of the continuous functions $\xi(\tau) = (\xi^0(\tau), \xi^1(\tau))$ defined on the $[0,T]$, with the values in $\mathbb{R}^2$. On this space, the Wiener measure is defined as follows

$$w_\sigma(d\xi) = \exp\left(-\frac{1}{2\sigma^2}\int_0^T d\tau \left[(\dot{\xi}^0(\tau))^2 + (\dot{\xi}^1(\tau))^2\right]\right) d\xi^0\, d\xi^1\,. \tag{20}$$

The measure (20) is invariant under the group $SO(2)$

$$w_\sigma(d(R_\gamma\xi)) = w_\sigma(d\xi)\,, \tag{21}$$

where $R_\gamma \in SO(2)$

$$R_\gamma\xi(\tau) \equiv \left(\xi^0(\tau)\cos\gamma - \xi^1(\tau)\sin\gamma,\ \xi^0(\tau)\sin\gamma + \xi^1(\tau)\cos\gamma\right)\,. \tag{22}$$

Group action

To define the action of the group of the diffeomorphisms $Diff_+^1([0,T])$ on $C([0,T],\mathbb{R}^2)$ it is convenient to use the polar coordinates defined by

$$\xi^0(\tau) = r(\tau)\cos\theta(\tau)\,, \qquad \xi^1(\tau) = r(\tau)\sin\theta(\tau)\,. \tag{23}$$

The group $Diff_+^1([0,T])$ acts as

$$(\varphi\xi)(\tau) = \left(\frac{r(\varphi^{-1}(\tau))}{\sqrt{\dot{\varphi}^{-1}(\tau)}}\cos\theta(\varphi^{-1}(\tau)), \frac{r(\varphi^{-1}(\tau))}{\sqrt{\dot{\varphi}^{-1}(\tau)}}\sin\theta(\varphi^{-1}(\tau))\right)\,. \tag{24}$$

Note that the action of the group $Diff_+^1([0,T])$ commutes with the action of the rotation group

$$R_\gamma\varphi\xi = \varphi R_\gamma\xi\,. \tag{25}$$

One-to-one correspondence

There is one to one correspondence

$$\xi \longleftrightarrow (\rho, \psi, \varphi, \alpha)\,, \tag{26}$$

where $\rho \in \mathbb{R}^+$, $\psi \in C_0([0,T])$, $\varphi \in Diff^1_+([0,T])$, $\alpha \in S^1$

$$\frac{1}{\rho^2} = \frac{1}{T}\int_0^T \frac{d\tau_1}{|\xi(\tau_1)|^2}\,, \qquad \varphi^{-1}(\tau) = \frac{\int_0^\tau \frac{d\tau_1}{|\xi(\tau_1)|^2}}{\frac{1}{T}\int_0^T \frac{d\tau_1}{|\xi(\tau_1)|^2}}\,, \tag{27}$$

$$\psi(s) = \arg z(\varphi(s)) - \arg z(0)\,, \qquad \alpha = \mathrm{Arg}\, z(0)\,, \tag{28}$$

where

$$z(\tau) \equiv \xi^0(\tau) + i\xi^1(\tau)\,, \qquad |\xi(\tau)|^2 \equiv (\xi^0(\tau))^2 + (\xi^1(\tau))^2\,. \tag{29}$$

Decomposition of the measure

The decomposition of the Wiener measure over the orbits of the group $Diff^1_+([0,T])$

$$w_\sigma(d\xi) = \rho \exp\left(-\frac{\sigma^2}{4\rho^2}\right) \dot{\varphi}(0)\dot{\varphi}(T)\, \mu_{\frac{2\sigma}{\rho}}(d\varphi)\, w^0_{\frac{\sigma}{\rho}}(d\psi)\, d\rho\, d\alpha\,. \tag{30}$$

V. V. Belokurov, E.T. Shavgulidze and N. E. Shavgulidze arXiv:2502.17201

The Cone

We define the future cone in the plane as follows

$$\mathrm{Cone} = \{(x^0, x^1) \in \mathbb{R}^2\,,\; x^0 > 0, |x^1| < x^0\}\,, \qquad \mathrm{Cone} \subset \mathbb{R}^2\,. \tag{31}$$

Let $C([0, T], \mathrm{Cone})$ be the space of continuous functions $\xi(\tau)$ with values in Cone. Consider the action of the group $SO(1,1)$ (the Lorentz group) on Cone

$$L_\gamma \xi(\tau) = (\,\xi^0(\tau)\,\cosh\gamma + \xi^1(\tau)\,\sinh\gamma,\; \xi^0(\tau)\,\sinh\gamma + \xi^1(\tau)\,\cosh\gamma)\,. \tag{32}$$

The action of $Diff_+^1([0,T])$ on $C([0,T],\mathrm{Cone})$

We define Minkowskian coordinates r, θ by the relations

$$\xi^0 = r(\tau)\cosh(\theta(\tau))\,, \qquad \xi^1 = r(\tau)\sinh(\theta(\tau))\,. \tag{33}$$

On the space $C([0,T],\mathrm{Cone})$, we define the action of the diffeomorphism group as follows

$$(\varphi\xi)(\tau) = \left(\frac{r(\varphi^{-1}(\tau))}{\sqrt{\dot{\varphi}^{-1}(\tau)}}\cosh\theta(\varphi^{-1}(\tau)),\ \frac{r(\varphi^{-1}(\tau))}{\sqrt{\dot{\varphi}^{-1}(\tau)}}\sinh\theta(\varphi^{-1}(\tau))\right)\,. \tag{34}$$

It commutes with the Lorentz transformations $L_\gamma\varphi = \varphi L_\gamma$

Orbit decomposition of $C([0, T], \mathrm{Cone})$

$$\xi \longleftrightarrow (\rho, \psi, \varphi)\,, \tag{35}$$

$$\frac{1}{\rho^2} = \frac{1}{T}\int_0^T \frac{d\tau_1}{(\xi^0(\tau_1))^2 - (\xi^1(\tau_1))^2}\,, \tag{36}$$

$$\varphi^{-1}(\tau) = \frac{\int_0^\tau \frac{d\tau_1}{(\xi^0(\tau_1))^2-(\xi^1(\tau_1))^2}}{\frac{1}{T}\int_0^T \frac{d\tau_1}{(\xi^0(\tau_1))^2-(\xi^1(\tau_1))^2}}\,,$$

$$\psi(s) = \mathrm{arctanh}\left(\frac{\xi^1(\varphi(s))}{\xi^0(\varphi(s))}\right)\,.$$

$$\xi(\tau) = \left(\frac{\rho}{\sqrt{\dot{\varphi}^{-1}(\tau)}}\cosh\psi(\varphi^{-1}(\tau))\,,\ \frac{\rho}{\sqrt{\dot{\varphi}^{-1}(\tau)}}\sinh\psi(\varphi^{-1}(\tau))\right)\,.$$

The measure on the $C([0,T],\mathrm{Cone})$

$$\tilde{w}_\sigma(d\xi) = \rho \exp\left(-\frac{\sigma^2}{4\rho^2}\right)\dot{\varphi}(0)\dot{\varphi}(T)\mu_{\frac{2\sigma}{\rho}}(d\varphi)w_{\frac{\sigma}{\rho}}(d\psi)d\rho\,. \qquad (37)$$

In Cartesian coordinates $\xi^0(\tau)$, $\xi^1(\tau)$, the measure $\tilde{w}_\sigma$ has the form

$$\tilde{w}_\sigma(d\xi) = \exp\left(-\frac{1}{2\sigma^2}\int_0^T \langle\dot{\xi},\dot{\xi}\rangle\; d\tau\right)d\xi\,, \qquad (38)$$

where

$$\langle\dot{\xi},\dot{\xi}\rangle \equiv (\dot{\xi}^0)^2 - (\dot{\xi}^1)^2 + 2\frac{(\xi^0\dot{\xi}^1 - \xi^1\dot{\xi}^0)^2}{(\xi^0)^2 - (\xi^1)^2}\,. \qquad (39)$$

The metric

$$ds^2 \equiv \langle dx, dx \rangle = (dx^0)^2 - (dx^1)^2 + 2\frac{(x^0 dx^1 - x^1 dx^0)^2}{(x^0)^2 - (x^1)^2} \,. \qquad (40)$$

In the Minkowskian coordinates (r, θ),

$$x^0 = r \cosh\theta \,, \qquad x^1 = r \sinh\theta \qquad (41)$$

it looks like the Euclidean metric on $\mathbb{R}^2$

$$ds^2 = dr^2 + r^2 d\theta^2 \,. \qquad (42)$$

Note that the "angular"coordinate θ is any real value ($\theta \in \mathbb{R}$).

The Cover

The space Cone with that metric admits an isometric mapping onto the infinite-sheeted covering of the plane. Consider a set of planes

$$\Pi_n = \{(r,\theta); r > 0, \theta \in [0, 2\pi)\} . \tag{43}$$

On every plane Π_n, the Euclidean metric of the form

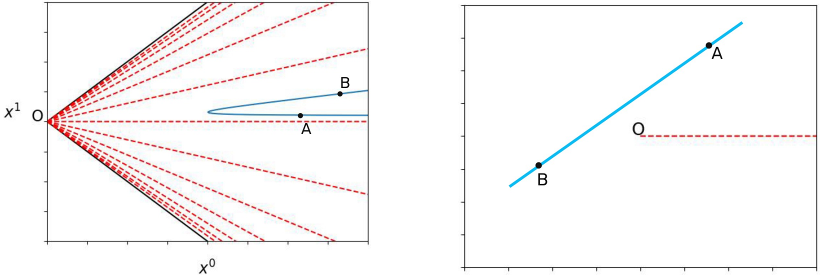
$$ds^2 = dr^2 + r^2 d\theta^2$$

is defined. On all planes Π_n, we make a cut along the direction $\theta = 0$ and glue the planes along these cuts, so

$$\lim_{\theta \to 2\pi} (r,\theta)_n = (r,0)_{n+1} . \tag{44}$$

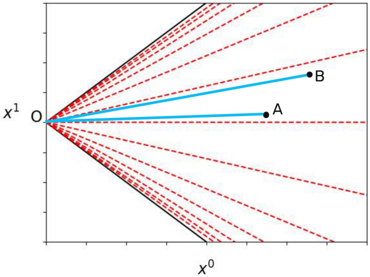
The collection of planes Π_n glued together in this manner will be denoted as Cover and called the infinite-sheeted covering of the plane.

The geodesics. $|\theta_A - \theta_B| < \pi$



(a) (b)

The geodesics. $\pi < |\theta_A - \theta_B| < 2\pi$



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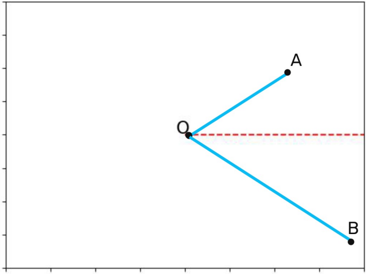
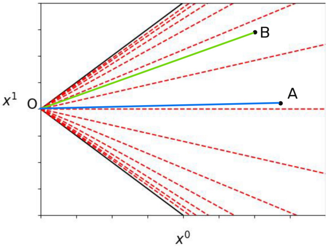


(b)

The geodesics. $2\pi < |\theta_A - \theta_B|$



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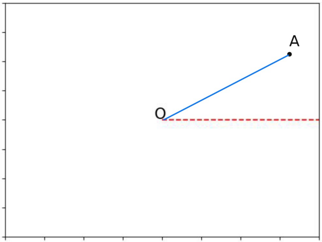

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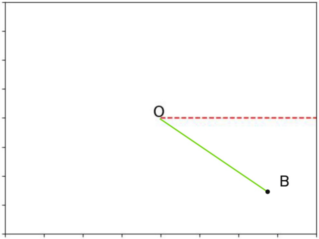
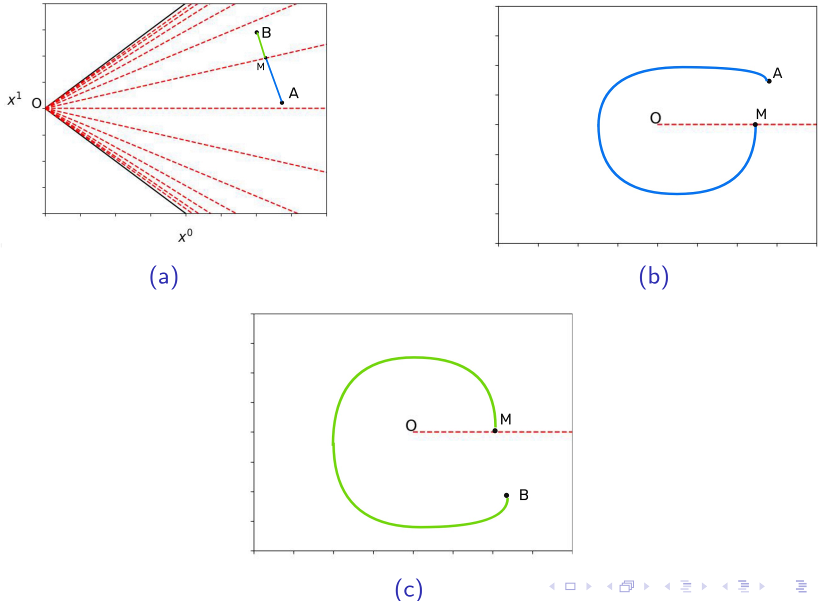
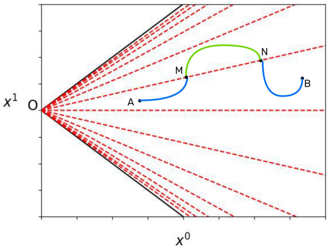


(c)

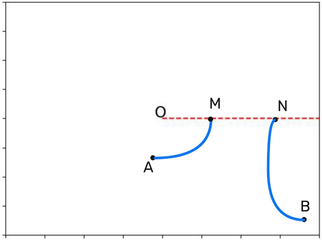
Arbitrary curve



Paths in the PI



(a)



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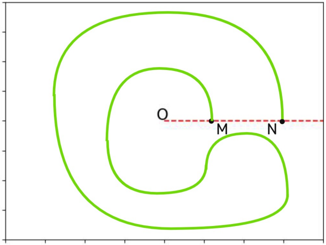

(c)

Directions for future research

- Extension to quantum field theory
- Complex rotation $\theta \to i\theta$ — Minkowskian space
- Euclidean QFT in the Rindler space

Thank you for your attention!

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