XXIII International Seminar on High-Energy Physics
"Quarks-2026"

High energy behaviour of Fermi theory

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Petrozavodsk, May 19, 2026

Motivation: non-renormalizable interactions

- Non-renormalizable theories naturally arise as low-energy limits of more fundamental models (effective field theories).
- Fermi's four-fermion theory describes weak interactions at low energies but is non-renormalizable ($[G] = -2$).
- It is important to find out whether radiative corrections can be computed and leading logarithms summed while staying within the framework of the model itself.

Goal

To present results of explicit loop calculations using the spinor-helicity formalism, the construction of recurrence relations based on the R'-operation, the derivation of generalized renormalization group equations, and the analysis of high-energy behaviour of amplitudes.

Fermi theory and summation of leading logarithms

- Lagrangian of the four-fermion interaction:

$$\mathcal{L} = i\bar{\Psi}\hat{\partial}\Psi - \frac{G}{\sqrt{2}}(\bar{\Psi}\hat{\mathcal{O}}\Psi)(\bar{\Psi}\hat{\mathcal{O}}\Psi)$$

- In dimensional regularization the leading logarithms $\log^n(E^2/\mu^2)$ correspond to leading poles $1/\epsilon^n$.
- Key idea: compute leading poles, then replace $1/\epsilon \to -\log(E^2/\mu^2)$.
- Goal: compute divergences up to three loops and construct recurrence relations for an arbitrary number of loops.

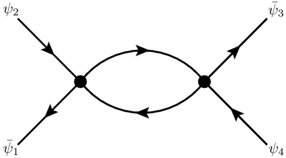
Spinor-helicity formalism and tree-level amplitude

- Spinor products: $\langle pk\rangle$, $[pk]$, $\langle pk\rangle[kp] = 2(p\cdot k)$.
- Fierz identity: $\langle p_1|\sigma^\mu|p_2]\,[p_3|\bar{\sigma}_\mu|p_4\rangle = 2\langle p_1 p_4\rangle[p_3 p_2]$.
- Tree-level amplitude in the scalar case ($\Gamma = I$):

$$A_4^{(0)} = \langle 12\rangle[34] - \langle 14\rangle[32]$$

- Only two independent spinor structures in all orders: $\langle 12\rangle[34]$ and $\langle 14\rangle[32]$.

Example: s-channel diagram computation



Product of two tree amplitudes:

$$A_4^{(0)}(\bar\psi_1\psi_2\bar\psi_5\psi_6) \times A_4^{(0)}(\bar\psi_8\psi_7\bar\psi_3\psi_4) = -i^2\left[\Gamma^{(1,2)}\Gamma^{(5,6)} - \Gamma^{(1,6)}\Gamma^{(5,2)}\right]$$
$$\times\left[\Gamma^{(8,7)}\Gamma^{(3,4)} - \Gamma^{(8,4)}\Gamma^{(3,7)}\right]$$
$$\times D^{\mu_1}_{(6,8)}(k)D^{\mu_2}_{(7,5)}(k+p)\,\bar{v}_L(p_1)u_R(p_2)\bar{v}_R(p_3)u_L(p_4).$$

Terms after expanding the brackets:

1) $-\langle 12\rangle\,\mathrm{Tr}[\sigma^{\mu_1}\bar\sigma^{\mu_2}]\,[34]\,I_1^{\mu_1\mu_2}$

2) $\langle 1|\sigma^{\mu_1}\bar\sigma^{\mu_2}|2\rangle\,[34]\,I_1^{\mu_1\mu_2}$,

3) $\langle 12\rangle\,[3|\sigma^{\mu_2}\bar\sigma^{\mu_1}|4]\,I_1^{\mu_1\mu_2}$

4) $-\langle 1|\sigma^{\mu_1}|4]\,[3|\bar\sigma^{\mu_2}|2\rangle\,I_1^{\mu_1\mu_2}$.

One-loop results

- Scalar operator: $A_4^{(1)} = \frac{5s}{6\epsilon}\langle 12\rangle[34] - \frac{5t}{6\epsilon}\langle 14\rangle[32]$.
- V–A operator (single structure $\langle 13\rangle[42]$):

$$A_4^{(1)} = -\frac{32}{3}\frac{s+t}{\epsilon}\langle 13\rangle[42].$$

Two-loop and three-loop corrections

- Two-loop scalar operator:

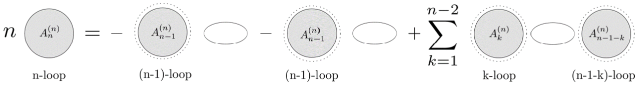
$$A_4^{(2)} = \frac{5}{12\epsilon^2}\left(s^2 + \tfrac{1}{2}t^2 + \tfrac{1}{2}u^2\right)\langle 12\rangle[34] - (t \leftrightarrow s).$$

- Three-loop scalar operator:

$$A_4^{(3)} = \frac{s}{432\epsilon^3}(196s^2 + 193st + 193t^2)\langle 12\rangle[34] - (t \leftrightarrow s).$$

- Structures $\langle 12\rangle[34]$ and $\langle 14\rangle[32]$ remain independent.

R'-operation and the leading pole



- R'-operation recursively removes UV subdivergences, leaving local counterterms (Bogoliubov–Parasiuk theorem).
- Requirement of locality links coefficients at different $1/\epsilon^n$ orders and yields:

$$A_n^{(n)} = (-1)^{n+1}\frac{A_1^{(n)}}{n}, \qquad \sum_{k=1}^{n} A_k^{(n)} = \frac{A_1^{(n)}}{n}.$$

- Linear term: $(n-1)$-loop counterterm + one-loop insertion. Nonlinear term (from three loops): two counterterms of orders k and $(n-1-k)$ connected by a one-loop diagram.
- Dashed lines = local counterterms (polynomials in momenta).

Recurrence relations: scalar operator

Scalar operator ($\langle 12\rangle[34]$):

$$nS_n(s,t) = \int_0^1 dx \sum_{k=0}^{n-1}\sum_{p=0}^{k} \frac{[s(-s-t)]^p[x(1-x)]^{p+1}}{p!(p+1)!}\left(s - t\left((p+1) + t'\tfrac{d}{dt'}\right)\right) \times \frac{d^pA_k(s,t',-s-t')}{dt'^p}\frac{d^pA_{n-1-k}(s,t',-s-t')}{dt'^p}\bigg|_{t'\to -sx},$$

$$nT_n(s,t) = -\int_0^1 dx \sum_{k=0}^{n-1}\sum_{p=0}^{k} \frac{[t(-s-t)]^p[x(1-x)]^{p+1}}{p!(p+1)!}\left(t\left((p+2) + s'\tfrac{d}{ds'}\right)\right) \times \frac{d^pA_k(s',t,-s'-t)}{ds'^p}\frac{d^pA_{n-1-k}(s',t,-s'-t)}{ds'^p}\bigg|_{s'\to -tx} + \text{similar terms with } u.$$

- The structure $\langle 14\rangle[32]$ is obtained by $p_2 \leftrightarrow p_4$ (with a sign).

Recurrence relations: V–A operator

V–A operator (⟨13⟩[42]):

$$nS_n(s,t,u) = -8s\int_0^1 dx \sum_{k=0}^{n-1}\sum_{p=0}^{k} \frac{[s(-s-u)]^p[x(1-x)]^{p+1}}{p!(p+1)!(p+2)^{-1}} \frac{d^pA_k}{du'^p}\frac{d^pA_{n-1-k}}{du'^p}\Big|_{u'\to -sx},$$

$$nU_n(s,t,u) = 8u\int_0^1 dx \sum_{k=0}^{n-1}\sum_{p=0}^{k} \frac{[u(-s-u)]^p[x(1-x)]^{p+1}}{p!(p+1)!(p+3)^{-1}} \frac{d^pA_k}{ds'^p}\frac{d^pA_{n-1-k}}{ds'^p}\Big|_{s'\to -ux}$$

$$+\,\text{similar contribution from } t, u.$$

- Single spinor structure, factorization $A_k = S_k + T_k + U_k$.

From recurrence relations to RG equations

- Introducing generating function:

$$A(s,t,u;z) = \sum_{n=0}^{\infty} A_n(s,t,u)(-z)^n, \quad z = \frac{\bar{G}}{\epsilon}$$

- Multiply recurrence relations by $(-z)^{n-1}$ and sum over n.
- Obtain **integro-differential equation** – generalized RG equation.
- Substituting $z \to -\bar{G}\log(s/\mu^2)$ yields high-energy behaviour.

Generalized RG equation: scalar operator

Equation for the amplitude generating function $A(s,t,u;z)$ in the case of scalar operator:

$$
\begin{aligned}
-\frac{dA(s,t,u)}{dz} = & \int_0^1 dx \sum_{p=0}^{\infty} \frac{\left[s(-s-t)\right]^p \left[x(1-x)\right]^{p+1}}{p!\,(p+1)!} \left(s - t\left((p+1) + t'\,\tfrac{d}{dt'}\right)\right) \left(\frac{d^p A(s,t',-s-t')}{dt'^p}\right)^2 \Bigg|_{t'\to -sx} \\
& + \text{ analogous term with } t \leftrightarrow u \\
& - \int_0^1 dx \sum_{p=0}^{\infty} \frac{\left[t(-s-t)\right]^p \left[x(1-x)\right]^{p+1}}{p!\,(p+1)!} \left(t\left((p+2) + s'\,\tfrac{d}{ds'}\right)\right) \left(\frac{d^p A(s',t,-s'-t)}{ds'^p}\right)^2 \Bigg|_{s'\to -tx} \\
& - \int_0^1 dx \sum_{p=0}^{\infty} \frac{\left[u(-s-u)\right]^p \left[x(1-x)\right]^{p+1}}{p!\,(p+1)!} \left(u\left((p+2) + s'\,\tfrac{d}{ds'}\right)\right) \left(\frac{d^p A(s',-s'-u,u)}{ds'^p}\right)^2 \Bigg|_{s'\to -ux} .
\end{aligned}
$$

- This equation sums leading poles to all orders of perturbation theory.

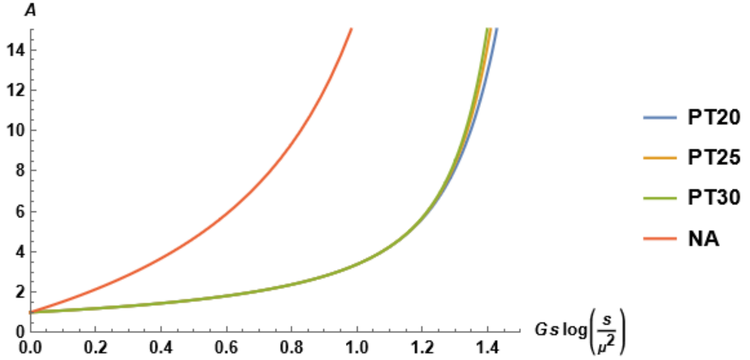
Generalized RG equation: V–A operator

Equation for the amplitude generating function $A(s,t,u;z)$ in the case of V–A operator:

$$
\begin{aligned}
-\frac{dA(s,t,u)}{dz} = &-8s\int_0^1 dx \sum_{p=0}^{\infty} \frac{[s(-s-u)]^p[x(1-x)]^{p+1}}{p!(p+1)!(p+2)^{-1}} \left(\frac{d^pA(s,-s-u',u')}{du'^p}\right)^2\Bigg|_{u'\to -sx} \\
&-8t\int_0^1 dx \sum_{p=0}^{\infty} \frac{[t(-t-u)]^p[x(1-x)]^{p+1}}{p!(p+1)!(p+2)^{-1}} \left(\frac{d^pA(-t-u',t,u')}{du'^p}\right)^2\Bigg|_{u'\to -tx} \\
&+8u\int_0^1 dx \sum_{p=0}^{\infty} \frac{[u(-s-u)]^p[x(1-x)]^{p+1}}{p!(p+1)!(p+3)^{-1}} \left(\frac{d^pA(s',-s'-u,u)}{ds'^p}\right)^2\Bigg|_{s'\to -ux} \\
&+8u\int_0^1 dx \sum_{p=0}^{\infty} \frac{[u(-t-u)]^p[x(1-x)]^{p+1}}{p!(p+1)!(p+3)^{-1}} \left(\frac{d^pA(-t'-u,t',u)}{dt'^p}\right)^2\Bigg|_{t'\to -ux} .
\end{aligned}
$$

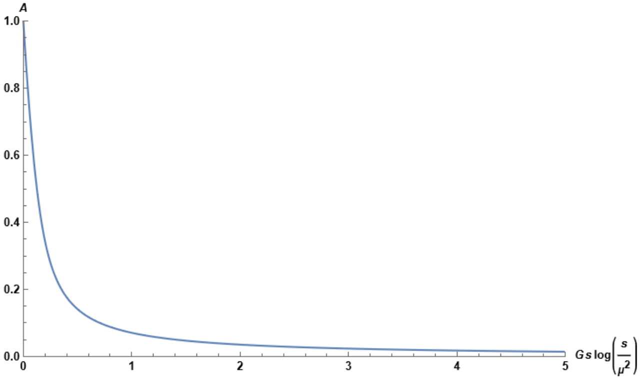
- After the substitution $z \to -\bar{G}\log(s/\mu^2)$ this equation describes the high-energy behaviour of the amplitude.

Numerical solutions: scalar operator



- Sign-constant series, growth as $y \to \infty$.
- Tendency toward a Landau pole.

Numerical solutions: V–A operator



- Alternating series, decrease, $A(y) \approx k/y$, $k \approx 1/15$.
- Asymptotic freedom.

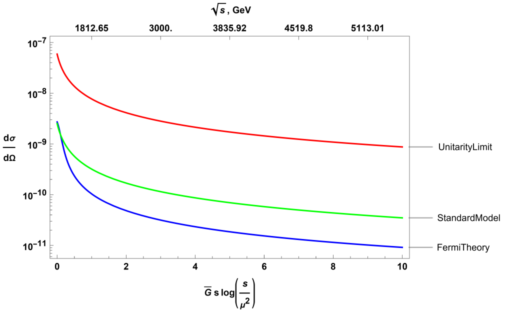
Restoration of unitarity

- Tree-level cross section violates the unitarity bound $\frac{d\sigma}{d\Omega} < \frac{1}{4\pi^2 s}$ for $s \gtrsim (1500\ \mathrm{GeV})^2$.
- Summation of leading logarithms in the V–A case gives:

$$\frac{d\sigma}{d\Omega} \approx \frac{C^2}{s \log^2(s/\mu^2)}, \quad C = \frac{\pi k}{2} \sim 0.1,$$

which satisfies the unitarity bound at high energies.

- Radiative corrections restore unitarity.



Main results

1. Radiative corrections in Fermi theory for scalar and V–A operators have been computed up to three loops using the spinor-helicity formalism.
2. Based on the R'-operation, recurrence relations have been constructed that allow one to find the leading divergence in any order.
3. Generalized integro-differential RG equations have been derived. Analysis of their solutions revealed a Landau pole for the scalar operator and asymptotic freedom for the V–A operator.
4. Summation of leading logarithms restores unitarity in the V–A theory; the result agrees with the Standard Model.

Thank you for your attention!