Applying geometrical methods for decomposition of single-qubit gates

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joint work with
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The problem setting

- Mathematical model
- Nonlinear optimization problem
- Geometrical approach

Physical system

In certain quantum systems, one-qubit gates can be performed using a controlling radio-frequency field or Raman excitation. In the mathematical model, this corresponds to rotating the qubit around certain axes, in our case, around $\mathcal{X}$ and $\mathcal{Y}$ axes, by an angle proportional to the irradiation time, i.e. all angles $\theta_x, \theta_y \geq 0$.

Often, to reduce thermal stresses, a single large pulse is replaced by a series of short pulses. In our case, this approach is applied to designing imperfect gates of a given accuracy by carrying out a sequence of unitary operations that correspond to successive rotations around two perpendicular axes.

Gates

Under a generic gate we assume any element $g \in SU(2)$ given by 2×2 matrix of the form

$$g = \begin{pmatrix} \alpha & \beta \\ -\overline{\beta} & \overline{\alpha} \end{pmatrix}, \quad |\alpha|^2 + |\beta|^2 = 1, \quad \alpha, \beta \in \mathbb{C}.$$

Each gate can be represented by the rotations around axes $\mathcal{X}, \mathcal{Y}, \mathcal{Z}$ (Euler convention 1-2-3) or around axes $\mathcal{X}, \mathcal{Y}, \mathcal{X}$, i.e. convention 1-2-1. We take the last one. Then for any $g \in SU(2)$ we obtain uniquely $\theta_1, \theta_2, \theta_3$ such that $g = U_3 U_2 U_1 = X_3 Y_2 X_1 = \exp\left(\frac{\imath}{2}\theta_3\sigma_1\right) \exp\left(\frac{\imath}{2}\theta_2\sigma_2\right) \exp\left(\frac{\imath}{2}\theta_1\sigma_1\right)$, where $\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma_2 = \begin{pmatrix} 0 & -\imath \\ \imath & 0 \end{pmatrix}$, $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ are Pauli matrices. Our goal to find for any gate the decomposition of the type $g = U_N U_{N-1} \cdots U_2 U_1$, $N \geq 6$.

Fidelity function for any two gates

Let g_0 be some element of SU(2) group, called afterwords the *perfect gate*. We approximate it with an *imperfect gate* g_ϵ which is experimentally prepared with a systematic error $0 \leq \varepsilon << 1$. To guess the quality of the approximation we define the gate fidelity for any two gates as

$$\mathcal{F}(g_\varepsilon, g_0) = \frac{1}{4}|\mathrm{tr}(g_\varepsilon g_0^\dagger)|^2, \quad 0 \leq \mathcal{F}(g_\varepsilon, g_0) \leq 1, \quad \mathcal{F}(g_0, g_0) = 1.$$

Definition

The gate $g_\varepsilon \in$ SU(2) is the gate of the n-th order accuracy if

$$\mathcal{F}(g_\varepsilon, g_0) = \frac{1}{4}|\mathrm{tr}(g_\varepsilon g_0^\dagger)|^2 = 1 - O(\varepsilon^n), \quad 0 \leq \varepsilon < 0$$

A basis and Euler parameters for a gate

The imperfect gate g_ε is constructed as the product of N successive SU(2) rotations about the $\mathcal{X}$ and $\mathcal{Y}$ axes by the angles whose values depend linearly on the systematic error, i.e., $(1+\varepsilon)\theta_1, \ldots, (1+\varepsilon)\theta_N$, for an odd or even number N correspondingly

$$g_\varepsilon = \begin{cases} X((1+\varepsilon)\theta_N)Y((1+\varepsilon)\theta_{N-1})\cdots Y((1+\varepsilon)\theta_2)X((1+\varepsilon)\theta_1)\,, \\ Y((1+\varepsilon)\theta_N)X((1+\varepsilon)\theta_{N-1})\cdots Y((1+\varepsilon)\theta_2)X((1+\varepsilon)\theta_1)\,. \end{cases}$$

Perfect and imperfect gates are uniquely defined by a corresponding set of four Euler parameters (u_1, u_2, u_3, u_4)

$$\begin{aligned} g_0 &= u_4\,\mathbb{I}_2 - \imath\, u_1\sigma_1 - \imath\, u_2\sigma_2 - \imath\, u_3\sigma_3\,, \\ g_\varepsilon &= v_4(\varepsilon)\,\mathbb{I}_2 - \imath\, v_1(\varepsilon)\sigma_1 - \imath\, v_2(\varepsilon)\sigma_2 - \imath\, v_3(\varepsilon)\sigma_3\,. \end{aligned}$$

We use them to compare two gates, then $g_\varepsilon|_{\varepsilon=0} = g_0$ if for all $v_\alpha|_{\varepsilon=0} = u_\alpha$, $\alpha = 1, 2, 3, 4$.

On the first step to compare the gates we used instead fidelity the Euler parameters. It narrowing the set of possible solutions but simplified our problem. The condition $g_\varepsilon|_{\varepsilon=0} = g_0$ if for all $v_\alpha|_{\varepsilon=0} = u_\alpha$, $\alpha = 1, 2, 3, 4$ guarantied us that at least in one point by $\varepsilon = 0$ the perfect and imperfect gates coincides. But we need also a stability of the solutions around this point. Because of that we introduce the square root of the fidelity $W_\varepsilon = \frac{1}{2}\mathrm{tr}\left(g_\varepsilon\, g_0^\dagger\right)$, $W_\varepsilon|_{\varepsilon=0} = W_0 = 1$, and study it Taylor series. W_ε is nothing else as the cosines between unit vectors $u = (u_1, u_2, u_3, u_4)$ and $v(\varepsilon) = (v_1(\varepsilon), v_2(\varepsilon), v_3(\varepsilon), v_4(\varepsilon))$. The derivative with respect to ε reduces to the following Lie derivative $\mathcal{L}_\Theta$, i.e. $\frac{d}{d\varepsilon} \quad \longrightarrow \quad \mathcal{L}_\Theta = \sum_{i=1}^{N} \theta_i \frac{\partial}{\partial \theta_i}$. If we demand that the derivatives of W_ε by $\varepsilon = 0$ vanishing up to the order k, then for the fidelity gilt $\mathcal{F}(g_\varepsilon, g_0) = 1 - O(\varepsilon^{2(k+1)})$ and we prove following Lemma.

The stability of the approximation

Lemma

The $2(k+1)$-order accuracy approximation of the gate g_0 by g_ε, designed according to the convention 1-2-1, is guaranteed if its Euler parameters $v_\alpha(\theta_1, \ldots, \theta_N)$ satisfy the following system of equations:

$$\begin{aligned} v_\alpha(\theta_1, \ldots, \theta_N)|_{\varepsilon=0} &= u_\alpha\,, & \alpha = 1,2,3,4 \\ (\mathcal{L}_\Theta)^m\, v_\alpha(\theta_1, \ldots, \theta_N)|_{\varepsilon=0} &= 0\,, & m = 1,2,\ldots,k\,. \end{aligned}$$

where $\mathcal{L}_\Theta$ denotes the following Lie derivative

$$\mathcal{L}_\Theta = \sum_{i=1}^{N} \theta_i \frac{\partial}{\partial \theta_i}\,.$$

Setting of the optimization problem

Now we can formulate the optimization problem. We introduce the utility (or objective) function Υ with properties $\Upsilon(\theta) \geq 0$, $\Upsilon(\theta) = 0$ on solutions, depending on $\theta = (\theta_1, \theta_2, \ldots, \theta_N)$, of the **Optimization Problem**

$$\Upsilon(\theta) = 1 - \mathcal{F}(g_\varepsilon, g_0)|_{\varepsilon=0} + \sum_{k=1}^{m} \left|\mathrm{tr}(\mathcal{L}_\Theta^k\, g_\varepsilon)\right|_{\varepsilon=0}^2$$

$$\theta = \arg\min(\Upsilon(\theta)), \quad \text{subject to } \min \sum_{i=1}^{N} \theta_i > 0, \quad i = 1, 2, \ldots, N,$$

$$\max m, \quad m \in \mathbb{N}, \quad \min N, \quad \text{by } N \geq 6, \quad \Upsilon(\theta) \geq 0, \quad \Upsilon(\theta) = 0.$$

It is a nonlinear optimization problem, and is difficult to solve. To simplify the Problem we fix first the N with $N = 6$, or $N = 7$. Then we fix also order of derivatives $m = 1$, and look for gates of the fours-order accuracy. Now we try to describe the set of feasible solutions for this optimization problem.

Problems by the numerical realization

G. Struchalin was the first who tried to solve the similar optimization problem (m=1, N=6,7) with numerical methods for different gates. For many gates decomposition in $N = 6$ was successful and optimal solutions were presented, but for some gates, like $g_s = X_3(\frac{\pi}{2})Y_2(\frac{\pi}{2})X_1(\frac{\pi}{2})$ also after 10^6 attempts with different solvers the optimal solution was not found. In the case $N = 7$ for all studied gates the optimal solutions were presented.
We hope to give description for a moduli space of feasible solutions and some criteria if the optimization problem solvable or not.

Decomposition of the gate

We look for feasible solutions in simplest case, where at least the condition $v_\alpha(\theta_1, \ldots, \theta_N)|_{\varepsilon=0} = u_\alpha\,, \alpha = 1, 2, 3, 4$ is satisfied. This is the system of the four equation, and in the case $N = 6$ it can be solved also with analytical methods. Our goal to obtain for any gate a decomposition of the type

$$g = \begin{cases} X_N Y_{N-1} \cdots Y_2 X_1\,, & \text{for N odd} \\ Y_N X_{N-1} \cdots Y_2 X_1\,, & \text{for N even,} \end{cases}$$

where for all rotations $\theta_\alpha > 0$.

The local isomorphism between $SU(2)$ and $SO(3)$

To formulate the conditions on the existence such decompositions we first use the local isomorphism between the groups $SU(2)$ and $SO(3)$. It is a two-to-one holomorphic mapping of the group $SU(2)$ onto the group SO(3), because of two elements $\pm g \in SU(2)$ have the same image. This mapping we present in the form

$$g \to R = \{R(g)|\ R_{lm} = \frac{1}{2} tr(\sigma_l g \sigma_m g^{-1}),\ l, m = 1, 2, 3\},$$
$$g \in SU(2), \quad R \in SO(3).$$

The decomposition of a gate $g \in SU(2)$ corresponds to the decomposition of the corresponding element $R \in SO(3)$

$$R = R_N R_{N-1} \dots R_2 R_1, \quad R_k \in SO(3),\ k = 1, 2, \dots N.$$

Geometrical way to construct feasible solutions

Our goal to define a set of the feasible solutions. Mostly gates are given as result of three rotations, if not then we use Euler theorem and present the target gate in this form.
Now we chopping angles and add parameters. Here on Figure is the original path $\mathcal{ABCD}$, and the resulting path $\mathcal{AEFGHD}$ after its chopping. We added two free parameters.

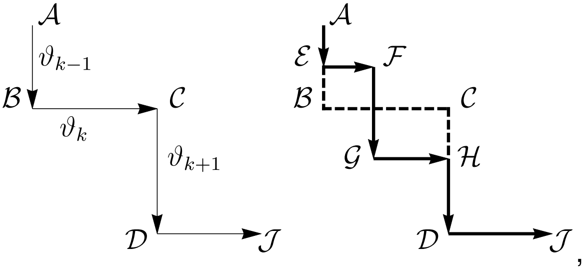

Figure: Method to change a decomposition from 3 to 5 segments.

Second way to get feasible solutions

In previous case with the chopping method we obtain feasible solutions with odd numbers of angles ($N = 3 \to 5 \to 7 \dots$). Now we proposed a method of identical right triangles.

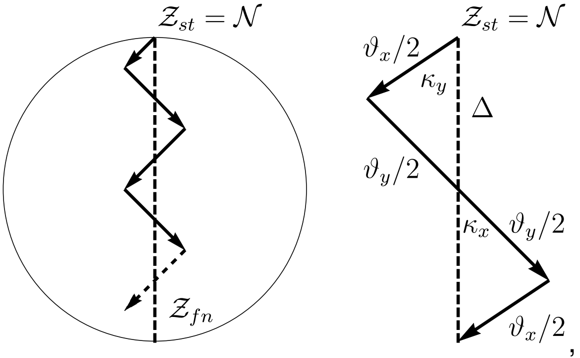

Figure: Path along the legs of identical right triangles

The problem of a stability described solutions

Both described methods guarantied us that the first condition of the optimization problem is satisfied. It means that in both cases the fidelity function equal to one. If we take in account the systematic error ε by preparation each of this angles, we see that the achieved accuracy is of the order ε^2. For the proper further work it is necessary to have at least the accuracy order of ε^4. All the 6 or 7 angles by described methods are smaller as $\pi/2$ and one can prove that the Lie derivative will be never zero by these angles. To get feasible solutions for optimization problem we should transform gates decomposition.
We look closer on rotations described with the group $SO(3)$.

The meaning of the first stability condition

In the simplest case for an imperfect gate $g_\varepsilon = v_4(\varepsilon)\,\mathbb{I}_2 - \imath\, v_1(\varepsilon)\sigma_1 - \imath\, v_2(\varepsilon)\sigma_2 - \imath\, v_3(\varepsilon)\sigma_3$ the first stability condition demands that the first derivative of the gate after ε vanishing

$$\sum_{i=1}^{N} \theta_i \frac{\partial}{\partial \theta_i} v_\alpha(\theta_1, \ldots, \theta_N)|_{\varepsilon=0} = 0\,.$$

We use a local isomorphismus of $SU(2)$ and $SO(3)$ and provide a geometrical interpretation of this condition on $SO(3)$. Each transformation with an element of $SO(3)$ means a rotation on a geodesic (a large circle on unit sphere), in positive directions about about $\mathcal{X}$ and $\mathcal{Y}$ axes, on positive angles $\theta_x, \theta_y \geq 0$.

A linear and angular velocity of a point on the unit sphere

We introduce a fixed frame $\mathbb{E} = (\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$, then each point on the sphere can be described as a column vector $\mathbf{X}(t) \in \mathbb{R}^3$. The rotating point on the sphere has a linear instant velocity $\frac{d\mathbf{X}}{dt}$ and an instant angular velocity omega $\mathbf{w}(t)$ which are related by

$$\frac{d\mathbf{X}}{dt} = [\mathbf{w}(t), \mathbf{X}(t)], \text{or} \qquad d\mathbf{X} = [\mathbf{w}(t)dt, \mathbf{X}(t)]$$

The middle angular velocity $\mathbf{P}_\Sigma$ over a segment of trajectory is equal to the integral

$$\mathbf{P}_\Sigma = \frac{1}{t_i - t_{i-1}} \int_{t_{i-1}}^{t_i} \mathbf{w}(t)dt,$$

if the instant angular velocity is constant then $\mathbf{P}_\Sigma = \mathbf{w}_i = |\mathbf{w}_i|\ \mathbf{p}_i$.

A middle angular velocity and the optimization problem

On a typical trajectory for an imperfect gate the instant angular velocity $\mathbf{w}(t)$ is a piece-wise constant function because of that the middle angular velocity take a form

$$\mathbf{P}_\Sigma(g_\varepsilon) = \sum_{i=1}^{N} \frac{1}{t_i - t_{i-1}} \int_{t_{i-1}}^{t_i} \mathbf{w}(t)dt = \sum_{i=1}^{N} |\mathbf{w}_i| \, \mathbf{p}_i = \sum_{i=1}^{N} \theta_i \, \mathbf{p}_i,$$

where θ_i is the absolute value of the angle of rotation on the i-th piece of trajectory, and $\mathbf{p}_i$ is unit vector of the rotation axis, polara of this rotation. But $\mathbf{P}_\Sigma(g_\varepsilon)$, the weighted sum of the trajectory polaras, is exact equivalent the value which should be vanish to get better decomposition stability.

Cyclic trajectories, compensating procedure

Before we hat a situation that g_ε- gate decomposition possess desired fidelity but not enough stability. Now we try to change stability without to destroy the achieved condition
$\mathcal{F}(g_\varepsilon, g_0)|_{\varepsilon=0} = 1$.
It is possible if we add to the decomposition cyclic rotations in a right sequence which formed closed trajectories. The fidelity will not be changed but the first derivative, or a middle angular velocity will be changed. First, we prove that such cyclic trajectories are exist with all the right properties. Let we take a chain of 6 rotations on the angles $\theta_i = \pi/2$ each, just about $\mathcal{X}$ and $\mathcal{Y}$ axes in positive direction, started and finished in the same point. We call it cyclic sextet, if all rotations follows in a right way: for each 3 segments the first and the last ones started , or finished in different half spheres.
Let it will be North pol **N**.

Manifold of cyclic sextets

Theorem. The cyclic sextets exist and build a 3 dimensional manifold (where as parameter we can use elements of $SO(3)$).
Existence. 1. Segment from **N** to the equator on Greenwich meridian, 2. Rotate on the equator till 90 degree meridian, 3. rotate from equator to South Pol **S**, 4. rotate from **S** on 180 degree meridian till equator, 5. rotate on equator till 270 degree meridian, 6. turn on 270 degree meridian and rotate till **N**. Now the the cyclic trajectory closed, and we proved existence of a cyclic sextet build in the right way.
It is easy to prove that this sextet is not unique and in neighborhood of the given sextet exist other ones.
We proved that one can choose sextet with $|\mathbf{P}_\Sigma(cyc)| \in (0, 2\pi)$ which lie in the same plane as $\mathbf{P}_\Sigma(g_\varepsilon)$, i.e. $\mathbf{P}_\Sigma(cyc) + \mathbf{P}_\Sigma(g_\varepsilon)$ can vanish for a special choice of parameters.

Conclusion

We prove that for any imperfect gate g_ε we can build a gate decomposition using just positive angles for rotations about $\mathcal{X}$ and $\mathcal{Y}$ axes in positive direction, such that both first conditions are satisfied

$$\mathcal{F}(g_\varepsilon, g_0)|_{\varepsilon=0} = 1;$$

$$\sum_{i=1}^{N} \theta_i \frac{\partial}{\partial \theta_i} v_\alpha(\theta_1, \ldots, \theta_N)|_{\varepsilon=0} = 0, \quad N \geq 7,$$

$$\text{or} \quad \mathcal{F}(g_\varepsilon, g_0) = \ 1 - O(\varepsilon^4)\,.$$

Literature

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