A brief introduction to the method of Orthogonal Amplitudes

Sergey Baranov

P.N.Lebedev Institute of Physics, Moscow, Russian Academy of Sciences

D.V.Skobeltzyn Institute of Nuclear Physics, Moscow State University

P L A N O F T H E T A L K

1. **Introduce basic vectors**
2. **Introduce basic amplitudes**
3. **General summation formula**
4. **Advantages for numeric calculations**
5. **Advantages for analytic calculations**
6. **Further extentions**

Step 1: basic vectors u and v

Consider we have a diagram with fermion line ending with particles with momenta p and q, and we calculate the square of the respective amplitude.

The first step is to introduce two 4-vectors, u and v, such that they are orthogonal to each other and to the fermion momenta p and q:

$$(u.v) = 0; \quad (u.p) = 0; \quad (u.q) = 0; \quad (v.p) = 0; \quad (v.q) = 0$$

In other respects these vectors are arbitrary. One particular choice is

$$u_0 = 0; \quad \vec{u} = [\vec{p} \times \vec{q}]; \qquad v_\alpha = \epsilon_{\alpha\beta\gamma\delta}\, u^\beta\, p^\gamma\, q^\delta$$

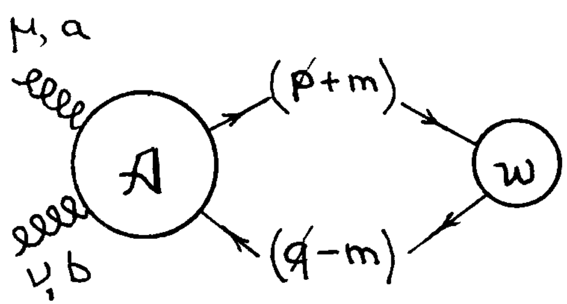
Step 2: Basic matrices

With the use of Dirac equation $(\hat{p} - m) = 0$ and the identity $\gamma_5 = i\gamma^0\,\gamma^1\,\gamma^2\,\gamma^3$, the number of independent operators is reduced from 16 to 4:

$$
\begin{array}{ccccccc}
I & \to & & & & & I \\
\gamma^\mu & \to & \hat{p},\ \hat{q},\ \hat{u},\ \hat{v} & & & \to & \hat{u},\ \hat{v} \\
\frac{1}{2}[\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu] & \to & \hat{p}\hat{q},\ \ \hat{p}\hat{u},\ \ \hat{p}\hat{v},\ \ \hat{q}\hat{u},\ \ \hat{q}\hat{v},\ \ \hat{u}\hat{v} & & & \to & \hat{u}\hat{v} \\
\gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3 & \to & \hat{p}\,\hat{q}\,\hat{u}\,\hat{v} & & & \to & \hat{u}\hat{v} \\
\gamma_5\hat{u} & \to & \hat{p}\,\hat{q}\,\hat{u}\,\hat{v}\,\hat{u} & \to & \hat{u}\,\hat{v}\,\hat{u} & \to & \hat{v} \\
\gamma_5\hat{v} & \to & \hat{p}\,\hat{q}\,\hat{u}\,\hat{v}\,\hat{v} & \to & \hat{u}\,\hat{v}\,\hat{v} & \to & \hat{u}
\end{array}
$$

Orthogonal Amplitudes

The following combinations form an orthogonal basis in the space of amplitudes:



$$
\begin{array}{llllll}
w_1 & = & I & \sim & \gamma_5 \hat{u}\hat{v} & (1) \\
w_2 & = & \hat{u} & \sim & \gamma_5 \hat{v} & (2) \\
w_3 & = & \hat{v} & \sim & \gamma_5 \hat{u} & (3) \\
w_4 & = & \hat{u}\hat{v} & \sim & \gamma_5 & (4)
\end{array}
$$

so that any amplitude can be presented as a linear combination of the four basic amplutudes:

$$\mathcal{A} = \sum_{w_i} c_i\, w_i/|w_i|^2 \quad \text{with} \quad |w_i|^2 = 4((p.q) \pm m_p m_q) \qquad (5)$$

where the projection coefficients c_i are found as

$$c_i^{\mu\nu,..ab,..} = tr\{(\hat{p} + m_p)\, \mathcal{A}^{\mu\nu,..ab,..}\, (\hat{q} - m_q)\, \bar{w}_i\} \qquad (6)$$

Step 3: summing the squares

If the amplitude $\mathcal{A}$ carries Lorentz and/or color indices, then the projection coefficients c_i do the same. But the coefficients c_i no longer carry Dirac indices. These coefficients are just complex numbers.

It follows from (5) that

$$\mathcal{A} \cdot \mathcal{A}^* = \sum_{\mathcal{W}_i} (c_i \cdot c_i^*)/|w_i|^2 \tag{7}$$

where the cross-terms disappear because of orthogonality of the basic amplitudes w_i. If one needs to take into account N amplitudes at a time, one simply collects the respective contributions to c_i by adding them together:

$$c_i^{\mu\nu,..ab,..} = \sum_{n=1}^{N} tr\{(\hat{p} + m_p)\, \mathcal{A}_n^{\mu\nu,..ab,..}\, (\hat{q} - m_q)\, \bar{w}_i\} \tag{8}$$

Advantages for numeric calculations

• Instead of multiplying every amplitude $\mathcal{A}$ by every conjugate $\mathcal{A}^*$, one only has to multiply every amplitude by four basic amplitudes w_i^*. The computational time is then proportional to the number of the amplitudes you consider, but not to the square of this number. That is, $4N$ operations against $N(N-1)/2$ operations.

• To calculate the projection coefficients c_i is much easier than to square even a single amplitude $\mathcal{A}$, because w_i contain much fewer Dirac matrices (at most two, that makes the trace much shorter).

Further advantages for analytic calculations

• Many terms drop out because of orthogonality of the basic vectors u and v to the particle momenta p and q.

• Choosing between two equivalent basic sets (with and without γ_5 matrix) one can completely remove Levi-Civita tensors from the output. For the terms in $\mathcal{A}$ containing even number of γ_5's, use left-hand side set of eqs (1)-(4). For the terms in $\mathcal{A}$ containing odd number of γ_5's, use right-hand side set of eqs (1)-(4). The expressions shown in the left- and right-hand sides are identical up to a purely imaginary numerical factor.

Further extentions: many fermion lines

If there are several fermion lines ending with momenta p_1, q_1; p_2, q_2; etc., one has to introduce several sets of basic amplitudes in a way similar to eqs (1)-(4), using the vectors u_1, v_1; u_2, v_2; etc., with the orthogonality conditions

$$(u_1.v_1) = 0; \quad (u_1.p_1) = 0; \quad (u_1.q_1) = 0; \quad (v_1.p_1) = 0; \quad (v_1.q_1) = 0$$

$$(u_2.v_2) = 0; \quad (u_2.p_2) = 0; \quad (u_2.q_2) = 0; \quad (v_2.p_2) = 0; \quad (v_2.q_2) = 0$$

$$\ldots \qquad\qquad \ldots \qquad\qquad \ldots$$

The overall basis for the entire system is the direct product of these:

$$w^{(1)}_{i_1} \otimes w^{(2)}_{i_2} \otimes \ldots \otimes w^{(n)}_{i_n}.$$

The dimension of the overall basis is 4^n.

No changes are required if the fermion lines are identical in flavor (just add permuted diagrams and sum them all together; see next slide.).

Further extentions: identical fermions

If there are identical fermions (say, with momenta p_1 and p_2), we have a 'normal' diagram $\mathcal{A}$ and a 'permuted' diagram $\mathcal{A}'$, for which the projection coefficients are calculated, respectively, as

$$\begin{aligned} c_{ij} &= tr\{\mathcal{A}(p_1 \leftrightarrow q_1)\,(\hat{q}_1 - m_q)\,\bar{w}_i^{(1)}(\hat{p}_1 + m_p)\} \\ &\times\; tr\{\mathcal{A}(p_2 \leftrightarrow q_2)\,(\hat{q}_2 - m_q)\,\bar{w}_j^{(2)}(\hat{p}_2 + m_p)\} \end{aligned}$$

and

$$\begin{aligned} c_{ij} &= tr\{\mathcal{A}'(p_2 \leftrightarrow q_1)\,(\hat{q}_1 - m_q)\,\bar{w}_i^{(1)}(\hat{p}_1 + m_p) \\ &\quad \mathcal{A}'(p_1 \leftrightarrow q_2)\,(\hat{q}_2 - m_q)\,\bar{w}_j^{(2)}(\hat{p}_2 + m_p)\} \end{aligned} \tag{9}$$

(here $\mathcal{A}(p \leftrightarrow q)$ denotes that part of the operators in the amplitude $\mathcal{A}$ that is located on the section of the fermion line between p and q.) For $\mathcal{A}'$, either calculate the long trace as it is, or use Fierz identity to swap the Dirac indices and split the trace in two.

Thank you!

This work was supported by Russian Science Foundation
under grant № 25-22-00066,
https://rscf.ru/en/project/25-22-00066/