Form factors in the sine-Gordon model at the reflectionless points

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1. **Background**
   - Sine-Gordon model
   - S-matrix and integrability
   - Form factors
2. **New free-field representation**
   - The algebra and the particles in the new representation
   - Form factors of vertices and of their descendants
   - Integrals of motion
   - Fusion of breathers
   - Connection to sinh-Gordon
3. **Conclusion**
   - Results
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Sine-Gordon model

A (1+1)d QFT with the action

$$S[\varphi] = \int d^2x \left\{ \frac{1}{16\pi} (\partial_\mu \varphi)^2 - 2\mu \cos \beta\varphi \right\}$$

- Perturbation of Liouville CFT by a primary operator
- Integrable model of QFT $\Leftrightarrow$ factorized S-matrix, found explicitly [ZZ'79]
- We introduce the parameter $\xi = \frac{\beta^2}{1-\beta^2}$ which is more convenient
- Spectrum of the model: kinks/antikinks ($\sigma = \pm 1$) and $\lfloor \frac{1}{\xi} \rfloor$ breathers, the bound states kink-antikink

Reflectionless points

Two particle S-matrix takes form (unique solution to bootstrap)

$$S(\theta) = \begin{pmatrix} a(\theta) & & & \\ & b(\theta) & c(\theta) & \\ & c(\theta) & b(\theta) & \\ & & & a(\theta) \end{pmatrix}$$

where the functions are $b(\theta) = \frac{\text{sh}\,\frac{\theta}{\xi}}{\text{sh}\,\frac{i\pi-\theta}{\xi}} a(\theta)$, $\quad c(\theta) = \frac{\text{sh}\,\frac{i\pi}{\xi}}{\text{sh}\,\frac{i\pi-\theta}{\xi}} a(\theta)$, and

$$a(\theta) = \exp\left(2i\int_0^\infty \frac{dt}{t}\cdot\frac{\text{sh}(\frac{\pi t}{2})\,\text{sh}(\frac{\pi t}{2}(\xi+1))}{\text{sh}(\pi t)\,\text{sh}(\frac{\pi t}{2}\xi)}\sin(\theta t)\right)$$

Notice that for $\xi = \frac{\beta^2}{1-\beta^2} = \frac{1}{\nu}$, $\nu \in \mathbb{N} \Rightarrow c(\theta) = 0$ and the matrix becomes **diagonal, i.e. no reflection**. We study this case

Form factors

Main task of QFT - calculation of correlation functions.

The way to do it for some local operator $\mathcal{O}$ - to know the form factors of this operator:

$$F^{\mathcal{O}}_{\sigma_{2n}\ldots\sigma_1}(\theta_{2n},\ldots\theta_1) \equiv \langle 0|\,\mathcal{O}(0)\,|A_{\sigma_{2n}}(\theta_{2n})\ldots A_{\sigma_1}(\theta_1)\rangle_{\mathrm{in}}$$

The correlation function would be then

$$\langle\mathcal{O}(x)\mathcal{O}(y)\rangle=\sum_{n=0}^{\infty}\frac{1}{n!}\int_{\mathbb{R}}\frac{d\theta^n}{(2\pi)^n}\cdot F_{\mathcal{O}}(\theta_1;\ldots;\theta_n)_{\sigma_1\ldots\sigma_n}F_{\mathcal{O}}(\theta_n;\ldots;\theta_1)^{\sigma_n\ldots\sigma_1}e^{\left(-rm\sum_{k=1}^{n}\mathrm{ch}\,\theta_k\right)}$$

Form factors for sine-Gordon model

There are two working approaches:

- The algebraic one by Lukyanov [L'97], **which we study**. The form factors of $e^{i\alpha\varphi}$ are given by the trace average over the oscillator modes of the free field $\phi(\theta)$ constructed by Lukyanov

$$F_{\sigma_{2n}\ldots\sigma_1}(\theta_{2n},\ldots\theta_1) = \mathcal{G}_\alpha \langle\langle \mathcal{Z}_{\sigma_{2n}}(\theta_{2n})\ldots\mathcal{Z}_{\sigma_1}(\theta_1)\rangle\rangle$$

  here $\mathcal{G}_\alpha = \langle e^{i\alpha\varphi}\rangle$, $\sigma_i = \pm 1$ and $\mathcal{Z}_\pm(\theta)$ are the kink/antikink creation operators, satisfying Faddeev-Zamolodchikov algebra
- The integral one by Smirnov [BBS'97](big formula which was postulated and somehow works)

The relation between those two approaches is unknown. We aim to connect them

Form factors at the reflectionless points (Lukyanov)

The form factors of $e^{i\alpha\varphi}$ are expressed through a free field $\phi(\theta)$ which bosonises the S-matrix

$$F_{\sigma_{2n}\dots\sigma_1}(\theta_{2n},\dots\theta_1) = \mathcal{G}_a\langle\langle \mathcal{Z}_{\sigma_{2n}}(\theta_{2n})\dots\mathcal{Z}_{\sigma_1}(\theta_1)\rangle\rangle$$

where

$$\mathcal{Z}_+(\theta) \sim :e^{i\phi(\theta)}:$$
$$\mathcal{Z}_-(\theta) \sim \oint_C \frac{d\gamma}{2\pi} e^{-\left(\frac{2a}{\beta}+\nu\right)(\gamma-\theta)} w(\gamma-\theta) :e^{-i\bar{\phi}(\gamma)+i\phi(\theta)}:$$

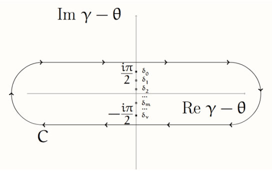

Figure 1: Contour of integration. Important result is that this contour is closed (for general ξ it isn't)

Form factors at the reflectionless points (Lukyanov)

- We get $\mathcal{Z}_-(\theta) = \sum_{m=0}^{\nu} \mathcal{Z}_-^m(\theta)$, where the operator $\mathcal{Z}_-^m(\theta) \sim: e^{i\phi(\theta) - i\bar{\phi}(\theta + \frac{i\pi}{2} - \frac{i\pi m}{\nu})}:$ is at the m'th pole
- The simplest case of 2 particles is

$$F_{+-}(\theta_2, \theta_1) = \mathcal{G}_\alpha \sum \langle\langle \mathcal{Z}_+(\theta_2) \mathcal{Z}_-^m(\theta_1) \rangle\rangle$$

Calculating this trace gives us

$$F_{+-}(\theta_2, \theta_1) \sim G_{++}(\theta_2 - \theta_1) \sum_{m=0}^{\nu} e^{\frac{2\alpha}{\beta} \cdot \frac{i\pi m}{\nu}} \tilde{c}_-^{\,m} h_{+-}^m(e^{\theta_1 - \theta_2})$$

with a rational function (minimal form factor)

$$h_{+-}^{(m)}(z) = h_{-+}^{(\nu-m)}(z) = q^{\frac{\nu+1}{4}(\nu - 2m)} z^{\frac{\nu+1}{2}} \prod_{n=0}^{\nu} \frac{1}{1 - q^{n-m} z}$$

New free-field representation

Idea: express the minimal form factors using another bosonisation

- We introduce the new Heisenberg algebra as

$$[\mathbf{P},\mathbf{Q}]=-\mathbf{i},\quad [\mathbf{P},\mathbf{a}_k]=[\mathbf{Q},\mathbf{a}_k]=\left[\mathbf{P},\bar{\mathbf{a}}_k^{(n)}\right]=\left[\mathbf{Q},\bar{\mathbf{a}}_k^{(n)}\right]=0$$

$$[\mathbf{a}_k,\mathbf{a}_l]=0,\quad [\mathbf{a}_k,\bar{\mathbf{a}}_l]=k\delta_{k+l,0},\quad [\bar{\mathbf{a}}_k,\bar{\mathbf{a}}_l]=\left(1+(-1)^k\right)k\sigma_k^{-1}\delta_{k+l,0}.$$

where $\sigma_k=\sum_{n=0}^{\nu}q^{nk}=(-1)^k\sigma_{-k},\qquad q=e^{\frac{i\pi}{\nu}}.$

- Let $Y_\pm(z)$ be the new operators (in terms of this algebra) corresponding to kinks/antikinks so that for example

$$Y_+(z')Y_-^{(m)}(z)=\left(\frac{z'}{z}\right)^{\frac{\nu+1}{4}}h_{+-}^{(m)}\left(\frac{z}{z'}\right):Y_+(z')Y_-^{(m)}(z):$$

Particles in the new representation

- Kink/antikink operators then look like

$$Y_+(z) = \exp\left(i\mathbf{Q} + \frac{\nu+1}{2}\mathbf{P}\log z + \sum_{k\neq 0}\frac{\mathbf{a}_k}{k}z^{-k}\right)$$

$$Y_-(z) = \frac{1}{\nu!}\sum_{m=0}^{\nu}\tilde{c}_-^m Y_-^m(z)$$

where

$$Y_-^m(z)=\exp\left(-i\mathbf{Q}-\tfrac{\nu+1}{2}\mathbf{P}\log(zq^{\nu-2m})+\textstyle\sum_{k\neq 0}\left(\mathbf{a}_k-q^{mk}\sigma_{-k}\bar{\mathbf{a}}_k\right)\tfrac{z^{-k}}{k}\right)$$

- Notice the asymmetry of this construction

Form factors in the new representation

In this representation the form factors have a simple form

$$F_\alpha(\theta_1,\ldots,\theta_{2N})_{\sigma_1\ldots\sigma_{2N}} \sim \prod_{i<j} e^{\frac{\nu+1}{4}(\theta_i-\theta_j)} G_{++}(\theta_i-\theta_j)$$
$$\times\ {}_\alpha\langle 1|Y_{\sigma_1}(e^{\theta_1})\cdots Y_{\sigma_{2N}}(e^{\theta_{2N}})|1\rangle_\alpha$$

- We get the form factors of descendants by replacing the vacuum $|1\rangle_\alpha \to |h\rangle_\alpha$ where $|h\rangle_\alpha$ is generated by $\bar{\mathbf{a}}_k$.

- The form factors are factorized into a universal part $G_{++}(\theta_i-\theta_j)$ (basically the kink-kink "form factor") and the operator dependent vacuum average which is easy to calculate
- For the universal part, after regularization (G-Barnes function)

$$G_{++}(\theta)=(2\pi)^{\frac{\nu+1}{2}} \prod_{k=0}^{\nu} \frac{G\left(\frac{1}{2}-\frac{i\theta}{2\pi}-\frac{k}{2\nu}\right)G\left(\frac{3}{2}+\frac{i\theta}{2\pi}-\frac{k}{2\nu}\right)}{G\left(\frac{1}{2}-\frac{i\theta}{2\pi}+\frac{k}{2\nu}\right)G\left(\frac{3}{2}+\frac{i\theta}{2\pi}+\frac{k}{2\nu}\right)}$$

Operators

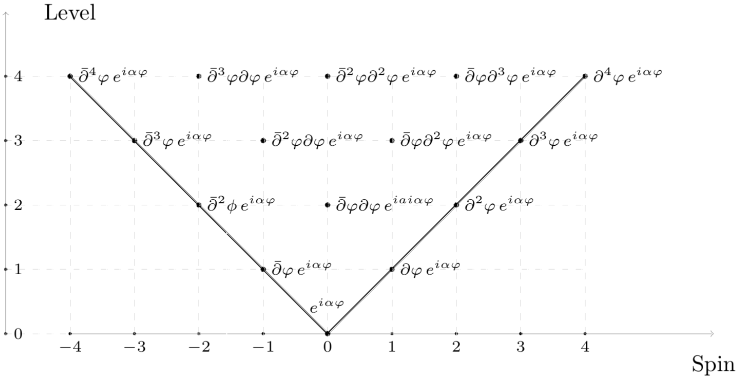

Figure 2: First descendants of an exponential operator

Integrals of motion

We introduced screening operators $S(z), \tilde{S}(z)$. Zero modes of these currents act on the Fock modules by shifting α as

$$S(z) : \mathcal{F}_\alpha \to \mathcal{F}_{\alpha-\beta}, \qquad \tilde{S}(z) : \mathcal{F}_\alpha \to \mathcal{F}_{\alpha+\beta^{-1}} \tag{1}$$

We then obtained for spin s integral of motion $I_s = \int dz \cdot T_{s+1} + \int d\bar{z} \cdot \Theta_{s-1}$ the corresponding form factors

$$F_{T_{s+1}} = {}_{31}\langle \mathbf{1}| \int du \cdot u^{s-(\nu+1)} \tilde{S}^{-1}(u) Y_+(z_{2n}) \ldots Y_+(z_{n+1})$$
$$Y_-(z_n) \ldots Y_-(z_1) \int dw \cdot w^{-\frac{2\nu+1}{\nu+1}} \cdot S(w) \, |\mathbf{1}\rangle_{13}$$

for the degenerate dimensions $\alpha_{mn} = -\frac{m-1}{2}\beta^{-1} + \frac{n-1}{2}\beta$

Breathers in this representation

- l'th breather is written as (coefficient b_l^m is known)

$$B_l(z) = \sum_{m=0}^{l} b_l^m s_l^{(m)}(z), \qquad l = 1, \dots, \nu - 1$$

- Operators are obtained by taking the product of kink and antikink with right rapidities

$$s_l^{(m)}(z) =: Y_+(zq^{\frac{(\nu-l)}{2}}) Y_-^{(m)}(zq^{\frac{(l-\nu)}{2}}) :$$

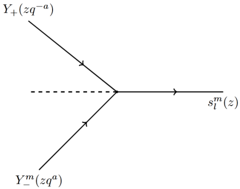

Figure 3: Breather formation

Fusion of breathers

- Breathers themselves can merge $B_l + B_{l'} \to B_{l''}$, for example $l'' = l + l'$,

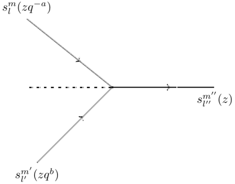

Figure 4: Fusion of breathers

- The product of our operators correctly reproduces the expected breather

$$\mathfrak{r}_{ll'}\left(\frac{z}{z'}\right)B_l(z'q^{-\frac{l'}{2}})B_{l'}(zq^{\frac{l}{2}}) = \frac{\mathfrak{C}_{ll'}}{1-\frac{z}{z'}} : B_{l+l'}(z)s(zq^{\frac{l-l'}{2}}) : +O(1)$$

Connection to sinh-Gordon

- Scattering matrices of the particles in sinh-Gordon model and of the 1st breathers **are the same**, so there is a correspondence
- The sinh-Gordon is formally the case of $\xi \in (-1, 0)$
- In free-field representation for sine-Gordon the 1st breather is

$$B_1(z) \sim s_1^0(z) - s_1^1(z)$$

- In free-field representation for sinh-Gordon the particle is

$$t(z) = e^{i\pi\hat{a}}\lambda^-(z) + e^{-i\pi\hat{a}}\lambda^+(z)$$

Direct identification of these free-field representations leaves all results in agreement, which proves the correctness of calculations

Results

- Constructed a free-field representation for form factors at reflectionless points
- Generalized the result to form factors of descendants
- Obtained form factors of conserved currents corresponding to an infinite set of integrals of motion
- Obtained a correct fusion of breathers
- Linked the free-field representations for the sine- and sinh-Gordon models

To do

- Obtain the equations of motion using a new representation
- Understand the relationship between Lukyanov's and Smirnov's approaches
- Find the recurrence relations for form factors
- Explain the appearance of branch points for screenings in the case of reflectionless points

Thank you for your attention!

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