Renormalization of Anisotropic Quantum Field Theories

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Content

Introduction

- There are a lot of physical systems with **anisotropy** : anisotropic crystals or solids, 2-components superfluids, anisotropy in plasma, cosmological perturbations with different sound speeds, astrophysical magnetic fields etc.
- That is why the *renormalization* problem deserves a very careful attention.
- **The main questions** : what survives, and what changes, in unitarity, in beta functions and in the one-loop effective potential when the sound speeds are different and other mixings are presented in the theory ?

Introduction

We will consider the following nice playground to study the properties of such systems (which are Lorentz–violating models) :

- The first one is

$$S_{\text{free}} = \sum_i \int d^4x \left[\frac{1}{2}\dot{\phi}_i^2 - \frac{1}{2}\partial_a\phi_i \, \mathscr{C}_i^{ab} \, \partial_b\phi_i - \frac{1}{2}m_i^2\phi_i^2 \right], \tag{1}$$

with $\mathscr{C}_i^{ab} = \mathscr{C}_i^{ba}$ real, symmetric, and positive definite. The dispersion relation reads

$$E_i(\mathrm{p})^2 = m_i^2 + \mathrm{p}^T \mathscr{C}_i \mathrm{p}. \tag{2}$$

- The second is

$$\mathcal{L} = \frac{1}{2}(\partial_0\boldsymbol{\varphi})^\top \mathrm{Z}_t (\partial_0\boldsymbol{\varphi}) + \frac{1}{2}(\nabla\phi)^\top \mathrm{C}_\phi(\nabla\phi) + \frac{1}{2}(\nabla\chi)^\top \mathrm{C}_\chi(\nabla\chi)$$
$$+ (\nabla\phi)^\top \mathrm{Y}(\nabla\chi) + \frac{1}{2}\boldsymbol{\varphi}^\top \mathrm{M}^2 \boldsymbol{\varphi} + \mathcal{L}_1 + \mathcal{L}_3 + \mathcal{L}_4, \tag{3}$$

with $\boldsymbol{\varphi} \equiv (\phi, \chi)^\top$ and $\boldsymbol{C}_{\phi,\chi}$ and etc will be discussed later on.

Model 1 : disparity in sound speeds

- So we begin with a model with a set of real scalars ϕ_i whose quadratic action is

$$S_{\rm free} = \sum_i \int d^4x \left[\frac{1}{2}\dot{\phi}_i^2 - \frac{1}{2}\partial_a\phi_i\,\mathscr{C}_i^{ab}\,\partial_b\phi_i - \frac{1}{2}m_i^2\phi_i^2\right], \tag{4}$$

with $\mathscr{C}_i^{ab} = \mathscr{C}_i^{ba}$ real, symmetric, and positive definite. The dispersion relation reads

$$E_i(\mathrm{p})^2 = m_i^2 + \mathrm{p}^T\mathscr{C}_i\mathrm{p}. \tag{5}$$

- For *two fields* one may always choose a basis in which one of the matrices is diagonal

$$\mathscr{C}_1 = I, \qquad \mathscr{C}_2 = \mathrm{diag}(v_x^2, v_y^2, v_z^2), \tag{6}$$

so the invariant anisotropy is *encoded in the spectrum* of the relative matrix

$$\mathscr{C}_1^{-1/2}\mathscr{C}_2\mathscr{C}_1^{-1/2}. \tag{7}$$

Model 1 : disparity in sound speeds

- Let us keep the basis-independent notation $\rightarrow$ it extends more naturally to larger multiplets and more general setups.
- For a two-body channel, i.e. two distinguishable particles in a state the *center-of-mass* (CM) kinematics is

$$\mathrm{p}_1 = \mathrm{p}, \qquad \mathrm{p}_2 = -\mathrm{p}, \qquad \mathrm{p} = p\,\hat{\mathrm{n}}, \qquad p \geq 0, \qquad \hat{\mathrm{n}}^2 = 1. \tag{8}$$

It is convenient to define

$$\omega_{1\beta}(\hat{\mathrm{n}}) \equiv \hat{\mathrm{n}}^T \mathscr{C}_i \hat{\mathrm{n}}, \qquad \omega_{2\beta}(\hat{\mathrm{n}}) \equiv \hat{\mathrm{n}}^T \mathscr{C}_j \hat{\mathrm{n}}, \tag{9}$$

where $\beta \equiv (\phi_i, \phi_j) \equiv (i, j)$. Let us clarify the notations here : index 1 relates to the first particle in the pair β which itself is a type i (so that we write index "i" in "$\mathscr{C}_i$") and 2 is for the second particle in the pair which in its turn is "j" type.

Model 1 : unitarity relation

- The two-body phase space becomes direction dependent !

$$p^2 dp d\Omega = dE d\Omega \frac{p E_{1\beta} E_{2\beta}}{E_{2\beta}\omega_{1\beta} + E_{1\beta}\omega_{2\beta}}. \tag{10}$$

- That is why the unitarity relation reads

$$-\frac{i}{2}(\mathbb{A} - \mathbb{A}^\dagger) = \mathbb{A}^\dagger\, G\, \mathbb{A} + X(E), \qquad X(E) \geq 0, \tag{11}$$

where $X(E)$ is the positive contribution of inelastic states and where

$$G^{\gamma}_{MN}(E) = \int d\Omega\, Y^*_M(\hat{n})\, g_\gamma(E, \hat{n})\, Y_N(\hat{n}), \tag{12}$$

Model 1 : unitarity relation and bound

- The quantity $\mathbb{A}_{L'L,\beta'\beta}(E)$ is therefore the anisotropic analogue of the usual partial-wave amplitude :

$$\mathbb{A}_{L'L,\beta'\beta}(E) = \frac{1}{64\pi^2} \int d\Omega' d\Omega \, Y^*_{L'}(\hat{n}') \, M_{\beta'\beta}(E; \hat{n}', \hat{n}) \, Y_L(\hat{n}), \tag{13}$$

each of its eigenvalues obeys

$$\Im\lambda_n = |\lambda_n|^2, \qquad |\Re\lambda_n| \leq \frac{1}{2}, \tag{14}$$

which is the *correct unitarity bound* in the anisotropic case.

Model 1 : effective potential

- As a next step let us consider the model of only two scalar fields with the interaction of the form :

$$V = \text{mass} + \frac{\lambda_1}{4!}\phi_1^4 + \frac{\lambda_2}{4!}\phi_2^4 + \frac{\lambda_3}{4}\phi_1^2\phi_2^2. \tag{15}$$

- Next we split each field into a constant background and a quantum fluctuation,

$$\phi_1 = \varphi_1 + \eta_1, \qquad \phi_2 = \varphi_2 + \eta_2, \tag{16}$$

- The one-loop effective potential then

$$V_1(\varphi_1, \varphi_2) = \frac{1}{2}\int \frac{d^3\mathrm{k}}{(2\pi)^3}\left[\Omega_+(\mathrm{k}) + \Omega_-(\mathrm{k})\right], \tag{17}$$

  where $\Omega_{+,-}$ contain the information about the directional matrix and about different masses.

- This representation is exact, but it also makes the obstruction to a simple closed form manifest !

Model 1 : beta functions

- The beta functions mirror the same geometry, and C_i do not run at one loop!
- From the counterterms we obtain (holding bare parameters fixed)

$$\beta_{\lambda_1} = \frac{3}{16\pi^2}\left(\frac{\lambda_1^2}{\Pi_1} + \frac{\lambda_3^2}{\Pi_2}\right), \qquad \beta_{m_1^2} = \frac{1}{16\pi^2}\left(\frac{\lambda_1 m_1^2}{\Pi_1} + \frac{\lambda_3 m_2^2}{\Pi_2}\right), \tag{18}$$

$$\beta_{\lambda_2} = \frac{3}{16\pi^2}\left(\frac{\lambda_2^2}{\Pi_2} + \frac{\lambda_3^2}{\Pi_1}\right), \qquad \beta_{m_2^2} = \frac{1}{16\pi^2}\left(\frac{\lambda_2 m_2^2}{\Pi_2} + \frac{\lambda_3 m_1^2}{\Pi_1}\right), \tag{19}$$

$$\beta_{\lambda_3} = \frac{\lambda_3}{16\pi^2}\left(\frac{\lambda_1}{\Pi_1} + \frac{\lambda_2}{\Pi_2} + 4\lambda_3 J_{12}\right). \tag{20}$$

And

$$\beta_{C_1} = 0, \qquad \beta_{C_2} = 0. \tag{21}$$

- So the anisotropic kinetic tensors affect the running of the couplings, but in the unbroken quartic theory they do not themselves run at one loop. That is because there is no momentum-dependent self-energy at this order.

Model 1 : scalon in anisotropic theory

- The *scalon* is the scalar excitation along a classically flat direction of the potential.
- Think of the two scalar fields as coordinates in a two-dimensional field space :

$$(\phi_1, \phi_2)\,.$$

- In the classically scale-invariant limit : $m_1 = m_2 = 0$, so the tree-level potential has no mass scale. It is purely quartic :

$$V_0\,(\phi_1, \phi_2) = \frac{\lambda_1}{24}\phi_1^4 + \frac{\lambda_2}{24}\phi_2^4 + \frac{\lambda_3}{4}\phi_1^2\phi_2^2.$$

- Because there are no mass terms, the potential is homogeneous :

$$V_0\,(t\phi_1, t\phi_2) = t^4\,V_0\,(\phi_1, \phi_2)\,.$$

- So if there is a direction in field space where the quartic coefficient vanishes, then the potential is flat along the whole ray. The excitation along this flat ray is called the scalon. It is the Gildener-Weinberg radial scalar mode.

Model 1 : scalon in anisotropic theory

- At tree level we have :

$$m^2_{\text{scalon,tree}} = 0.$$

- But one-loop corrections lift the flat ray and the scalon gets a radiative mass :

$$m^2_{\text{scalon}} = 8Bv^2.$$

- Important : to obtain flat direction we impose $\lambda_1\lambda_2 = 9\lambda_3^2, \quad \lambda_3 < 0\,!$
- Anisotropy enters only through B :

$$B = \frac{1}{64\pi^2\rho^4}\left[\frac{M_1^4}{\Pi_1} + \frac{M_2^4}{\Pi_2} + 2\mathcal{J}_{12}s^2\right],$$

where ρ, $v =< \rho >$ are just convenient new variables instead of $\phi_{1,2}$.

Model 2 : general anisotropic QFT

- To keep the theory RG complete if we allow cubic couplings (RG flow does not generate any new operators) $\rightarrow$ must add :

$$\frac{1}{2}\left(\partial_0\boldsymbol{\varphi}\right)^\top \boldsymbol{Z}_t \left(\partial_0\boldsymbol{\varphi}\right) + \left(\nabla\phi\right)^\top \boldsymbol{Y}\left(\nabla\chi\right).$$

- Allow a general symmetric mass matrix $\boldsymbol{M}^2$:

$$\frac{1}{2}\boldsymbol{\varphi}^\top \boldsymbol{M}^2 \boldsymbol{\varphi}.$$

- All together :

$$\begin{aligned}\mathcal{L} = &\frac{1}{2}\left(\partial_0\boldsymbol{\varphi}\right)^\top \boldsymbol{Z}_t \left(\partial_0\boldsymbol{\varphi}\right) \\ &+ \frac{1}{2}\left(\nabla\phi\right)^\top \boldsymbol{C}_\phi\left(\nabla\phi\right) + \frac{1}{2}\left(\nabla\chi\right)^\top \boldsymbol{C}_\chi\left(\nabla\chi\right) \\ &+ \left(\nabla\phi\right)^\top \boldsymbol{Y}\left(\nabla\chi\right) + \frac{1}{2}\boldsymbol{\varphi}^\top \boldsymbol{M}^2 \boldsymbol{\varphi} + \mathcal{L}_{\text{int}}.\end{aligned}$$

Model 2 : general anisotropic QFT

- The Lagrangian of interactions reads

$$
\begin{aligned}
\mathcal{L}_1 &= \mu^{\frac{D+2}{2}} (j_\phi\, \phi + j_\chi\, \chi), \\
\mathcal{L}_3 &= \mu^{\frac{6-D}{2}} \Big(\frac{h_1}{3!}\, \phi^3 + \frac{h_2}{3!}\, \chi^3 + \frac{h_3}{2}\, \phi^2 \chi + \frac{h_4}{2}\, \phi \chi^2 \Big), \\
\mathcal{L}_4 &= \mu^{4-D} \Big(\frac{\lambda_1}{4!}\, \phi^4 + \frac{\lambda_2}{4!}\, \chi^4 + \frac{\lambda_3}{4}\, \phi^2 \chi^2 \\
&\quad + \frac{\lambda_4}{6}\, \phi^3 \chi + \frac{\lambda_5}{6}\, \phi \chi^3 \Big).
\end{aligned}
$$

Preliminaries

- It turns out, that everything (propagator, all beta functions, etc) can be written in terms of **direction-dependent eigenvalues/determinant** of some matrix, which is the “combination” of initial matrices $\boldsymbol{C}_\phi$, $\boldsymbol{C}_\chi$, mass matrix etc.

Preliminaries

- Recall

$$\mathcal{S}_{\text{free}} = \int d^D x\, \mathcal{L}_{\text{free}} = \frac{1}{2} \int_p \varphi(-p)^\top \boldsymbol{D}(p)\, \varphi(p),$$

where

$$\boldsymbol{D}(p) \equiv \omega^2\, \boldsymbol{Z}_t \;+\; \boldsymbol{Q}(\boldsymbol{p}) \;+\; \boldsymbol{M}^2,$$

and where

$$\boldsymbol{Q}(\boldsymbol{p}) \equiv \sum_{a,b=1}^{d} p_a p_b \begin{pmatrix} C_{ab}^{(\phi)} & Y_{ab} \\ Y_{ab} & C_{ab}^{(\chi)} \end{pmatrix}.$$

- It is convenient to define $\hat{\boldsymbol{n}} \equiv \boldsymbol{p}/|\boldsymbol{p}|$, $\hat{\boldsymbol{n}} \in S^{d-1}$

$$\boldsymbol{C}(\hat{\boldsymbol{n}}) \;\equiv\; \hat{n}_a \begin{pmatrix} C_{ab}^{(\phi)} & Y_{ab} \\ Y_{ab} & C_{ab}^{(\chi)} \end{pmatrix} \hat{n}_b, \quad \boldsymbol{Q}(\boldsymbol{p}) = |\boldsymbol{p}|^2\, \boldsymbol{C}(\hat{\boldsymbol{n}}),$$

$$\hat{\boldsymbol{n}}^\top \boldsymbol{C}_\phi\, \hat{\boldsymbol{n}} > 0, \qquad \hat{\boldsymbol{n}}^\top \boldsymbol{C}_\chi\, \hat{\boldsymbol{n}} > 0, \qquad \det \boldsymbol{C}(\hat{\boldsymbol{n}}) > 0.$$

Preliminaries

- In these notations the propagator reads :

$$\boldsymbol{G}(p) = \boldsymbol{Z}_t^{-1/2} \left(\omega^2 1 + \boldsymbol{K}(\boldsymbol{p})\right)^{-1} \boldsymbol{Z}_t^{-1/2},$$

where

$$\boldsymbol{K}(\boldsymbol{p}) \equiv \boldsymbol{Z}_t^{-1/2} \left(\boldsymbol{Q}(\boldsymbol{p}) + \boldsymbol{M}^2\right) \boldsymbol{Z}_t^{-1/2}.$$

Define

$$\boldsymbol{S}(\hat{\boldsymbol{n}}) \equiv \boldsymbol{Z}_t^{-1/2} \boldsymbol{C}(\hat{\boldsymbol{n}}) \boldsymbol{Z}_t^{-1/2}, \qquad \boldsymbol{L} \equiv \boldsymbol{Z}_t^{-1/2} \boldsymbol{M}^2 \boldsymbol{Z}_t^{-1/2},$$

so

$$\boldsymbol{K}(\boldsymbol{p}) = |\boldsymbol{p}|^2 \, \boldsymbol{S}(\hat{\boldsymbol{n}}) + \boldsymbol{L}.$$

Preliminaries

- In UV (massless limit), decompose $\boldsymbol{S}(\hat{\boldsymbol{n}})$:

$$\boldsymbol{S}(\hat{\boldsymbol{n}}) = \sum_{a=1}^{2} s_a(\hat{\boldsymbol{n}})\, \Pi_a(\hat{\boldsymbol{n}}), \quad s_a(\hat{\boldsymbol{n}}) > 0.$$

- Then

$$v_a(\hat{\boldsymbol{n}}) \equiv \sqrt{s_a(\hat{\boldsymbol{n}})}$$

are directional sound velocities of normal modes.

- Define also

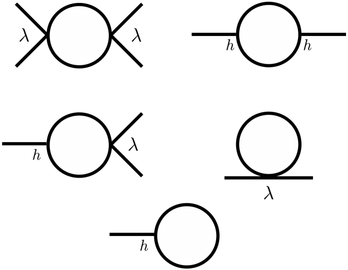
$$\boldsymbol{P}_a(\hat{\boldsymbol{n}}) \equiv \boldsymbol{Z}_t^{-1/2} \Pi_a(\hat{\boldsymbol{n}}) \boldsymbol{Z}_t^{-1/2},$$

so we get eigenvalue decomposition of propagator

$$\boldsymbol{G}(p) = \sum_{a=1}^{2} \frac{\boldsymbol{P}_a(\hat{\boldsymbol{n}})}{\omega^2 + |\boldsymbol{p}|^2 s_a(\hat{\boldsymbol{n}})}.$$

Preliminaries

- We calculate 1-loop level **beta functions**. Diagrams set is well-known



- All diagrams reduces to master-integrals : **tadpole** $\mathbb{T}_{mn}$ (one propagator) and **bubble** $\mathbb{B}_{ij;kl}$ (two propagator) $\rightarrow$ expressed in terms of **direction-dependent eigenvalues/determinant** as well.

Preliminaries

Everything are expressed via such matrix integrals

- Tadpole :

$$\mathbb{T}_{mn} \equiv \int \frac{d\omega}{2\pi} \frac{d^d \boldsymbol{p}}{(2\pi)^d} \, \boldsymbol{G}_{mn}(p).$$

- Bubble at zero external momentum :

$$\mathbb{B}_{ij;kl}(0) \equiv \int \frac{d\omega}{2\pi} \frac{d^d \boldsymbol{p}}{(2\pi)^d} \, \boldsymbol{G}_{ik}(p) \, \boldsymbol{G}_{jl}(p).$$

Preliminaries

To calculate beta functions we need only divergent parts of these diagrams (DR+MS scheme). They are given by

- Tadpole :

$$[\boldsymbol{T}]_{\text{pole}} = -\frac{1}{4\pi^2\epsilon}\left\langle \sum_{a,b} \mathcal{K}(s_a, s_b)\boldsymbol{Z}_t^{-1/2}\boldsymbol{\Pi}_a \boldsymbol{L}\boldsymbol{\Pi}_b \boldsymbol{Z}_t^{-1/2} \right\rangle_{\hat{\boldsymbol{n}}},$$

where $\langle f(\hat{\mathrm{n}})\rangle_{\hat{\mathrm{n}}} \equiv \frac{1}{\Omega_{d-\mathbf{1}}}\int_{S^{d-\mathbf{1}}} d\Omega_{d-1} f(\hat{\mathrm{n}})$, and

$$\mathcal{K}\left(s_a, s_b\right) \equiv \frac{1}{\sqrt{s_a}\sqrt{s_b}\left(\sqrt{s_a}+\sqrt{s_b}\right)}.$$

- Bubble at zero external momentum :

$$[\mathbb{B}_{ij;kl}]_{\text{pole}} = \frac{1}{4\pi^2\epsilon}\left\langle \sum_{a,b} \mathcal{K}(s_a, s_b)\boldsymbol{P}_a(\hat{\boldsymbol{n}})_{ik}\boldsymbol{P}_b(\hat{\boldsymbol{n}})_{jl} \right\rangle_{\hat{\boldsymbol{n}}}.$$

Beta functions

- Remind that we stick to DR + minimal subtraction scheme $D = 4 - \epsilon$.
- At fixed bare couplings beta function are defined

$$\beta_g \equiv \mu \frac{dg}{d\mu}\bigg|_{g_0=\text{fixed}},$$

with

$$g_0 = \mu^{d_g} \left(g + \delta g\right).$$

- Only the couplings $\{\lambda, h, M^2, j\}$ are renormalized at one loop.
- The kinetic and anisotropy matrices do **not** run in DR+MS at this order.

Beta function

- The simple-pole coefficients are

$$\delta\lambda_{ijkl} = \frac{A_{ijkl}}{\epsilon}, \quad \delta h_{ijk} = \frac{B_{ijk}}{\epsilon}, \quad \delta j_i = \frac{C_i}{\epsilon}, \quad \delta M^2_{ij} = \frac{D_{ij}}{\epsilon}.$$

- The beta function for λ quartic couplings takes the universal tensor form

$$\begin{aligned}\beta_{\lambda_{ijkl}} &= -\epsilon\,\lambda_{ijkl} \\ &+ \frac{1}{2}\Big(\lambda_{ijmn}\,\mathcal{B}_{mn;pq}\,\lambda_{klpq} + \lambda_{ikmn}\,\mathcal{B}_{mn;pq}\,\lambda_{jlpq} \\ &+ \lambda_{ilmn}\,\mathcal{B}_{mn;pq}\,\lambda_{jkpq}\Big).\end{aligned}$$

- All anisotropy dependence enters exclusively through the bubble residue $\mathcal{B}_{ij;kl} \equiv \epsilon\,\mathbb{B}_{ij;kl}$, $\mathcal{T}_{mn} \equiv \epsilon\,\mathbb{T}_{mn}$.
- Similar expressions we have for beta functions for cubic couplings, mass term, and linear j_i.

Beta function

Cubic couplings and mass/linear running are also expressed as

$$\beta_{h_{ijk}} = -\left(1+\frac{\epsilon}{2}\right) h_{ijk} + \frac{1}{2}\Big(\lambda_{ijmn}\,\mathcal{B}_{mn;pq}\,h_{kpq} + \lambda_{ikmn}\,\mathcal{B}_{mn;pq}\,h_{jpq} + \lambda_{jkmn}\,\mathcal{B}_{mn;pq}\,h_{ipq}\Big).$$

$$\beta_{m^2_{ij}} = -2\,m^2_{ij} - \frac{1}{2}\,\lambda_{ijmn}\,\widetilde{\mathcal{T}}_{mn} + \frac{1}{2}\,h_{imn}\,\mathcal{B}_{mn;pq}\,h_{jpq},$$

where

$$m^2_{ij} \equiv \frac{M^2_{ij}}{\mu^2}.$$

$$\beta_{j_i} = -\left(3-\frac{\epsilon}{2}\right) j_i - \frac{1}{2}\,h_{imn}\,\widetilde{\mathcal{T}}_{mn}.$$

Explicit form of beta functions and fixed points

Case 1 : $\boldsymbol{Z}_t = 1, \boldsymbol{Y} = 0$

- Directional "speeds" :

$$s_1(\hat{\boldsymbol{n}}) = \hat{\boldsymbol{n}}^\top \boldsymbol{C}_1 \hat{\boldsymbol{n}}, \quad s_2(\hat{\boldsymbol{n}}) = \hat{\boldsymbol{n}}^\top \boldsymbol{C}_2 \hat{\boldsymbol{n}},$$

  where $\boldsymbol{C}_1 \equiv \boldsymbol{C}_\phi, \ \boldsymbol{C}_2 \equiv \boldsymbol{C}_\chi$.

- All anisotropy dependence collapses into three scalar angular weights :

$$J_{11} = \left\langle \left[\hat{n}^\top \boldsymbol{C}_1 \hat{n}\right]^{-3/2} \right\rangle_{\hat{n}}, \quad J_{22} = \left\langle \left[\hat{n}^\top \boldsymbol{C}_2 \hat{n}\right]^{-3/2} \right\rangle_{\hat{n}}$$

$$J_{12} = \int_0^1 \left(\det\left[(1-\tau)\boldsymbol{C}_1 + \tau \boldsymbol{C}_2\right]\right)^{-1/2} d\tau,$$

- Then tadpole residue becomes

$$\mathcal{T} \equiv \epsilon\, \boldsymbol{T} = -\frac{1}{8\pi^2} \begin{pmatrix} J_{11} M_{11}^2 & J_{12} M_{12}^2 \\ J_{12} M_{12}^2 & J_{22} M_{22}^2 \end{pmatrix}.$$

Explicit form of beta functions and fixed points

Case 1 : $\boldsymbol{Z}_t = 1, \boldsymbol{Y} = 0$

- One can show that $J_{aa} = \left\langle \left(\hat{n}^\top \boldsymbol{C}_a \hat{n}\right)^{-3/2} \right\rangle_{S^2} = (\det \boldsymbol{C_a})^{-1/2}$ in our case.
- The cross-term J_{12} is defined by the generalized eigenvalues of the pair $(\boldsymbol{C}_1, \boldsymbol{C}_2)$ denoted by $\gamma_1, \gamma_2, \gamma_3 > 0$, i.e. $\mathrm{C}_2 v = \gamma \mathrm{C}_1 v$.
- Then one can show that $J_{12}^2 \leq J_{11} J_{22}$ and

$$J_{12} = (\det \boldsymbol{C}_1)^{-1/2} \int_0^1 \prod_{i=1}^{3} ((1-\tau) + \tau\gamma_i)^{-1/2} \, d\tau$$

$$= (\det \boldsymbol{C}_1)^{-1/2} F_D^{(3)} \left(1; \frac{1}{2}, \frac{1}{2}, \frac{1}{2}; 2; 1-\gamma_1, 1-\gamma_2, 1-\gamma_3\right),$$

where F_D is a hypergeometric function.

Explicit form of beta functions and fixed points

Case 1 : $\boldsymbol{Z}_t = 1, \boldsymbol{Y} = 0$

- For quartic couplings we then have

$$\beta_{\lambda_1} = -\varepsilon\lambda_1 + \frac{3}{16\pi^2}\left[J_{11}\lambda_1^2 + 2J_{12}\lambda_4^2 + J_{22}\lambda_3^2\right],$$

$$\beta_{\lambda_2} = -\varepsilon\lambda_2 + \frac{3}{16\pi^2}\left[J_{22}\lambda_2^2 + 2J_{12}\lambda_5^2 + J_{11}\lambda_3^2\right],$$

$$\begin{aligned}\beta_{\lambda_3} &= -\varepsilon\lambda_3 \\ &+ \frac{1}{16\pi^2}\Big[J_{11}\left(\lambda_1\lambda_3 + 2\lambda_4^2\right) + 4J_{12}\lambda_3^2 + J_{22}\left(\lambda_2\lambda_3 + 2\lambda_5^2\right) \\ &+ 2J_{12}\lambda_4\lambda_5\Big],\end{aligned}$$

$$\beta_{\lambda_4} = -\varepsilon\lambda_4 + \frac{3}{16\pi^2}\left[J_{11}\lambda_1\lambda_4 + 2J_{12}\lambda_3\lambda_4 + J_{22}\lambda_3\lambda_5\right],$$

$$\beta_{\lambda_5} = -\varepsilon\lambda_5 + \frac{3}{16\pi^2}\left[J_{22}\lambda_2\lambda_5 + 2J_{12}\lambda_3\lambda_5 + J_{11}\lambda_3\lambda_4\right],$$

- Similar expressions $\rightarrow$ for other couplings.

Explicit form of beta functions and fixed points

Case 1 : $\boldsymbol{Z_t} = 1, \boldsymbol{Y} = 0$

- Sub-case : $\lambda_{4,5} = 0$ and $h_i = 0$
- Beta functions are

$$\beta_{\lambda_1} = -\varepsilon\lambda_1 + \frac{3}{16\pi^2}\left[J_{11}\lambda_1^2 + J_{22}\lambda_3^2\right],$$

$$\beta_{\lambda_2} = -\varepsilon\lambda_2 + \frac{3}{16\pi^2}\left[J_{22}\lambda_2^2 + J_{11}\lambda_3^2\right],$$

$$\beta_{\lambda_3} = -\varepsilon\lambda_3 + \frac{1}{16\pi^2}\left[J_{11}\lambda_1\lambda_3 + 4J_{12}\lambda_3^2 + J_{22}\lambda_2\lambda_3\right].$$

Generalized Wilson-Fisher fixed point

Case 1 : $\boldsymbol{Z_t} = 1, \boldsymbol{Y} = 0$

- Sub-case : $\lambda_{4,5} = 0$ and $h_i = 0$.
- Formal solutions to set equations $\beta_{\lambda_i} = 0$ corresponds to different **fixed points** of our model :

$$\lambda_1^* = \frac{\frac{16\pi^2}{3}\varepsilon \cdot 3 - 4J_{12}\lambda_3^*}{2J_{11}}, \quad \lambda_2^* = \frac{\frac{16\pi^2}{3}\varepsilon \cdot 3 - 4J_{12}\lambda_3^*}{2J_{22}},$$

$$\lambda_3^* = \frac{\frac{16\pi^2}{3}\varepsilon\left(4J_{12} \pm \sqrt{4J_{12}^2 - 3J_{11}J_{22}}\right)}{2\left(4J_{12}^2 + J_{11}J_{22}\right)},$$

Generalized Wilson-Fisher fixed point exists when mismatch parameter ρ obeys

$$\rho^2 = \frac{J_{12}^2}{J_{11}J_{22}} \in [3/4, 1]$$

i.e. if anisotropy mismatch is too strong then WF point disappears. If speed matrices are isotropic, i.e. just two sound speeds this reduces to simple restriction of sound speeds ratio $c_2/c_1 \in [1/9, 9]$

Explicit form of beta functions and fixed points

Fixed points important for :

- Describes real continuous phase transitions ;
- Critical exponents and stability.

Physical interpretation

- In our considerations everything is controlled by

$$\mathcal{K}(s_a, s_b) \equiv \frac{1}{\sqrt{s_a}\sqrt{s_b}\left(\sqrt{s_a}+\sqrt{s_b}\right)} \to \frac{1}{v_a v_b (v_a + v_b)},$$

so what is the interpretation ?

- Actually, it is a phase space

$$\frac{d\Phi_2}{d\Omega_{\hat{\boldsymbol{n}}}} = \frac{1}{16\pi^2} \frac{1}{v_a(\hat{\boldsymbol{n}}) v_b(\hat{\boldsymbol{n}}) \left[v_a(\hat{\boldsymbol{n}}) + v_b(\hat{\boldsymbol{n}})\right]} = \frac{1}{16\pi^2}\mathcal{K}(\hat{\boldsymbol{n}}).$$

- Next, let us recall the well-known spectral representation or Kallen–Lehmann representation of the two-point function :

$$\Pi\left(p^2\right) \equiv \int_0^\infty dq^2 \frac{\rho\left(q^2\right)}{p^2 - q^2 + i\varepsilon}.$$

- It turns out, that our bubble master integral has the form of

$$B = \frac{1}{\pi}\int_0^\infty dE \frac{E}{p_0^2 + E^2}\Phi_2(E),$$

with $\Phi_2(E)$ being two-particle density of states per solid angle.

Explicit form of beta functions and fixed points

Case 2 : $\boldsymbol{Z}_t = 1$, $\boldsymbol{Y} \neq 0$

- Similar things $\to$ but 6 J-weights ! We do not have simple RG-closed $\lambda_{1,2,3}$ sector, all couplings are generated.
- Here we have

$$J^{(Y)}_{11} = 2\Big\langle \sum_{a,b} \mathcal{K}(s_a, s_b)\, p_a\, p_b \Big\rangle_{S^2}, \qquad J^{(Y)}_{22} = 2\Big\langle \sum_{a,b} \mathcal{K}(s_a, s_b)\, q_a\, q_b \Big\rangle_{S^2},$$

$$J^{(Y)}_{12} = 2\Big\langle \sum_{a,b} \mathcal{K}(s_a, s_b)\, p_a\, q_b \Big\rangle_{S^2}, \qquad J^{(Y)}_{1122} = 2\Big\langle \sum_{a,b} \mathcal{K}(s_a, s_b)\, r_a\, r_b \Big\rangle_{S^2},$$

$$J^{(Y)}_{1112} = 2\Big\langle \sum_{a,b} \mathcal{K}(s_a, s_b)\, p_a\, r_b \Big\rangle_{S^2}, \qquad J^{(Y)}_{2212} = 2\Big\langle \sum_{a,b} \mathcal{K}(s_a, s_b)\, q_a\, r_b \Big\rangle_{S^2},$$

  where p_a, q_a, etc are just some numbers.

- So J_{11}, J_{22}, J_{12} are present in previous slides and measure how many pure ϕ or χ run in phase space, while new weights like J_{2212} are responsible for off-diagonal correlations involving coherent mixture of both fields.

$$[\mathbb{B}_{ij;kl}]_{\mathbf{pole}} = \frac{1}{4\pi^2\epsilon}\Big\langle \sum_{a,b} \mathcal{K}(s_a, s_b) \boldsymbol{P}_a(\hat{\boldsymbol{n}})_{ik} \boldsymbol{P}_b(\hat{\boldsymbol{n}})_{jl} \Big\rangle_{\hat{\boldsymbol{n}}}.$$

Conclusion and outlook

- Plan to go beyond one loop ;
- Froissart-Martin bound modification in such theories ?
- What happen in other renormalization schemes ?
- Stability and critical exponents ;
- Take into considerations more fields as well as other types of matter : fermions, gauge theories, etc ;
- All comments and ideas are welcomed !

THANK YOU FOR YOUR ATTENTION !
AND HAPPY QUARKS-2026 ! ♡