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based on works:

[1] PRL 129(15): 151601, (2022)
[2] PLB 840, 137839, (2023)
[3] PRD 108, 12, L121701 (2023)
[4] PRD 109, 10, 105001 (2024)
[5] 2304.13151

Echoes of gravity and chiral anomaly in a relativistic vortical fluid

Quarks-2024, Ivanisovo,
19 to 24 May 2024

Contents

Part 1

Introduction

Gravity in flat spacetime: Cheshire cat grin

“Well! I've often seen a cat without a grin,' thought Alice 'but a grin without a cat! It's the most curious thing i ever saw in my life!”

— Lewis Carroll, Alice in Wonderland

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CVE and CME – new anomalous transport

Consistency with **quantum anomaly** modifies hydrodynamic equations

[V. I. Zakharov, Lect. Notes Phys.871,295(2013)]

[D.T. Son, P. Surowka, PRL, 103 (2009) 191601]

Quantum chiral anomaly (gauge part)

$$\langle \partial_\mu \hat{j}^\mu_A \rangle = -\frac{C}{8}\varepsilon^{\mu\nu\alpha\beta}F_{\mu\nu}F_{\alpha\beta}$$

Entropy current satisfies **second law** of thermodynamics $\partial_\mu s^\mu \geqslant 0$

Chiral magnetic effect (CME)

CME: $j^\mu = C\mu_5 B^\mu$

Current flows along the **magnetic field**

Chiral vortical effect (CVE)

$$\omega^\mu = \frac{1}{2}\varepsilon^{\mu\nu\alpha\beta}u_\nu\partial_\alpha u_\beta$$

CVE: $j^\mu_A = C\mu^2\omega^\mu$

Current flows along the **vorticity**

Derivation **without entropy** current and generalization to the **second order** in gradients:

[Shi-Zheng Yang, Jian-Hua Gao, Zuo-Tang Liang, Symmetry 14 (2022) 5, 948]

[M. Buzzegoli, Lect. Notes Phys. 987, 53-93 (2021)]

Use **global equilibrium**

Modern development and the problem

What about the **gravitational chiral anomaly**?

- The gravitational chiral anomaly (unlike gauge part) grows **rapidly** with **spin**:

$$\langle \nabla_\mu \hat{j}^\mu_A \rangle_S = \frac{(S - 2S^3)}{96\pi^2\sqrt{-g}} \varepsilon^{\mu\nu\rho\sigma} R_{\mu\nu\kappa\lambda} R_{\rho\sigma}{}^{\kappa\lambda}$$

[M. J. Duff, in First School on Supergravity (1982) arXiv:1201.0386]

[S. M. Christensen, M. J. Duff, Nucl. Phys. B 154, 301–342 (1979)]

- How does the **gravitational** chiral anomaly manifest itself in **hydrodynamics**?
- Is it possible to see the **cubic factor** $S - 2S^3$ in hydrodynamics?

I. Gravitational chiral anomaly in hydrodynamics

General derivation

Decomposition of the tensors

- **Components of the thermal vorticity tensor**

6 components

[M. Buzzegoli, E. Grossi, F. Becattini, JHEP 10 (2017) 091]

$$\varpi_{\mu\nu} = \epsilon_{\mu\nu\alpha\beta} w^\alpha u^\beta + \alpha_\mu u_\nu - \alpha_\nu u_\mu$$

Similar to the expansion for the electromagnetic field

- We also decompose the **Riemann tensor** into the **components:**

$$\begin{aligned} R_{\mu\nu\alpha\beta} = & \; u_\mu u_\alpha A_{\nu\beta} + u_\nu u_\beta A_{\mu\alpha} - u_\nu u_\alpha A_{\mu\beta} - u_\mu u_\beta A_{\nu\alpha} \\ & + \epsilon_{\mu\nu\lambda\rho} u^\rho (u_\alpha B^\lambda{}_\beta - u_\beta B^\lambda{}_\alpha) \\ & + \epsilon_{\alpha\beta\lambda\rho} u^\rho (u_\mu B^\lambda{}_\nu - u_\nu B^\lambda{}_\mu) \\ & + \epsilon_{\mu\nu\lambda\rho} \epsilon_{\alpha\beta\eta\sigma} u^\rho u^\sigma C^{\lambda\eta} \end{aligned}$$

20 components

Coincide with **3d** tensors in the fluid rest frame:

[L. D. Landau and E. M. Lifschits, The Classical Theory of Fields, Vol. 2, 1975]

[A. Z. Petrov, 1950]

- We consider **Ricci-flat** spaces $R_{\mu\nu} = 0$

Gradient expansion in the curved spacetime

The **gravitational chiral anomaly** has the **4th order** in gradients – it is to be related to the **3rd order** terms in gradient expansion of the axial **current**.

Hydrodynamic expansion of the axial current up to the **3rd order** in gradients:

Survive in flat spacetime

$$j_\mu^{A(3)} = \xi_1(T) w^2 w_\mu + \xi_2(T) \alpha^2 w_\mu + \xi_3(T)(\alpha w) w_\mu + \xi_4(T) A_{\mu\nu} w^\nu + \xi_5(T) B_{\mu\nu} \alpha^\nu$$

arbitrary coefficients

"gravitational" currents

See also gradient expansion for the fluid in the gravitational field, e.g.:

[P. Romatschke, Class. Quant. Grav. 27, 025006 (2010)]

[S. M. Diles, L. A. H. Mamani, A. S. Miranda, V. T. Zanchin, JHEP 2020, 1–40 (2020)]

Anomaly matching: principle

Following [D.T. Son, P. Surowka, PRL, 103 (2009) 191601]
- it would be necessary to construct the **entropy current**.

However in [Shi-Zheng Yang, Jian-Hua Gao, and Zuo-Tang Liang, Symmetry 14, 948 (2022)]

it is shown that it is possible to use the **global equilibrium** condition

$$\nabla_\mu \beta_\nu + \nabla_\nu \beta_\mu = 0$$

After that it is enough to consider ***only*** the equation for the current.

- Good for gravity, which is complicated in general case!

We use only:

$$\nabla_\mu j^\mu_A = \mathcal{N} \epsilon^{\mu\nu\alpha\beta} R_{\mu\nu\lambda\rho} R_{\alpha\beta}{}^{\lambda\rho}$$

Substitute the gradient expansion:

$$\nabla^\mu \Big(\xi_1(T) w^2 w_\mu + \xi_2(T) \alpha^2 w_\mu + \xi_3(T)(\alpha w) w_\mu + \xi_4(T) A_{\mu\nu} w^\nu + \xi_5(T) B_{\mu\nu} \alpha^\nu \Big) = 32\mathcal{N} A_{\mu\nu} B^{\mu\nu}$$

Anomaly matching: system of equations

This **system** of linear **differential** equations has the form:

$$\left.\begin{aligned} -3T\xi_1 + T^2\xi_1' + 2T\xi_3 &= 0 \\ -3T\xi_2 + T^2\xi_2' - T\xi_3 + T^2\xi_3' &= 0 \\ T^2\xi_4' + 3T\xi_5 + 2T^{-1}\xi_2 + T^{-1}\xi_3 &= 0 \\ -2T^{-1}\xi_1 - 3T\xi_4 - T\xi_5 &= 0 \\ T^2\xi_5' - T\xi_5 - T^{-1}\xi_3 &= 0 \\ -T^{-1}\xi_4 + T^{-1}\xi_5 - 32\mathscr{N} &= 0 \end{aligned}\right\}$$

Can be solved!

Includes the factor from the **gravitational chiral anomaly**

Solution

Let's move on to the limit of **flat space-time**. Despite the absence of a gravitational field, there **remains** a contribution to the axial current induced by the **gravitational chiral anomaly:**

Zero gravitational field

Limit $R_{\mu\nu\alpha\beta} = 0$

$$j_\mu^A = \lambda_1(\omega_\nu\omega^\nu)\omega_\mu + \lambda_2(a_\nu a^\nu)w_\mu$$

$$\frac{\lambda_1 - \lambda_2}{32} = \mathcal{N}$$

Nonzero gravitational field

$$R_{\mu\nu\alpha\beta} \neq 0$$

$$\nabla_\mu j_A^\mu = \mathcal{N}\epsilon^{\mu\nu\alpha\beta}R_{\mu\nu\lambda\rho}R_{\alpha\beta}{}^{\lambda\rho}$$

- A new type of anomalous transport – the **Kinematical Vortical Effect (KVE)**. Does not explicitly depend on temperature and density → determined only by the **kinematics** of the flow.

Direct verification: spin 1/2

Transport coefficients and anomaly: spin 1/2

- In [GP, O.V. Teryaev, and V.I. Zakharov, JHEP, 02:146, 2019], [V. E. Ambrus, JHEP, 08:016, 2020], [A. Palermo, et al. JHEP 10 (2021) 077] and for ω^3 in [A. Vilenkin, Phys. Rev., D20:1807-1812, 1979] the following expression was obtained:

$$j_\mu^A = \left(\frac{T^2}{6} + \frac{\mu^2}{2\pi^2} \underbrace{- \frac{\omega^2}{24\pi^2} - \frac{a^2}{8\pi^2}}_{\text{KVE}}\right)\omega_\mu$$

- Comparing it with the well-known anomaly [L. Alvarez-Gaume, E. Witten, Nucl. Phys., B234:269, 1984]:

$$\nabla_\mu j_A^\mu = \frac{1}{384\pi^2\sqrt{-g}}\varepsilon^{\mu\nu\alpha\beta}R_{\mu\nu\lambda\rho}R_{\alpha\beta}{}^{\lambda\rho}$$

We see that the formula is **fulfilled**:

$$\boxed{\frac{\lambda_1 - \lambda_2}{32} = \mathcal{N}}$$

$$\Rightarrow \left(-\frac{1}{24\pi^2} + \frac{1}{8\pi^2}\right)/32 = \frac{1}{384\pi^2}$$

Correspondence between **gravity** and **hydrodynamics** is **confirmed!**

Direct verification: spin 3/2

Rarita-Schwinger-Adler model of spin 3/2

The **Rarita-Schwinger theory** – well-known theory of spin 3/2.
But this theory has a number of **pathologies.**

- For example, it **doesn't allow to construct perturbation theory!**

Solved in **[Stephen L. Adler. Phys. Rev. D, 97(4):045014, 2018]** by introducing of interaction with additional spin ½ field:

Action:

additional spin ½ field

$$S = \int d^4x \, (-\varepsilon^{\lambda\rho\mu\nu}\bar{\psi}_\lambda\gamma_5\gamma_\mu\partial_\nu\psi_\rho + i\bar{\lambda}\gamma^\mu\partial_\mu\lambda - im\bar{\lambda}\gamma^\mu\psi_\mu + im\bar{\psi}_\mu\gamma^\mu\lambda)$$

"coupling mass"

Anomaly was found in **[Prokhorov, Teryaev, Zakharov, Phys.Rev.D 106 (2022) 2, 025022]**

$$\nabla_\mu j^\mu_A = -\frac{19}{384\pi^2\sqrt{-g}}\varepsilon^{\mu\nu\rho\sigma}R_{\mu\nu\kappa\lambda}R_{\rho\sigma}{}^{\kappa\lambda}$$

-19 times different from the anomaly for spin ½

Zubarev density operator

Global Equilibrium Conditions

$$\nabla_\mu \beta_\nu + \nabla_\nu \beta_\mu = 0 \qquad \nabla_\mu \zeta = 0$$

Form of the density operator for a medium with rotation and acceleration

$$\widehat{\rho} = \frac{1}{Z} \exp\left[-b_\mu \widehat{P}^\mu + \frac{1}{2} \varpi_{\mu\nu} \widehat{J}^{\mu\nu} + \zeta \widehat{Q} \right]$$

Thermal vorticity tensor

Lorentz Transform Generators

$$\widehat{J}^{\mu\nu} = \int_\Sigma \mathrm{d}\Sigma_\lambda \left(x^\mu \widehat{T}^{\lambda\nu} - x^\nu \widehat{T}^{\lambda\mu} \right)$$

$$\varpi_{\mu\nu} \hat{J}^{\mu\nu} = -2\alpha^\rho \hat{K}_\rho - 2 w^\rho \hat{J}_\rho$$

$\hat{K}^\mu$ – boost (related to **acceleration**)

$\hat{J}^\mu$ – angular momentum (related to **vorticity**)

KVE IN RSA THEORY: CALCULATION

- Our *goal* is to calculate the conductivities λ_1 and λ_2 in the KVE current:

$$j^{\mu}_{A,KVE} = \lambda_1(\omega_\nu\omega^\nu)\omega^\mu + \lambda_2(a_\nu a^\nu)\omega^\mu$$

- Using the perturbation theory, we obtain:

$$\lambda_1 = -\frac{1}{6}\int_0^{|\beta|}[d\tau]\langle T_\tau \hat{J}^3_{-i\tau_x}\hat{J}^3_{-i\tau_y}\hat{J}^3_{-i\tau_z}\hat{j}^3_A(0)\rangle_{T,c}$$

- The matrix element has the form of a product of **vertices** and **propagators**.

Vertices

$$\mathcal{D}^{\mu\nu}_{(ij)} = -\frac{1}{2}(-i)^{\delta_{0\mu}+\delta_{0\nu}}\varepsilon^{ij\nu\beta}\left(\gamma_5\tilde{\gamma}_\mu\tilde{\partial}^{X_2}_\beta - \frac{1}{4}\gamma_5\tilde{\gamma}_\beta[\tilde{\gamma}_\vartheta,\tilde{\gamma}_\mu]\left(\tilde{\partial}^{X_1}_\vartheta + \tilde{\partial}^{X_2}_\vartheta\right)\right) + (\mu \leftrightarrow \nu)$$

$0 \leq (i,j) < 4$

Propagators

$$\langle T_\tau \tilde{\psi}_{a\mu}(X_1)\bar{\tilde{\psi}}_{b\nu}(X_2)\rangle_T = \sum_P e^{iP_\alpha(X_1-X_2)^\alpha}\frac{i}{2P^2}\left(\tilde{\gamma}_\nu \not{P}\tilde{\gamma}_\mu + 2\left[\frac{1}{m^2} - \frac{2}{P^2}\right]P_\mu P_\nu \not{P}\right)_{ab}$$

$$\langle T_\tau \tilde{\psi}_{a\mu}(X_1)\bar{\lambda}_b(X_2)\rangle_T = \sum_P e^{iP_\alpha(X_1-X_2)^\alpha}\frac{-P_\mu \not{P}_{ab}}{mP^2}$$

$$\langle T_\tau \lambda_a(X_1)\bar{\lambda}_b(X_2)\rangle_T = 0$$

Mixed terms are non-zero

here $P_\mu = (p_n, -\mathbf{p})\,,\ p_n = (2n+1)\pi T$

Field λ is **non-propagating**

KVE vs Gravitational Anomaly

The obtained formula for cubic gradients (KVE):

$$j_\mu^{A(3)} = \left(-\frac{53}{24\pi^2}\omega^2 - \frac{5}{8\pi^2}a^2\right)\omega_\mu$$

Gravitational chiral **anomaly**:

$$\nabla_\mu j_A^\mu = \frac{-19}{384\pi^2\sqrt{-g}}\varepsilon^{\mu\nu\alpha\beta}R_{\mu\nu\lambda\rho}R_{\alpha\beta}{}^{\lambda\rho}$$

$$\frac{\lambda_1 - \lambda_2}{32} = \mathcal{N}$$

Direct **verification**:

$$\left(-\frac{53}{24\pi^2} + \frac{5}{8\pi^2}\right)\Big/ 32 = -\frac{19}{384\pi^2}$$

Coincidence of hydrodynamics and gravitational anomaly!

- The relationship between the transport coefficients in a **vortical accelerated fluid** and the **gravitational** chiral **anomaly** is shown!
- Verification in a **nontrivial** case with higher spins and interaction.

Recent Development

Generalization to (anti)de Sitter space

- Going **beyond approximation** $R_{\mu\nu} = 0$ **[Khakimov, Prokhorov, Teryaev, Zakharov,2401.09247 (2024)]**

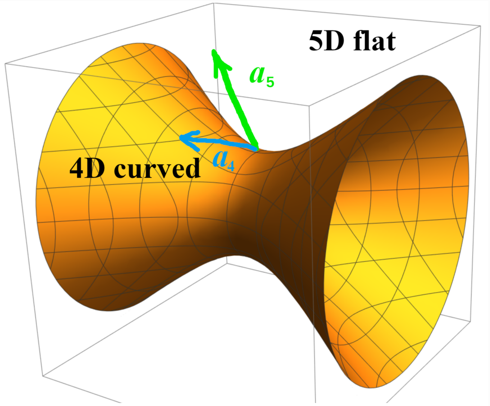
$$\frac{\lambda_1 - \lambda_2}{32} = \mathcal{N}$$ -- **anomaly-hydro** relation remains valid

$$j^A_\mu \sim a^2 \omega_\mu \Longleftrightarrow j^A_\mu \sim R\omega_\mu$$ -- equivalence principle in higher orders

- **5-dimensional Unruh effect**: **[Khakimov, Prokhorov, Teryaev, Zakharov, Phys.Rev.D 108, 12, L121701 (2023)]**

The temperature measured by an accelerated observer in **(A)dS space** is determined by the 5-dimensional acceleration!

[S. Deser and O. Levin, Phys. Rev. D, 59:064004, 1999]



Hydrodynamic expansion for the stress-energy tensor:

$$\langle \hat{T}^{\mu\nu}\rangle_{s=1/2} = \left[\frac{7\pi^2}{180}T^4 + \frac{1}{72}\left(|a|^2 + \frac{R}{12}\right)T^2 - \frac{17}{2880\pi^2}\left(|a|^2 + \frac{R}{12}\right)^2\right]\left(4u^\mu u^\nu - g^{\mu\nu}\right) + \frac{11}{960\pi^2}\left(\frac{R}{12}\right)^2 g^{\mu\nu}$$

$$\langle \hat{T}^{\mu\nu}\rangle(T = T_{UR}) = \frac{k}{4}R^2 g^{\mu\nu}$$ has a vacuum form ⇒ $$T_{UR} = \frac{a_5}{2\pi} = \sqrt{T_U^2 + T_R^2} = \frac{\sqrt{|a|^2 + R/12}}{2\pi}$$

II. Novel phase transitions in the accelerated matter

Euclidean Rindler space

The effects of acceleration can also be investigated from the point of view of an **accelerated observer**. In this case, the **Rindler coordinates** are to be used:

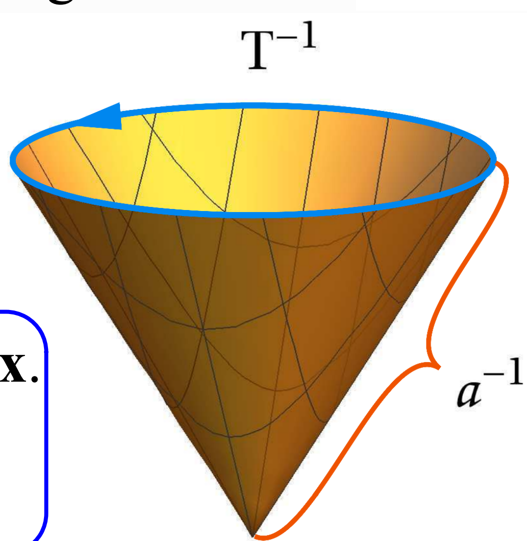
$$ds^2 = -\rho^2 d\theta^2 + d\mathrm{x}^2 + d\mathrm{y}^2 + d\rho^2$$

Passing to imaginary time:

$$ds^2 = \rho^2 d\theta^2 + d\rho^2 + d\mathbf{x}_\perp^2$$

$\mathcal{M} = \mathbb{R}^2 \otimes \mathcal{C}_\nu^2$

It describes a flat two-dimensional cone with an angular deficit $2\pi - a/T$.
This metric contains a **conical singularity** at $\rho = 0$



Dictionary for translation
Thermodynamic characteristics in *Geometrical*:

Inverse **acceleration** $\Longleftrightarrow$ **distance** from the **vertex**.

Inverse proper **temperature** $\Longleftrightarrow$ **circumference**.

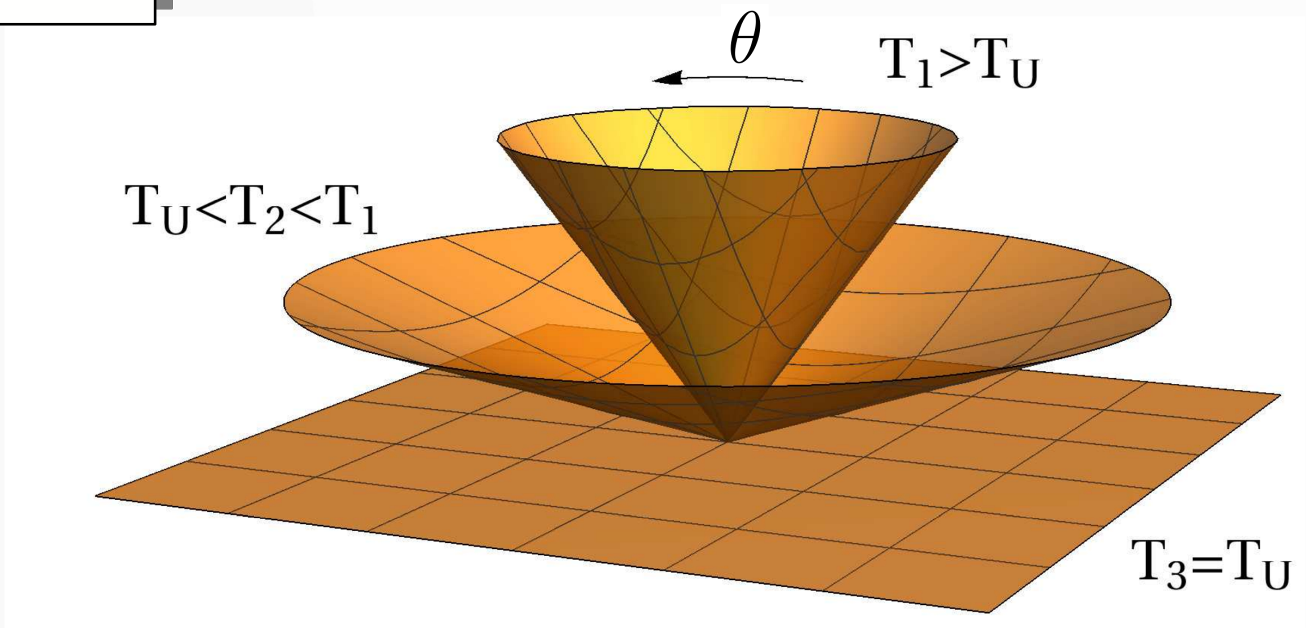
Euclidean Rindler space

- We will show **critical behaviour** at the **Unruh temperature:**

$$T_U = \frac{\hbar|a|}{2\pi k_B c}$$

[GP, O. Teryaev, V. Zakharov, 2304.13151]

[GP, O. Teryaev, V. Zakharov Phys. Rev. D 100 (2019) 12, 125009]



- At $T < |a|/2\pi$ the angle becomes **greater than 360°.**
- Similar effects were discussed: **[GP, O. Teryaev, V. Zakharov Phys.Rev.D 98 (2018) 7, 071901]** **[E.T. Akhmedov, D.V. Diakonov Phys.Rev.D 105 (2022) 10, 105003]**

Matsubara modes on the horizon

- Consider the **eigenmodes** of the square of the Dirac operator

$$\not{D}_x^2 \varphi(x) = -\lambda^2 \phi(x)$$

- There are **two solutions**!

$$\phi_q^{\pm}(x) = \frac{\sqrt{\nu}}{4\pi^{3/2}}\, e^{ip_{\rm x}{\rm x}+ip_{\rm y}{\rm y}+i(n+\frac{1}{2})\varphi} J_{\pm\beta_{s_1}}(\xi\rho)\, w_{(s_1,s_2)}$$

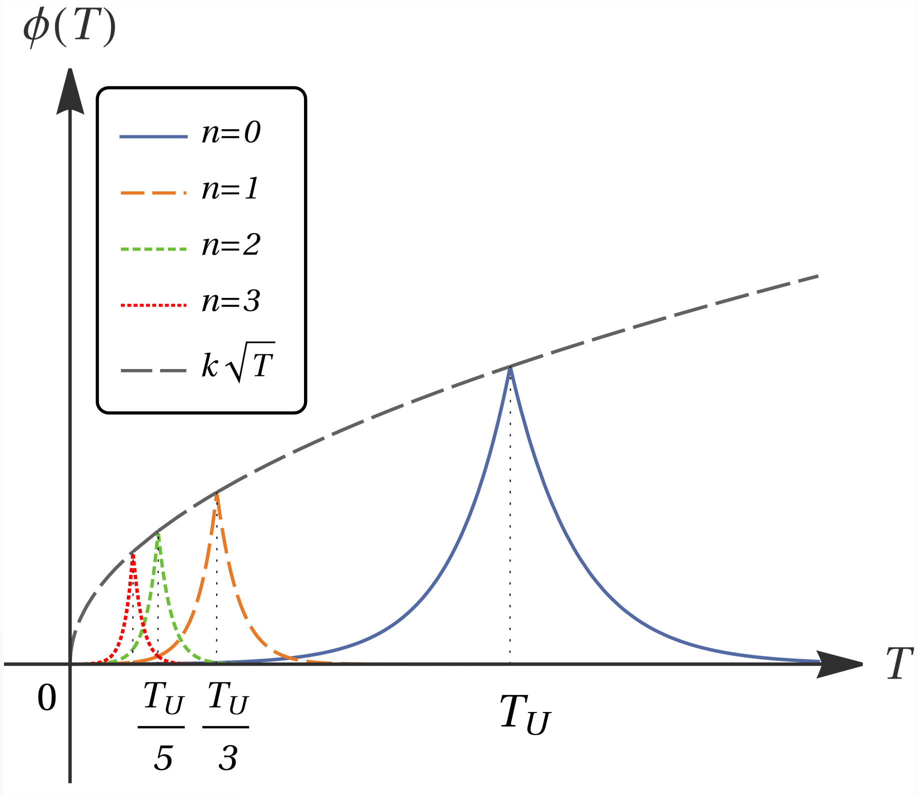
- **On the horizon** only one of the two solutions is **finite**!

- When passing through each point

$$T_k = T_U/(2k+1)$$

two **lowest** Matsubara modes **change the solution**

- Lead to **peaks** in the behavior of the modes:



Stress-energy tensor: phase transition

- **Different** mean values below and above $T_k = T_U/(2k+1)$

$$T > \frac{|a|}{2\pi} : \langle \hat{T}^\alpha_\beta \rangle = \left(\frac{7\pi^2 T^4}{60} + \frac{|a|^2 T^2}{24} - \frac{17|a|^4}{960\pi^2} \right) \left(u^\alpha u_\beta - \frac{1}{3} \Delta^\alpha_\beta \right)$$

$$\frac{|a|}{6\pi} < T < \frac{|a|}{2\pi} : \langle \hat{T}^\alpha_\beta \rangle = \left(\frac{127\pi^2 T^4}{60} - \frac{11|a|^2 T^2}{24} - \frac{17|a|^4}{960\pi^2} \right) \left(u^\alpha u_\beta - \frac{1}{3} \Delta^\alpha_\beta \right) + \left(\pi|a|T^3 - \frac{T|a|^3}{4\pi} \right) \widetilde{\Delta}^\alpha_\beta ,$$

$$\widetilde{\Delta}^\alpha_\beta = \Delta^\alpha_\beta + \frac{a^\alpha a_\beta}{|a|^2} \qquad \Delta^\alpha_\beta = \delta^\alpha_\beta - u^\alpha u_\beta$$

- Related to the **anti-Hermiticity** of spin part of the **boost operator**:

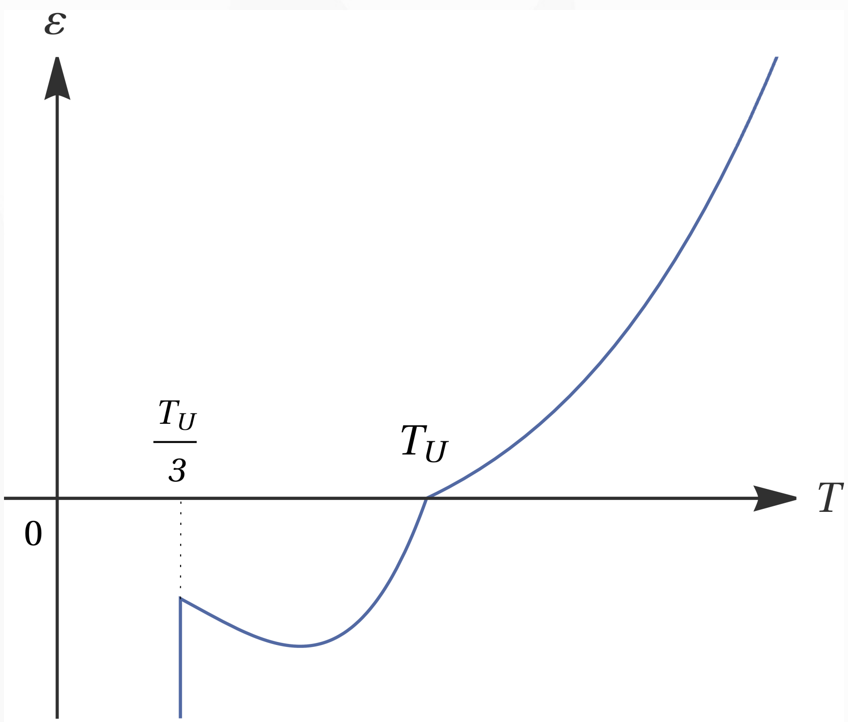
Effective interaction with acceleration in statistics:

$$H \to H - \mathbf{a} \cdot \mathbf{K}$$

For spin part of the boost:

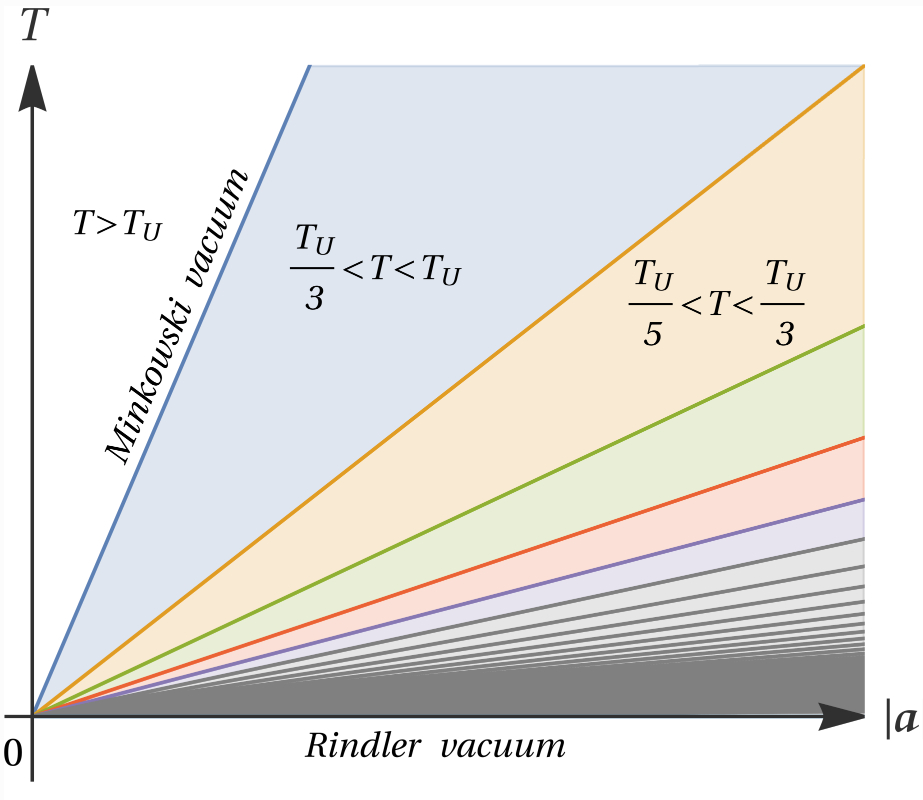
$$\Sigma_0 \phi(x) = s_1 \frac{i}{2} \phi(x) , \quad s_1 = \pm 1$$

Stress-energy tensor: phase transition



Phase transitions at the points:

$$T_k = T_U/(2k+1)$$



phase diagram

Conclusion

Conclusion

- The relationship between the hydrodynamic current in the third order of gradient expansion $\lambda_1(\omega_\nu\omega^\nu)\omega_\mu$ and $\lambda_2(a_\nu a^\nu)w_\mu$, the Kinematic Vortical Effect (KVE), and the gravitational chiral anomaly has been established.

- The obtained formula has been verified directly for spins 1/2 and 3/2.

- It is shown that the effects survive when there is also constant curvature. The role of five-dimensional acceleration is demonstrated for the case of an accelerated observer in (A)dS space.

- There is a novel second order phase transition at the Unruh temperature in the accelerated medium and subsequent first order phase transitions at the Unruh temperature divided by the odd number.

- These phase transitions are related to the singularity of the lowest Matsubara modes on the horizon and anti-Hermiticity of the spin boost.

Additional slides

Exact Green Function

- **Green function** of the square operator can be constructed from the modes:

$$G^E_{ij}(x;x') = \sum_{s_1,s_2}\sum_n \iiint dp_x\, dp_y\, \xi\, d\xi \frac{\phi^{reg}_{q,i}(x)\phi^{reg\,\dagger}_{q,j}(x')}{\xi^2+p_x^2+p_y^2}$$

- **Green function** of the Dirac operator can be constructed from the Green function of square operator:

$$S_E(x;x') = \not{D}_x G_E(x;x')$$

should be renormalized

$$S_E^{ren} = S_E - S_E^0$$

$$\nu = 1$$
$$T = T_U$$

- The **result** looks like:

$$G^E(x;x'|N_0) = \frac{\nu\left[\sinh\left(\frac{1+\nu}{2}\vartheta - \vartheta\nu N_0\right)e^{(2N_0+1)\Delta\phi\Sigma_0} - \sinh\left(\frac{1-\nu}{2}\vartheta - \vartheta\nu N_0\right)e^{(2N_0-1)\Delta\phi\Sigma_0}\right]}{8\pi^2\rho\rho'\sinh\vartheta\left[\cosh(\nu\vartheta) - \cos\Delta\varphi\right]}$$

- The number of **complete revolutions** per angle 2π on the cone and simultaneously the number of **pairs of Matsubara** modes that have changed their solutions

$$N_0 = \left\lfloor \frac{1}{2\nu} + \frac{1}{2} \right\rfloor = \left\lfloor \frac{a}{4\pi T} + \frac{1}{2} \right\rfloor$$

Stress-energy tensor: phase transition

- **Second order** phase transition:

$$\frac{\partial \varepsilon}{\partial T}\Big|_{T\to T_U+0} = \frac{4\pi T_U^3}{5}, \qquad \frac{\partial \varepsilon}{\partial T}\Big|_{T\to T_U-0} = \frac{24\pi T_U^3}{5}$$

$T > T_U$ $\qquad$ 0

$T < T_U$ $\qquad$ $\langle \hat{T}^\beta_\beta \rangle = \frac{\nu(\nu^2-1)}{4\pi^2\rho^4} = 2\pi T|a|\left(T^2 - \frac{|a|^2}{4\pi^2}\right)$

- Trace as an **order parameter**?

- The change in energy is associated with a change in the energy of the two lowest modes:

$$\Delta\varepsilon = \Delta\varepsilon_{(n=0,\, s_1=1)} + \Delta\varepsilon_{(n=-1,\, s_1=-1)} = 2\pi^2 T^4 - \frac{T^2|a|^2}{2}$$