Renormalon-chain contributions to two-point correlators of nonlocal quark currents and light-meson distribution amplitudes

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PRD **108** (2023) 096015

Outline

Introduction: What are QCD composite vertices and renormalon chains?

The correlator of two composite operators

$(x, \underline{0})$ moment of the correlator and mesonic distribution amplitudes

Renormalon chains in light-meson distribution amplitudes (DA)

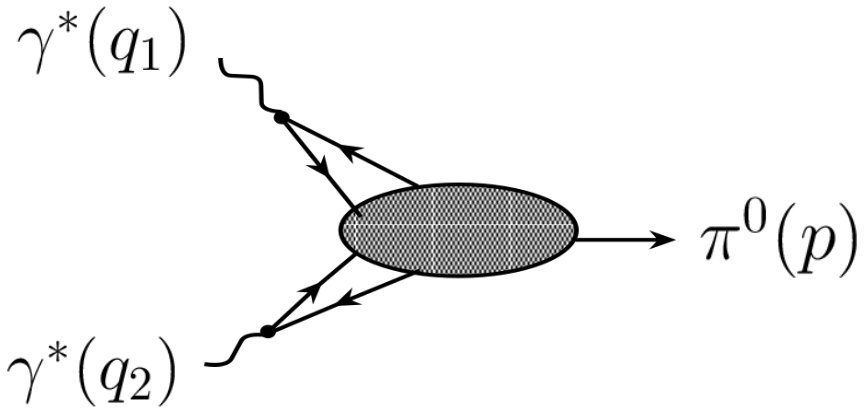
Summary

Nonlocal composite vertices in QCD

Q: In what physical setting do the **composite vertices** emerge?

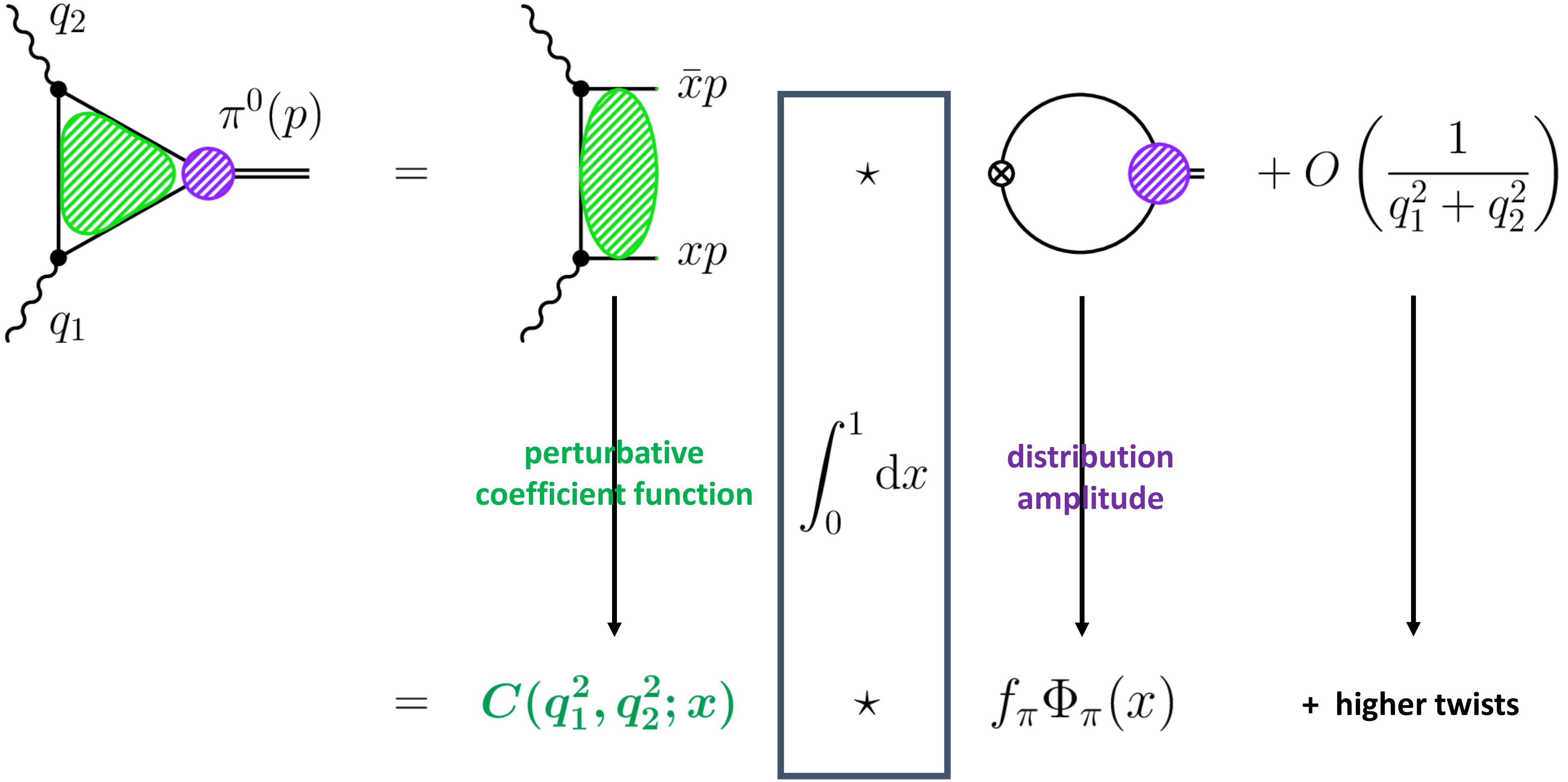
A: When the **QCD FACTORIZATION** of large and small scales is observed in a hard process

Example: $\gamma^*(q_1)\gamma^*(q_2) \to \pi^0(p)$
with $-q_1^2$, $-q_2^2 \gg 1$ GeV^2
and p^2 is at the hadron scale



$$\langle 0|\mathrm{T}\left[V_\mu(q_1)V_\nu(q_2)\right]|\pi(p)\rangle \equiv i\int d^4\mathbf{z}\, e^{iq_1\mathbf{z}}\langle 0|\mathrm{T}\left[j_\mu(\mathbf{z})j_\nu(0)\right]|\pi(p)\rangle$$
$$= e_{\mu\nu\lambda\sigma}q_1^\lambda q_2^\sigma \boldsymbol{F^{\gamma^*\gamma^*\pi}(q_1^2, q_2^2)}$$

Nonlocal composite vertices in QCD



Nonlocal composite vertices in QCD

Meson distribution amplitude (DA)

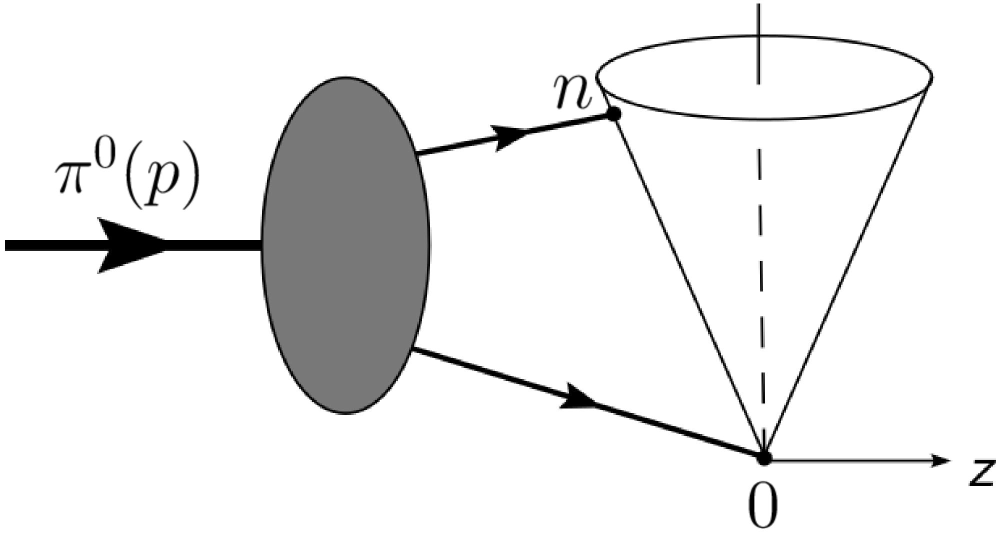
$$f_\pi \int_0^1 \mathrm{d}x e^{i(np)x} \boldsymbol{\Phi_\pi(x)} = \langle 0| \bar{d}(n) \frac{\hat{\Gamma}}{(np)} [n, 0]\, u(0) |\pi(p)\rangle$$

Bilocal current (on the light cone)

Fourier transform

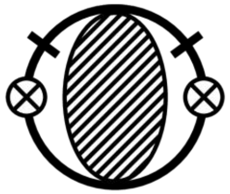
$$\hat{O}(x, 0)$$

Composite operator



$$[n, 0] = \mathrm{P} \exp \left[i g t_a \int_0^n \mathrm{d}\mathbf{z}^\mu\, A_\mu^a(\mathbf{z}) \right]$$

Correlators of composite vertices in QCD

x  y

$$= \Pi(x,y;p^2) = \int \mathrm{d}^D\mathbf{z}\, e^{ip\mathbf{z}} \langle 0|\mathrm{T}\left[\hat{O}(x;0)\hat{O}(y;\mathbf{z})\right]|0\rangle$$

The correlator of composite operators describes the perturbative content of DAs

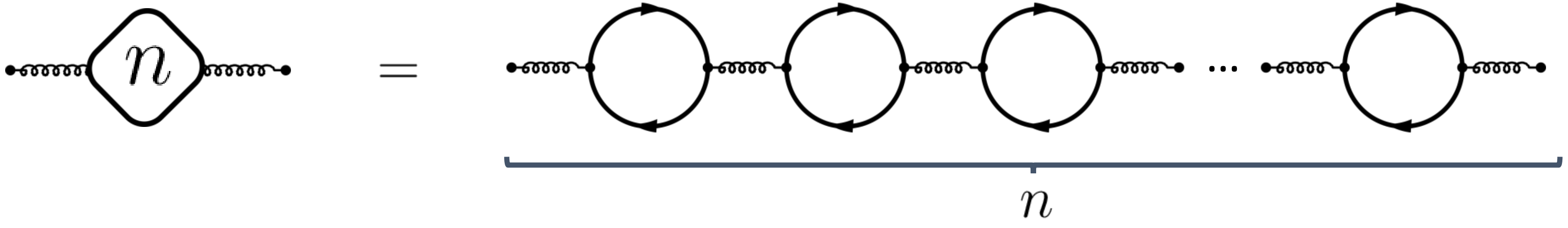
QCD SR

$$\Phi_{\text{meson}}(x) \sim \underset{-p^2\to M^2}{\textbf{Borel transform}} \left[\int_0^1 \mathrm{d}y\, \Pi\big(x,y;p^2\big)\right]$$

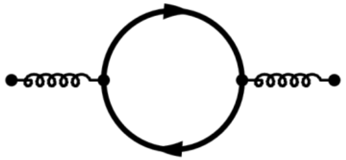
Feynman integrals in QCD after factorization:

$$f_1(x) \star \Pi(x,y;p^2) \star f_2(y) \qquad\qquad f_1(x)\star f_2(x) = \int_0^1 \mathrm{d}x\, f_1(x) f_2(x)$$

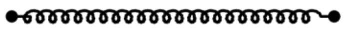
Renormalon-chain correlators



$$\sim O(a_s^n \beta_0^n)$$



$$\sim \frac{4}{3} a_s T_F n_f \frac{h_1(\varepsilon)}{\varepsilon} \left(\frac{-p^2}{\mu^2} \right)^{D/2-2}$$



$$= a_s \gamma_3^{(q)}$$

NNA

naive nonabelianization

$$-a_s \beta_0$$

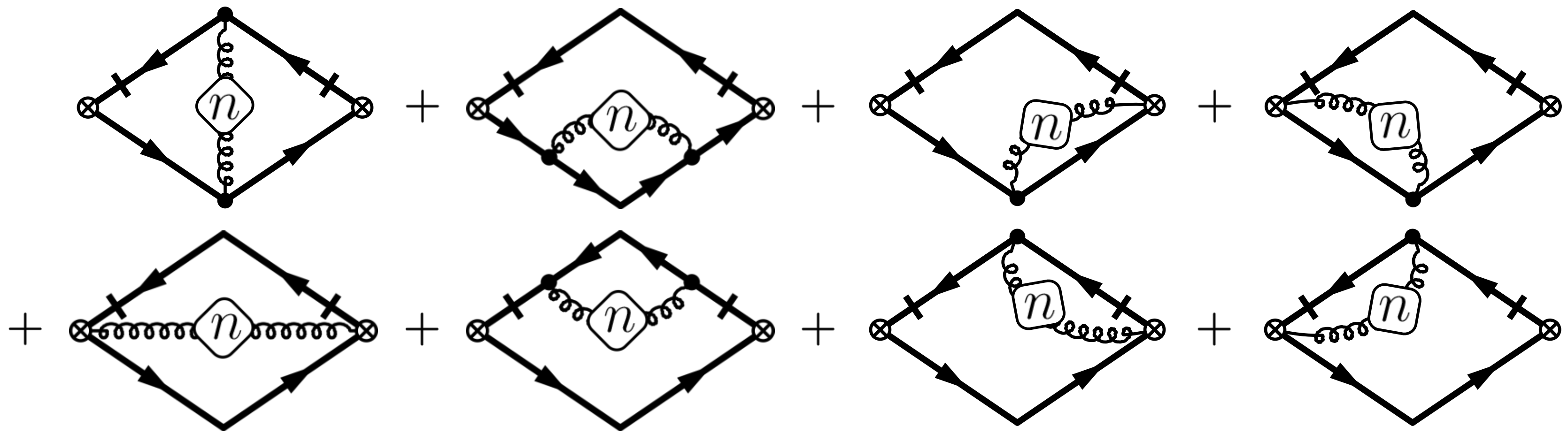
$$h_1(\varepsilon) = \frac{(1-\varepsilon)\Gamma(1+\varepsilon)\Gamma^3(1-\varepsilon)}{(1-2\varepsilon/3)(1-2\varepsilon)\Gamma(1-2\varepsilon)}$$

$$n_f = 3, \qquad \beta_0 = \frac{11}{3} C_A - \frac{4}{3} T_F n_f = 9$$

$$h_1(0) = 1, \qquad h_1'(0) = -C = -\frac{5}{3}$$

The correlator $\Pi_n(x, y)$

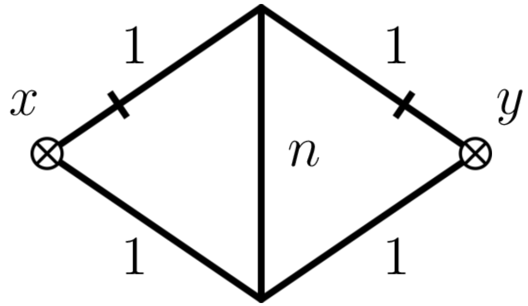
$$-i\frac{a_s}{\pi^2}N_c C_F \boldsymbol{A^n \Pi_n(x, y; L)} =$$



$$L = \ln\frac{-p^2}{\mu^2} \qquad a_s = \frac{\alpha_s}{4\pi} \qquad \boldsymbol{A = a_s\gamma_3^{(q)} = \frac{4}{3}a_s T_F n_f} \text{ or } \boldsymbol{-a_s\beta_0}$$

Two-loop master integral

$\tilde{n}$ is a lightlike vector, $\tilde{n}^2 = 0$, $\tilde{n}p = 1$



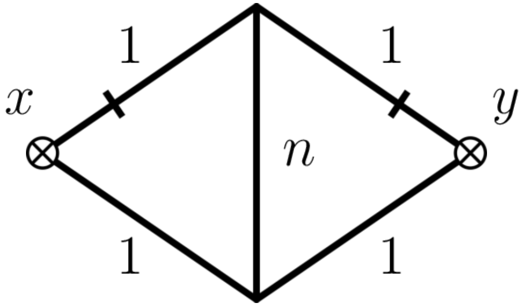
$$= \int \frac{\mathrm{d}^D k_1\, \mathrm{d}^D k_2\, \delta\left(x - \tilde{n}k_1\right) \delta\left(y - \tilde{n}k_2\right)}{k_1^2 k_2^2 (k_1 - p)^2 (k_2 - p)^2 \left[(k_1 - k_2)^2\right]^n}$$

$$= (-)^{n+1} \pi^D \left(-p^2\right)^{\omega/2} G(1, 1, 1, 1, n; x, y; D)$$

$$-\omega/2 = 4 + n - D$$

Mikhailov, Volchanskiy, JHEP (2019) 202

Two-loop master integral



$$= (-)^{n+1}\pi^D \left(-p^2\right)^{\omega/2} \frac{\hat{\mathbf{S}} f(n; \boldsymbol{z}; D)}{|\boldsymbol{x}-\boldsymbol{y}|^{-\omega/2}}$$

$\boldsymbol{z} = \frac{\bar{x}y}{x\bar{y}}$ is the conformal ratio

This special case of the kite integral and its Mellin moments can be expressed in terms of generalized hypergeometric functions not higher than $_3f_2$.

$$f(n;z;D) = \Gamma\begin{bmatrix} 2+\ddot{n},\ \dot{n},\ 1-\dot{n} \\ n,\ \lambda \end{bmatrix} z^{\lambda-1}\bar{z}^{2-\lambda}\theta(\bar{z}) \times \left[{}_3f_2\left(\begin{matrix} 1,\ 1,\ \lambda \\ 1-\dot{n},\ \dot{n}+2 \end{matrix}\middle|\, \bar{z}\right) - \bar{z}^{\dot{n}}\,{}_2f_1\left(\begin{matrix} n,\ \dot{n}+1 \\ 2(\dot{n}+1) \end{matrix}\middle|\, \bar{z}\right)\right]$$

where $\lambda = \frac{D}{2}-1, \qquad \dot{n} = n-\lambda, \qquad \ddot{n} = n-2\lambda$

$$\Gamma_r\begin{bmatrix} a_1,\ldots,\ a_p \\ b_1,\ldots,\ b_q \end{bmatrix} = \frac{\prod_{i=1}^{p}\Gamma(a_i+r)}{\prod_{i=1}^{q}\Gamma(b_i+r)}, \qquad \Gamma\begin{bmatrix} \mathbf{a} \\ \mathbf{b} \end{bmatrix} = \Gamma_0\begin{bmatrix} \mathbf{a} \\ \mathbf{b} \end{bmatrix}$$

$${}_pf_q\left(\begin{matrix} \mathbf{a} \\ \mathbf{b} \end{matrix}\middle|\, z\right) = \Gamma\begin{bmatrix} \mathbf{a} \\ \mathbf{b} \end{bmatrix} {}_pF_q\left(\begin{matrix} \mathbf{a} \\ \mathbf{b} \end{matrix}\middle|\, z\right) = \sum_{r\geqslant 0}\Gamma_r\begin{bmatrix} \mathbf{a} \\ \mathbf{b} \end{bmatrix}\frac{z^r}{r!}$$

Mikhailov, Volchanskiy, JHEP (2019) 202

$(x,\underline{0})$ moment of the correlator

purely finite part

$$\dot{\boldsymbol{\Pi}}_n(\boldsymbol{x},\underline{0};\boldsymbol{L}) = \int_0^1 \mathrm{d}y\,\dot{\Pi}_n(x,y;L) = \dot{\boldsymbol{\Pi}}'_n(\boldsymbol{x},\underline{0};\boldsymbol{L}) + \dot{\boldsymbol{\Pi}}''_n(\boldsymbol{x},\underline{0};\boldsymbol{L})$$

from counterterms to the nonlocal vertices

Exponential generating function:

$$\sum_{n\geqslant 0}\frac{A^n}{n!}\dot{\boldsymbol{\Pi}}'_n(\boldsymbol{x},\underline{0};\boldsymbol{L}) = \hat{\mathrm{S}}\left\{\frac{e^{A(L-5/3)}x^A}{A^2(1+A)(2+A)}\left[-\bar{x}(A+4x)+2x\bar{x}\frac{(\pi A)^2\cot(\pi A)}{x^A\sin(\pi A)} + Ax(2\bar{x}+A)\mathsf{B}_{\bar{x}}(A,1-A) + \frac{2x^2\bar{x}A^2}{(1+A)^2}\,{}_3\boldsymbol{F}_2\left(\left.\begin{matrix}1,\,1,\,1+A\\ 2+A,\,2+A\end{matrix}\right| x\right)\right]\right\}$$

Ordinary generating function:

$$\sum_{n\geqslant 0}A^n\dot{\boldsymbol{\Pi}}''_n(\boldsymbol{x},\underline{0};\boldsymbol{L}) = -\frac{1}{2A}\int_0^A \mathrm{d}s\int_0^1 \mathrm{d}y\,\frac{y\bar{y}V(x,y;s)_{+(x)}}{sh_1(s)}$$

Mellin moments $f(\underline{a}) = \int_0^1 \mathrm{d}x\,x^a f(x)$

generalized ERBL evolution kernel

$$V(x,y;\varepsilon) = 2\,\hat{\mathbf{S}}\left[\theta(y-x)\left(\frac{x}{y}\right)^{1-\varepsilon}\left(1-\varepsilon+\frac{1}{y-x}\right)\right]$$

Generalized hypergeometric function

The epsilon-expansion of the hypergeometric function can be written in terms of HPLs as

$$
{}_pF_{p-1}\left(\begin{matrix}\mathbf{a}\varepsilon \\ 1+\mathbf{b}\varepsilon\end{matrix}\,\middle|\, z\right) = 1 + \hat{\omega}_0^{p-1}\frac{\varepsilon^p e_p^a}{1-\sum_{n=1}^{p}\varepsilon^n\left[(e_n^a - e_n^b)\hat{\omega}_1 - e_n^b\hat{\omega}_0\right]\hat{\omega}_0^{n-1}}\hat{\omega}_1 1
$$

The elementary symmetric polynomials are defined as $e_n^a = \sum_{1\leqslant j_1<\cdots<j_n\leqslant p}\prod_{k=1}^{n} a_{j_k}$

The harmonic polylogarithms (HPLs) are $\mathrm{H}_{\mathbf{k}}(z) = \hat{\omega}_0^{|k_1|-1}\hat{\omega}_{\pm 1}\cdots\hat{\omega}_0^{|k_n|-1}\hat{\omega}_{\pm 1}1$,

where $\hat{\omega}_0 = \int_0^z \frac{\mathrm{d}t}{t}$ and $\hat{\omega}_1 = \int_0^z \frac{\mathrm{d}t}{\bar{t}} = \int_0^z \frac{\mathrm{d}t}{1-t}$

Remiddi, Vermaseren, IJMPA 15 (2000) 725

Kalmykov, Ward, Yost, JHEP 11 (2007) 009

Puhlfürst, Stieberger, NPB 902 (2015) 186

$(x,\underline{0})$ moment of the correlator

$$\dot{\Pi}_n(x,\underline{0};L) = \frac{\mathrm{d}}{\mathrm{d}L}\Pi_n(x,\underline{0};L) = (-1)^{n+1} n! \sum_{k=0}^{n} \frac{(-L)^k}{k!} \Pi_n^{k+1}(x,\underline{0})$$

$$\boldsymbol{\Pi_n^{n+1}(x,\underline{0})} = \frac{1}{2}\,\hat{\mathbf{S}}\,(x\ln x) + \delta_{0,n}\left[-\frac{1}{2}\,\hat{\mathbf{S}}\,(x\ln x) + \frac{1}{2}x\bar{x}\left(\frac{\pi^2}{3} - 5 - \boldsymbol{\ln^2\frac{x}{\bar{x}}}\right)\right]$$

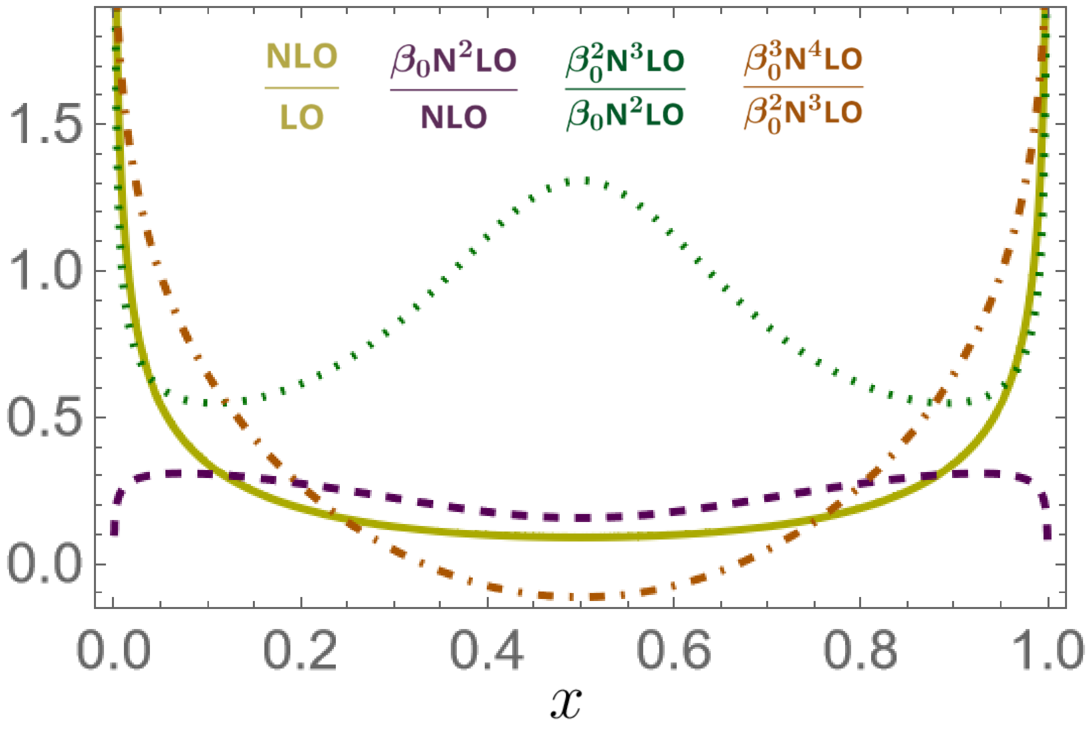
$$\boldsymbol{\Pi_n^{n}(x,\underline{0})} = \hat{\mathbf{S}}\Bigg\{x\bar{x}\left(-3\boldsymbol{\mathrm{Li}_3\,x} + \ln x\,\mathrm{Li}_2\,x + \frac{\pi^2}{6}\ln x\right) - \frac{x}{2}\left(\mathrm{Li}_2\,x - \frac{\pi^2}{6} - \frac{1}{2}\ln^2 x + \frac{19}{6}\ln x\right) + \delta_{1,n}\frac{1}{24}x\ln x(7+3\ln x)$$
$$+ \delta_{1,n}\frac{1}{2}x\bar{x}\left[\boldsymbol{\mathrm{Li}_3\,x} - \ln x\,\mathrm{Li}_2\,x + \frac{1}{6}\ln^3 x - \frac{1}{2}\ln x\ln^2\bar{x} - \frac{5}{6}\ln^2 x - \frac{5}{3}\ln x\ln\bar{x} - \frac{5\pi^2}{36} + \frac{7}{12}\right]\Bigg\}$$

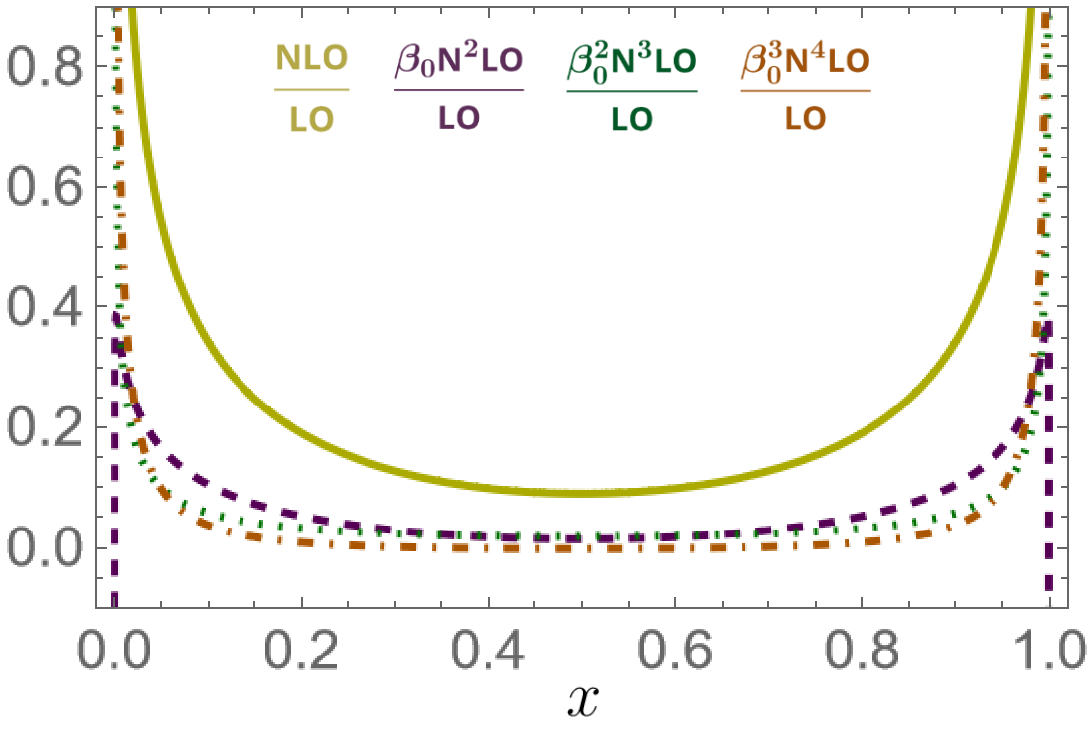
$$\boldsymbol{\Pi_n^{n-1}(x,\underline{0})} \sim \hat{\mathbf{S}}\,\boldsymbol{\mathrm{Li}_4\,x} \qquad \boldsymbol{\Pi_n^{n-2}(x,\underline{0})} \sim \hat{\mathbf{S}}\,\boldsymbol{\mathrm{H}_{3,2}(x)}$$

$$\Pi_n^{k+1}(x,\underline{0}) \sim \hat{\mathrm{S}}\,\mathrm{H}_\mu(x), \qquad \mu = m_1,\dots m_r : \sum_{i=1}^{r} m_i = n-k+2$$

harmonic polylogarithm

$(x, \underline{0})$ moment of the correlator





$$a_s = \frac{\alpha_s(\mu^2 = 1\ \text{GeV}^2)}{4\pi} = \frac{0.494}{4\pi} \qquad n_f = 3, \qquad \beta_0 = \frac{11}{3}C_A - \frac{4}{3}T_F n_f = 9$$

Agaev et al, PRD 83 (2011) 054020

$(\underline{-1},\underline{0})$ moment of the correlator

$$\dot{\Pi}_n(\underline{-1},\underline{0};L) = \int_0^1 \frac{\mathrm{d}x}{x}\,\dot{\Pi}_n(x,\underline{0};L)$$

Exponential generating function:

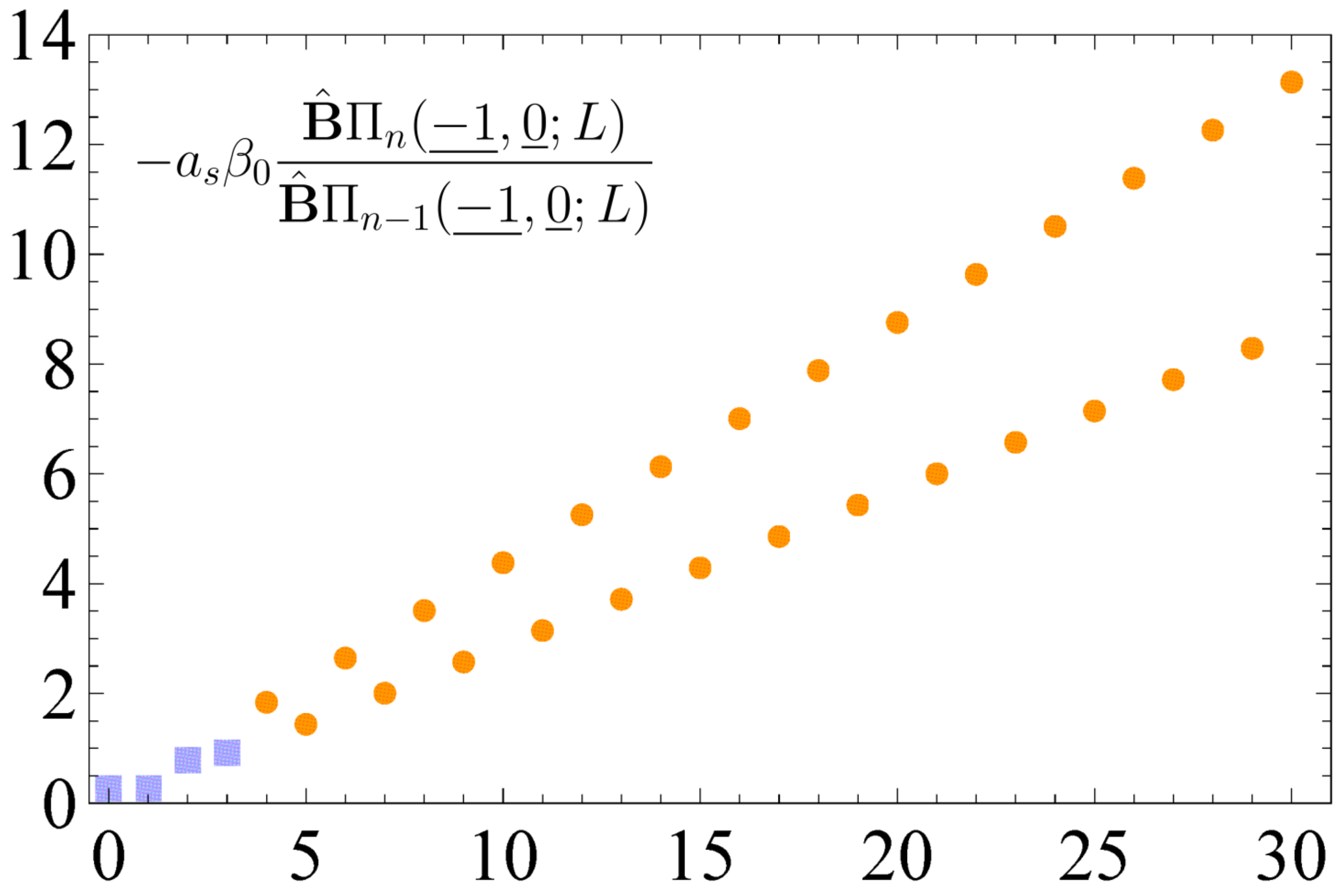
$$\sum_{n\geqslant 0} \frac{A^n}{n!}\dot{\Pi}'_n(\underline{-1},\underline{0};L) = \frac{e^{A(L-5/3)}}{2(1+A)(2+A)}\left[\psi_1\left(\frac{2-A}{2}\right) - \psi_1\left(\frac{1-A}{2}\right)\right]$$

Ordinary generating function:

$$\sum_{n\geqslant 0} A^n\dot{\Pi}''_n(\underline{-1},\underline{0};L) = -\frac{1}{A}\int_0^A \frac{\mathrm{d}a}{h_1(a)}\left\{\frac{5+6a-5a^2}{(1-a^2)(4-a^2)} + \frac{(1+2a)[\gamma_{\text{EM}}+\psi(1-a)]}{a(1+a)(2+a)}\right\}$$

$(\underline{-1}, \underline{0})$ moment of the correlator

$\frac{\text{NLO}}{\text{LO}}$	26%
$\frac{\beta_0 \text{N}^2\text{LO}}{\text{LO}}$	7%
$\frac{\beta_0^2 \text{N}^3\text{LO}}{\text{LO}}$	5%
$\frac{\beta_0^3 \text{N}^4\text{LO}}{\text{LO}}$	5%
$\frac{\beta_0^4 \text{N}^5\text{LO}}{\text{LO}}$	9%

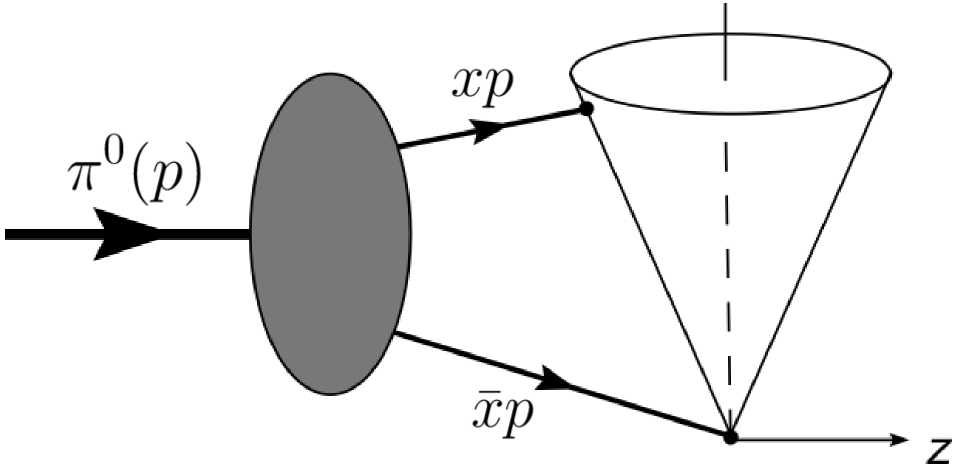


Renormalon chains in light-meson DAs

QCD SR

$$\varphi_\pi(x) \sim \underset{P^2\to M^2}{\textbf{Borel transform}} \left[\Pi\left(x, \underline{0}; L = \ln\frac{-p^2}{\mu^2}\right)\right]$$

φ_{meson}, meson $= \pi,\ \rho$ is a distribution amplitude (DA) of twist-2.



DA behavior at endpoints $x = 0$ and 1 is crucially important for

the $\pi^\pm$ electromagnetic FF and **the π^0 transition FF**

$\gamma^*(q)\pi^\pm \to \pi^\pm$ $\gamma^*(q)\gamma \to \pi^0$

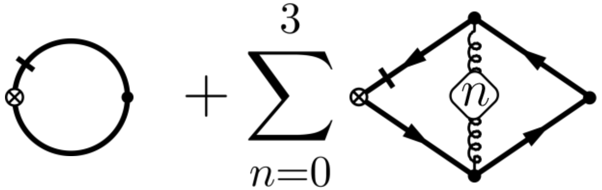
Renormalon chains in light-meson DAs

QCD SR

$$f^2_{\text{mes}}\varphi_{\text{mes}}(x)e^{-m^2_{\text{mes}}/M^2} = \Phi^{\text{PT}}_{\text{mes}}(x) + \Phi^{\text{NP}}_{\text{mes}}(x), \qquad \text{mes} = \rho\text{L} \text{ or } \pi$$

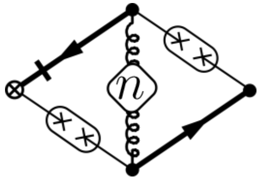
QCD sum rules for DA is an interplay between **perturbative**

$$\Phi^{\text{PT}}(x) \sim \hat{\text{B}}\Pi^{\text{PT}}(x) = \hat{\text{B}}\left[\quad + \sum_{n=0}^{3} \quad \right]$$



and **nonperturbative** contributions

$$\Phi^{\text{NP}}(x) \sim \hat{\text{B}}\Pi^{\text{NP}}(x) = \overset{\rho\text{L}}{\underset{\pi}{\pm}} \hat{\text{B}} \sum_{n=0}^{3} \quad + \text{ other terms (the same for both mesons)}$$



ρL / π

dominant at endpoints

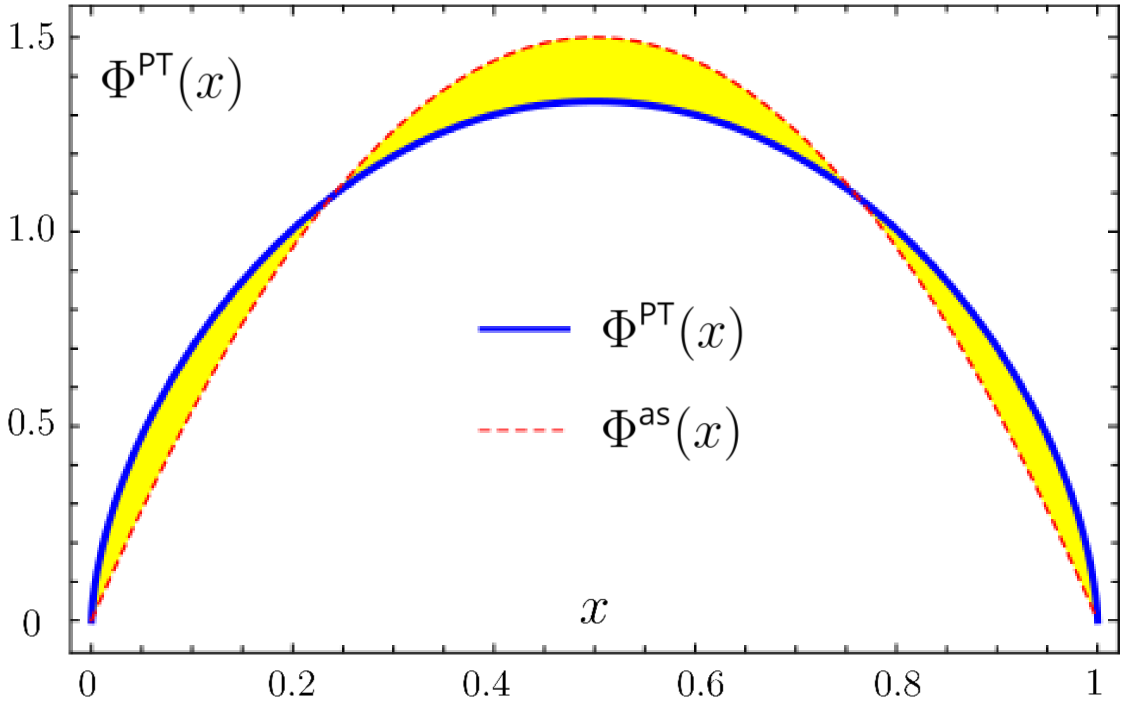
a few %

Renormalon chains in light-meson DAs

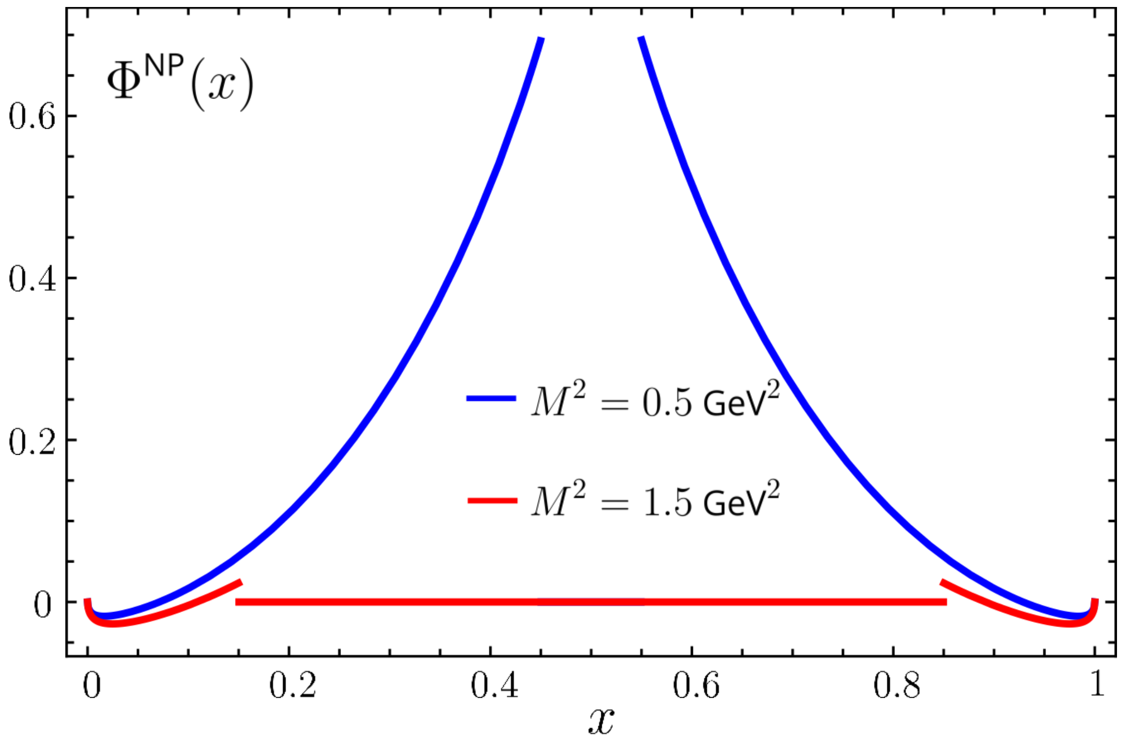
QCD SR

$$f^2_{\text{mes}}\varphi_{\text{mes}}(x)e^{-m^2_{\text{mes}}/M^2} = \Phi^{\text{PT}}_{\text{mes}}(x) + \Phi^{\text{NP}}_{\text{mes}}(x)$$

mes = ρL or π



perturbative contributions

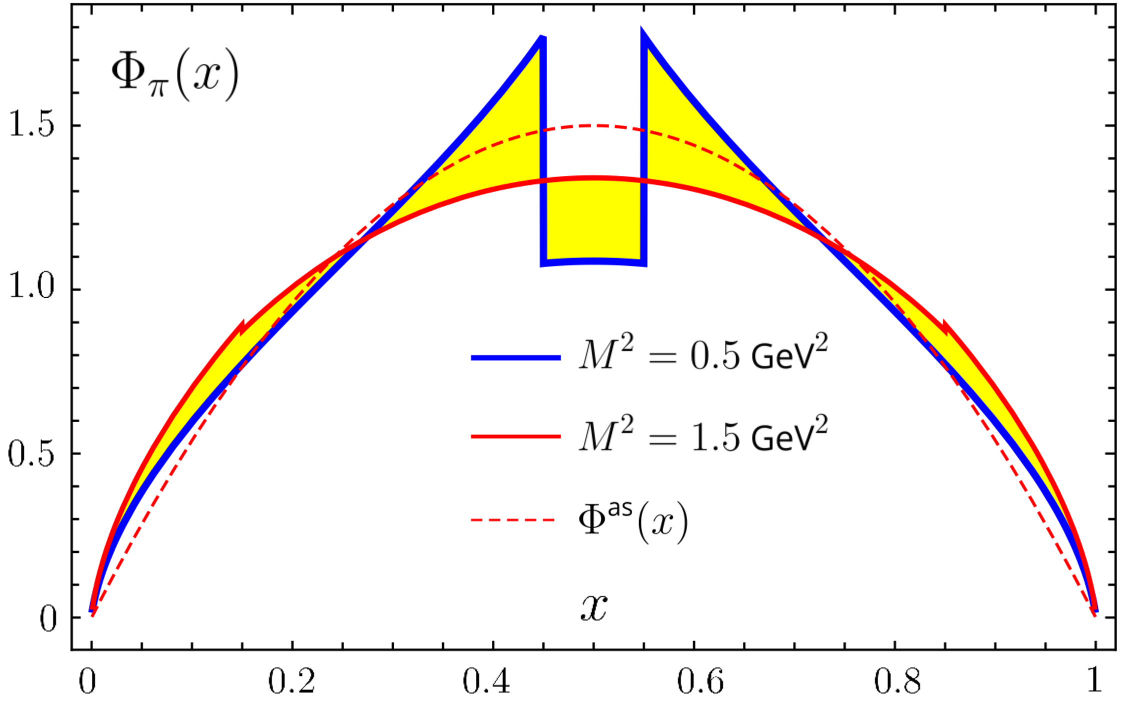


scalar nonlocal vacuum condensate

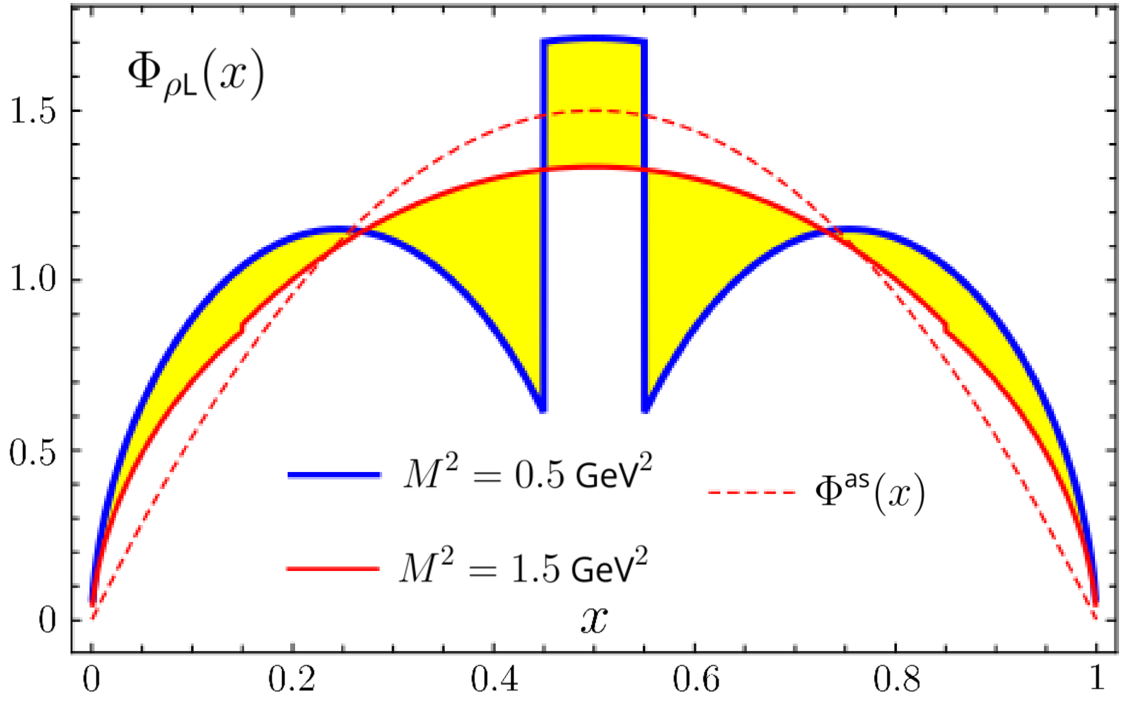
Renormalon chains in light-meson DAs

QCD SR

The r.h.s. of SR $\Phi_{\text{mes}}(x) = \Phi^{\text{PT}}_{\text{mes}}(x) \pm \Phi^{\text{NP}}_{\text{mes}}(x)$ mes $=$ ρL or π



pion



longitudinally polarized rho meson

Renormalon chains in pion DA

The **perturbative** part of the QCD sum rules increases the steepness of DA at the endpoints up to order $a_s(a_s\beta_0)^3$ (The asymptotic series should be truncated at this order.)

$$\int_0^1 \frac{\mathrm{d}x}{x}\,\Phi_\pi^{\mathsf{PT}}(x) \approx 3.5$$

at tension with the experimental data available

The **condensate** contributions compensate the perturbative effect, which eases the tension with the data:

$$\int_0^1 \frac{\mathrm{d}x}{x}\left[\Phi_\pi^{\mathsf{PT}}(x) + \Phi_\pi^{\mathsf{NP}}(x)\right] \approx 3.34$$

The slope is 6.2 ± 0.5 (asymptotic value — 6)

The global fit to the experimental data is $\int_0^1 \frac{\mathrm{d}x}{x}\Phi_\pi(x) \approx 3.25 \pm 0.20$ **(CELLO, CLEO, BaBar, Belle in LCSR)**

Higher-order renormalon corrections do not spoil the agreement with the data and prefer DA of a moderate width.

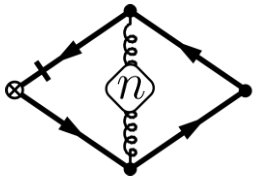
Renormalon chains in light-meson DAs

QCD SR

$$f^2_{\text{mes}}\Phi_{\text{mes}}(x)e^{-m^2_{\text{mes}}/M^2} = \Phi^{\text{PT}}_{\text{mes}}(x) + \Phi^{\text{NP}}_{\text{mes}}(x), \qquad \text{mes} = \rho\text{L} \ \text{or} \ \pi$$

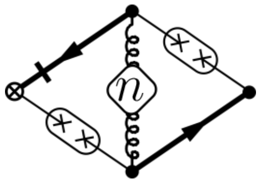
QCD sum rules for DA is an interplay between **perturbative**

$$\Phi^{\text{PT}}(x) \sim \hat{\text{B}}\Pi^{\text{PT}}(x) = \hat{\text{B}}\Big[\quad + \sum_{n=0}^{3} \quad \Big]$$



and **nonperturbative** contributions

$$\Phi^{\text{NP}}(x) \sim \hat{\text{B}}\Pi^{\text{NP}}(x) = \overset{\rho\text{L}}{\underset{\pi}{\pm}}\hat{\text{B}}\sum_{n=0}^{3} \quad + \ \text{other terms (the same for both mesons)}$$



dominant at endpoints

a few %

Renormalon chains in light-meson DAs

QCD SR

$$\varphi_{\rho\text{L}}(x) \approx \left[\varphi_\pi(x) - \frac{2}{f_\pi^2}\Phi_{\text{sNLC}}(x,M) + \text{contributions of higher resonances (negligible)}\right]\frac{f_\pi^2}{f_\rho^2}e^{m_\rho^2/M^2}$$

Mikhailov, Stefanis, PRD 104 (2021) 096013

BEFORE — **NLO QCD SR**

$$0.032(46) = a_2^{\rho\text{L}} < a_2^\pi = 0.149^{+0.052}_{-0.043}$$

Mikhailov, Stefanis, PRD 104 (2021) 096013

lattice QCD

$$0.132(27) = a_2^{\rho\text{L}} > a_2^\pi = 0.116(20)$$

Braun et al., JHEP (2017) 082 Bali et al., JHEP (2020) 037

AFTER — **large-β_0 N^{4}LO QCD SR**

$$a_2^{\rho\text{L}} > a_2^\pi$$

Mikhailov, Volchanskiy, PRD 108 (2023) 096015

$\lambda_q^2 \approx 0.45$ GeV^2 $\mu^2 = 1$ GeV^2

Summary

We have evaluated correlators $\Pi(x, y; L)$ of two vector composite quark currents of order β_0^nN^{n+1}LO in QCD, $n \geqslant 0$. The double-zeroth moment as well as some other fixed-order special cases agree with previous calculations in the literature. **Generating functions** for the correlator have been constructed. The correlator $\Pi_n(x, y)$ and $\Pi_n(x, \underline{0})$ at any fixed order $a_s^{n+1}\beta_0^n$ can be expressed in terms of **harmonic polylogarithms** of weight $n+2$.

We have estimated quantitative significance of the lower-order $a_s^{n+1}\beta_0^n$ contributions to the QCD sum rules for the light-meson distribution amplitudes (pion and longitudinal rho).

Higher-order renormalon corrections **do not spoil the agreement** with the data and prefer pion DA of a moderate width.

Higher-order renormalon corrections **significantly change** the endpoint behavior of longitudinal rho DA reconciling QCD SR with the lQCD results for the hierarchy of the 2nd Gegenbauer moment $a_2^{\rho L} > a_2^{\pi} > 0$

Borel transform

QCD SR

$$\Phi_{\text{meson}}(x) \sim \underset{-p^2\to M^2}{\textbf{Borel transform}} \left[\Pi\left(x, \underline{0}; L = \ln\frac{-p^2}{\mu^2}\right)\right]$$

$$\hat{\mathsf{B}}\left[f(t)\right](\mu) = \lim_{\substack{t=n\mu \\ n\to\infty}} \frac{(-t)^n}{\Gamma(n)} \frac{\mathrm{d}^n}{\mathrm{d}t^n} f(t),$$

$$\hat{\mathsf{B}}\left[t^{-a}\right](\mu) = \frac{\mu^{-a}}{\Gamma(a)}, \quad a > 0, \qquad \hat{\mathsf{B}}\left[e^{-at}\right](\mu) = \delta(1-\mu a), \quad a > 0.$$

$$\hat{\mathsf{B}}\left[\ln^m(t)\right](\mu) = m(-1)^m \left(\frac{d}{da}\right)^{m-1} \frac{e^{-al}}{\Gamma(1+a)}\bigg|_{a=0} = -m! \sum_{s=0}^{m-1} \frac{1}{s!} \left[\ln\left(\mu e^{\gamma_{\mathsf{E}}}\right)\right]^s \sum_{\forall\Pi} \prod_{i=1}^{N} \frac{(-\zeta_{p_i})^{r_i}}{p_i^{r_i} r_i!}$$

Here, $\Pi = (p_1^{r_1}, p_2^{r_2}, \ldots, p_N^{r_N})$ is a partition of $n \in \mathbb{N}$,

i.e. $n = \sum_{i=1}^{N} p_i r_i$: $1 < p_1 < p_2 < \cdots < p_N$ with p_i, $r_i \in \mathbb{N}$.