Semiclassical exponent for multiparticle production in $\lambda\phi^4$ theory

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Toy model

Consider

$$A_n(\lambda) \equiv \int_{-\infty}^{+\infty} dx\, x^n \mathrm{e}^{-\frac{x^2}{2}-\frac{\lambda x^4}{4}} = \lambda^{\frac{n+1}{2}} \int_{-\infty}^{+\infty} dx\, x^n \mathrm{e}^{-\frac{1}{\lambda}\left(\frac{x^2}{2}+\frac{x^4}{4}\right)} \tag{1}$$

n is even

For $\lambda \ll 1$ the integral $A_n(\lambda)$ can be esimated

- Perturbatively
- Via steepest descend method

Let's compare both approaches!

Perturbation theory

Expand $\exp\left(-\frac{\lambda x^4}{4}\right)$ and integrate series terms

$$A_n(\lambda) = \sum_{k=0}^{+\infty} \frac{(-\lambda)^k 2^{\frac{n+1}{2}}}{k!} \Gamma\left(2k + \frac{n}{2} + \frac{1}{2}\right) \tag{2}$$

Asymptotic series can be rewiritten

$$A_n(\lambda) = A_n^0 \sum_{k=0}^{+\infty} \frac{(-\lambda)^k}{4^k k!} (n+1)\cdot(n+3)\cdot\ldots\cdot(n+4k-1) \tag{3}$$

$$A_n^0 \equiv \sqrt{2\pi}\, 2^{-\frac{n}{2}} \frac{n!}{\left(\frac{n}{2}\right)!} \tag{4}$$

k-th order contribution is A_n^0 multiplied by polynomial of order n^{2k}

Perturbation theory resummation

One can resum $\lambda^k n^{2k}$ and $\lambda^k n^{2k-1}$ terms in the series

$$A_n(\lambda) = A_n^0 \sum_{k=0}^{+\infty} \sum_{l=0}^{2k} C_{kl} \lambda^k n^l \quad (5)$$

and obtain

$$\lambda^k n^{2k} : \qquad \sum_{k=0}^{+\infty} \frac{(-\lambda)^k}{4^k k!} n^{2k} = \mathrm{e}^{-\frac{\lambda n^2}{4}} \equiv \mathrm{e}^{\frac{F_1(\lambda n)}{\lambda}} \quad (6)$$

$$\lambda^k n^{2k-1} : \quad \sum_{k=0}^{+\infty} \frac{(-\lambda)^k}{4^k k!} n^{2k-1} \sum_{j=0}^{2k-1} (2j+1) = \mathrm{e}^{-\frac{\lambda n^2}{4}} \left(-\lambda n + \frac{\lambda^2 n^3}{4} \right) \quad (7)$$

Appearance of $\mathrm{e}^{\frac{F_1(\lambda n)}{\lambda}}$ and corrections $\propto F_2(\lambda n)/\lambda$ and $\propto G_1(\lambda n)$

Steepest descend approximation

$$A_n(\lambda) = \lambda^{\frac{n+1}{2}} \int_{-\infty}^{+\infty} dx\, \mathrm{e}^{-\frac{S(x)}{\lambda}} \quad (8)$$

$$S(x) = \frac{x^2}{2} + \frac{x^4}{4} + \lambda n \ln x \quad (9)$$

We consider the limit $\lambda \ll 1$, λn - fixed

Saddle-point equation

$$x_s^2 + x_s^4 - \lambda n = 0 \quad (10)$$

$$x_s = \pm \left[\frac{\pm\sqrt{1+4\lambda n} - 1}{2} \right]^{1/2} \quad (11)$$

Two relevant real x_s-s have the form $x_s = \sqrt{\lambda n}\, g(\lambda n)$, $g(\lambda n)$ – analytical

Perturbation theory for the saddle point

If we rescale $x_s = \sqrt{\lambda n}\tilde{x}_s$
(or consider $n \gg 1$, λn - fixed instead of $\lambda \ll 1$, λn - fixed)

$$\tilde{x}_s^2 + \lambda n \tilde{x}_s^4 - 1 = 0 \tag{12}$$

Can be solved perturbatively for $\lambda n \ll 1$ and we obtain both relevant saddle points!

Limit $\lambda n \ll 1$ is applicable

- In perturbation theory
- In steepest descend

We can map these asymptotic methods with each other!

Steepest descend answer as function of λ, n

Saddle-point exponent

$$S(x_s) = \frac{\sqrt{1+4\lambda n} - 1 + 2\lambda n}{8} - \frac{\lambda n}{2}\ln(\lambda n) - \frac{\lambda n}{2}\ln\left(\frac{\sqrt{1+4\lambda n}-1}{2\lambda n}\right) \tag{13}$$

$$-\frac{S(x_s)}{\lambda} + \frac{n}{2}\ln\lambda \xrightarrow{n\gg 1} \ln A_n^0 + \frac{F_1(\lambda n)}{\lambda} \tag{14}$$

Second derivative

$$S''(x_s) = 2\left(\frac{4\lambda n}{\sqrt{1+4\lambda n}-1} - 1\right) \tag{15}$$

Higher derivatives $S^{(m)}(x_s)$ will contain $\propto \lambda n x_s^{-m}$ – non-analytical in n

Combining two approaches

Perturbation theory

$$A_n(\lambda) = A_n^0 \mathrm{e}^{\frac{F(\lambda n)}{\lambda}} \left(1 + \lambda G_1(\lambda n) + \ldots\right) \tag{16}$$

Steepest descend

$$A_n(\lambda) \sqrt{\frac{2\pi}{S''(x_s)}} \mathrm{e}^{-\frac{S(x_s)}{\lambda} + \frac{n}{2} \ln \lambda} \left(1 + \lambda \Delta_1 + \ldots\right) \tag{17}$$

Asymptotic expansions must coincide at $n \gg 1$, $\lambda \ll 1$ and all non-analytical in n behavior is encoded in A_n^0.

Full perturbative answer can be obtained with A_n^0 and steepest descend method!

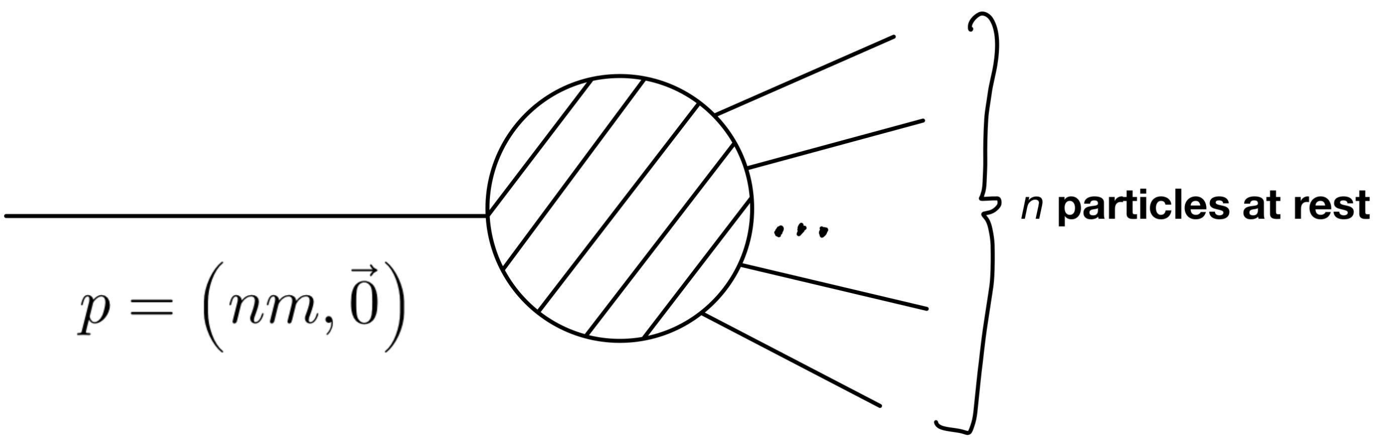
Multiparticle production at threshold

We consider scalar field theory in $3+1$ dimensions

$$S\left[\phi\right] = \int d^4x \left[\frac{(\partial_\mu \phi)^2}{2} - \frac{m^2\phi^2}{2} - \frac{\lambda\phi^4}{4}\right] \tag{18}$$

Our first aim is

$$\mathcal{A}_{1\to n} = \langle n,\, E = nm|\hat{\phi}(0)|0\rangle,\; n - \mathrm{odd} \tag{19}$$



Perturbation theory

Known results from the literature for perturbative expansion

$$\mathcal{A}_{1\to n} = \mathcal{A}^{\text{tree}}_{1\to n} + \lambda \mathcal{A}^{1-\text{loop}}_{1\to n} + \dots \tag{20}$$

Tree-level result [Brown, 1992]

$$\mathcal{A}^{\text{tree}}_{1\to n} = n! \left(\frac{\lambda}{8m^2}\right)^{\frac{n-1}{2}} \tag{21}$$

1-loop correction [Voloshin, 1992]

$$\mathcal{A}^{1-\text{loop}}_{1\to n} = \mathcal{A}^{\text{tree}}_{1\to n} B(n-1)(n-3),\ B \in \mathbb{C} \tag{22}$$

Renormalization conditions: $\mathcal{A}^{1-\text{loop}}_{1\to 1}, \mathcal{A}^{1-\text{loop}}_{1\to 3} = 0$

Partial series resummation and low-energy corrections

Loop corrections to $\mathcal{A}_{1\to n}$ dependence on n [Argyres, 1993]

$$\mathcal{A}_{1\to n} = \mathcal{A}^{\text{tree}}_{1\to n}(1 + \#_1 \lambda(n^2 + \ldots) + \#_2 \lambda^2 (n^4 + \ldots) + \ldots \quad (23)$$

Contributions $\propto \lambda^k n^{2k}$ can be resummed in all orders [Libanov et al., 1994]

$$\sum_{k=0}^{+\infty} \#_k \lambda^k n^{2k} = \mathrm{e}^{B\lambda n^2} \quad (24)$$

The same-type resummation $\exp(F_1(\lambda n)/\lambda)$ as in toy model!

Energy corrections near the threshold are also exponential [Libanov et al., 1994]

$$\mathcal{A}^{\text{tree}}_{1\to n}(\varepsilon) = \mathcal{A}^{\text{tree}}_{1\to n} \mathrm{e}^{-\frac{5}{6}\varepsilon n} = \mathcal{A}^{\text{tree}}_{1\to n} \mathrm{e}^{F_{1E}(\lambda n, \varepsilon)/\lambda} \quad (25)$$

$$\varepsilon \equiv \frac{E}{n} - m \quad (26)$$

Similarities between $\lambda\phi^4$ and the toy model

In the toy model

- Perturbative $A_n(\lambda)$ is the product of non-analytical in n A_n^0 and polynomial in all orders
- Perturbation series can be resummed into saddle-point exponent $\exp(F(\lambda n)/\lambda)$, pre-factor and other steepest descend corrections for $n \gg 1$

In $\lambda\phi^4$

- Perturbative $\mathcal{A}_{1\to n}$ is a product of non-analytical in n $\mathcal{A}_{1\to n}^{\text{tree}}$ and a function with dominating term $\propto \lambda^k n^{2k}$ at k loops for $n \gg 1$ near the threshold
- Perturbation series can be partially resummed into an exponent $\exp(F(\lambda n, \varepsilon)/\lambda)$

Maybe one can obtain perturbative $\mathcal{A}_{1\to n}$ using tree-level + semiclassics?

Semiclassical approach at threshold

$\mathcal{A}_{1\to n}$ can be represented as a Cauchy-type integral for matrix element between a coherent state $|z_0\rangle$ and vacuum

$$\mathcal{A}_{1\to n} = \frac{n!}{2\pi i}\oint \frac{dz_0}{z_0^{n+1}}\langle z_0|\hat{\mathcal{S}}\hat{\phi}(0)|0\rangle = \frac{n!}{2\pi i}\oint \frac{dz_0}{z_0^{n+1}}\int \mathcal{D}\phi\Big|_{BC}\phi(0)\mathrm{e}^{iS_{BP}+B_f} \tag{27}$$

After rewriting this value in the path integral form redefinitions

$$z_0 = \left(\frac{8m^2}{\lambda}\right)^{\frac{1}{2}}\mathrm{e}^{-\tau_\infty},\ \phi = \tilde{\phi}/\sqrt{\lambda} \tag{28}$$

$\mathcal{A}_{1\to n}$ will have saddle-point form with factorized $\mathcal{A}^{\text{tree}}_{1\to n}$

What we know about semiclassical exponent and saddle-point ϕ_s?

Semiclassical exponent from method of singular solutions

We consider inclusive probability at fixed multiplicity n and energy E

$$\mathcal{P}_n(E) \equiv \sum_f \left| \langle f; E, n \left| \hat{\mathcal{S}} \hat{\mathcal{O}} \right| 0 \rangle \right|^2 \sim \mathrm{e}^{F(\lambda n, \varepsilon)/\lambda}, \ \varepsilon \equiv \frac{E}{n} - m \qquad (29)$$

Method of singular solutions [Son, 1995]

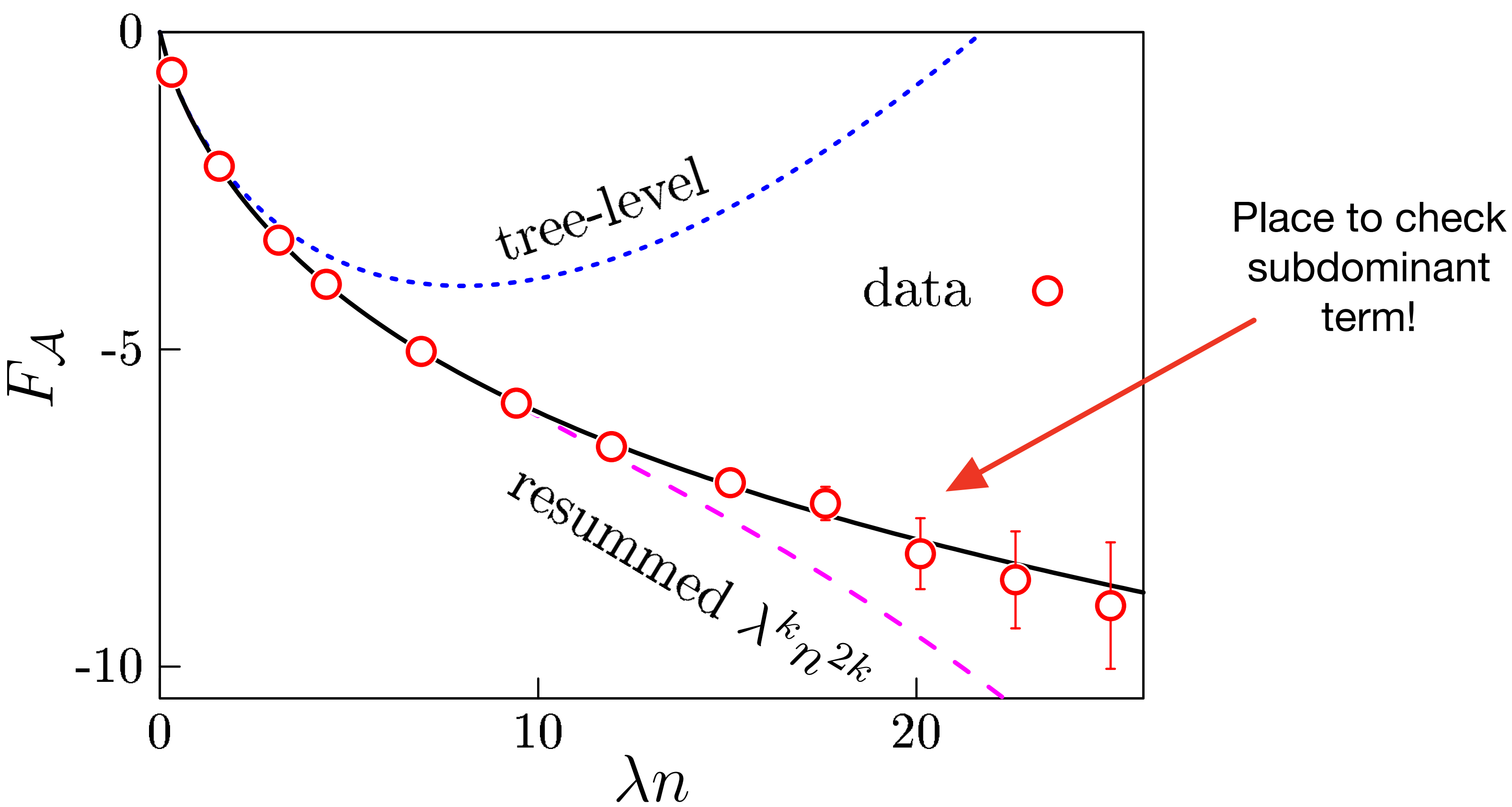
- $\hat{\mathcal{O}}$ is a few-particle operator that doesn't affect the exponent F [Libanov et al., 1994]
- $\hat{\mathcal{O}} = \exp\left(- \int d^3\mathbf{x} J(\mathbf{x}) \hat{\phi}(0, \mathbf{x}) / \sqrt{\lambda} \right)$
- Find saddle-point solution ϕ_s with nonzero J
- Calculate $F(\lambda n, \varepsilon)$ on ϕ_s and extrapolate $J \to 0$

Method showed agreement with tree-level and resummed loop correction

Numerical $\mathcal{A}_{1\to n}$

In [Demidov, et at., 2023] we implemented method of singular solutions numerically and obtained $|\mathcal{A}_{1\to n}|$

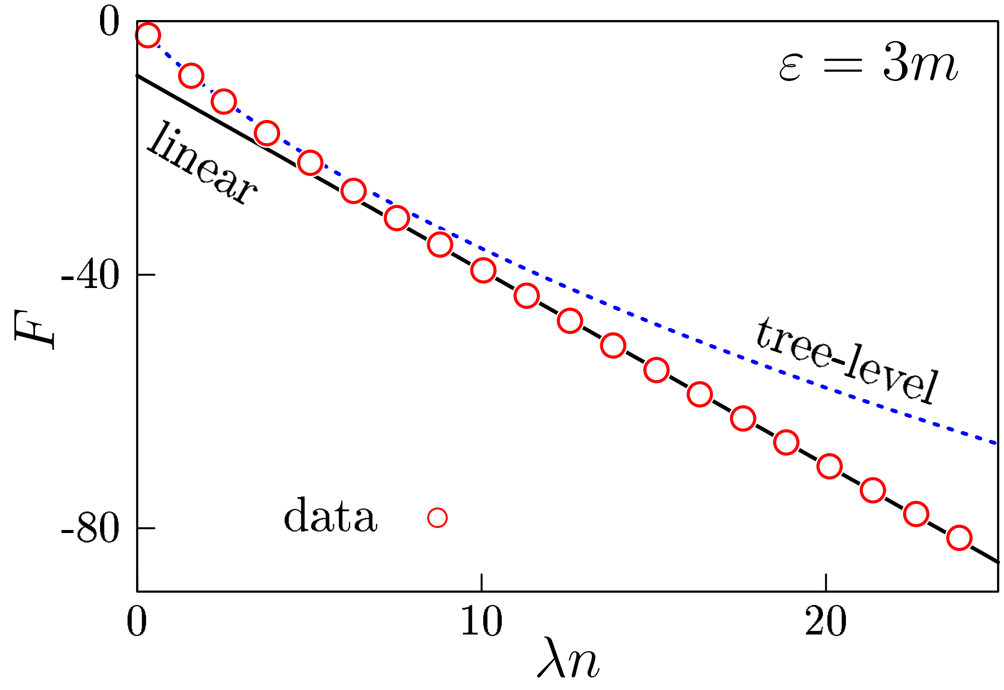
$$|\mathcal{A}_{1\to n}|^2 \sim \lim_{\varepsilon\to 0} \frac{n!}{\mathcal{V}_n} \mathrm{e}^{F/\lambda} \sim n! m^{4-2n} \mathrm{e}^{2F_{\mathcal{A}}(\lambda n)/\lambda} \tag{30}$$



Typical dependence of exponent on λn

$$F \to \lambda n f_\infty(\varepsilon) + g_\infty(\varepsilon) \quad \text{or} \quad \mathcal{P}_n(E) \to \mathrm{e}^{n f_\infty(\varepsilon) + g_\infty(\varepsilon)/\lambda} \tag{31}$$

for all ε



The same behavior in anharmonic oscillator in QM

$$\mathcal{P}_n^{(QM)} \equiv |\langle n|\hat{\mathcal{O}}|0\rangle|^2 \sim \exp(-\pi n) \quad \text{at} \quad n \gg O(\lambda_{(QM)}^{-1}) \tag{32}$$

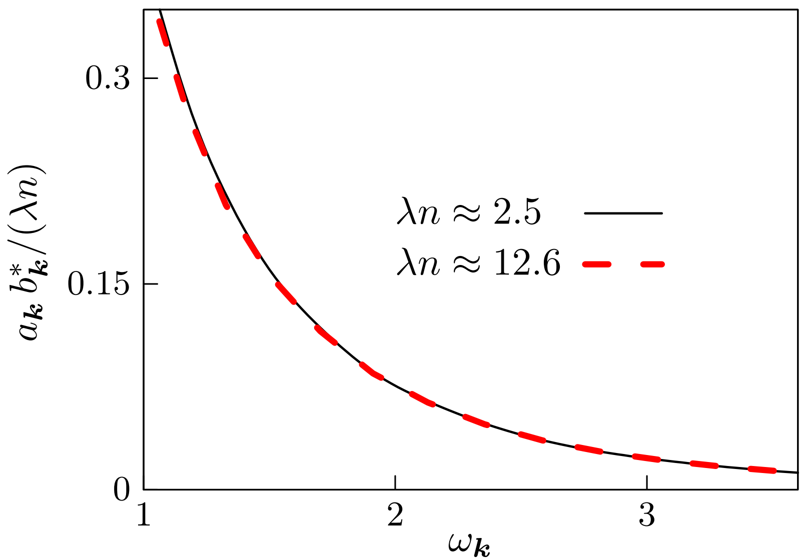
Scaling in the limit $\lambda n \gg 1$

In the toy model $A_n(\lambda) \sim \sqrt{n!}\exp(\alpha n + \beta)$ as $\lambda n \gg 1$.
For $x_s = (\lambda n)^{1/4} \tilde{x}_s$ saddle-point equation becomes

$$\tilde{x}_s^4 - 1 + \frac{\tilde{x}_s^2}{\sqrt{\lambda n}} = 0 \qquad (33)$$

Some sort of scaling was obtained in $\lambda\phi^4$

$$\phi_s(t, r) \approx \sqrt{\lambda n}\,\tilde{\phi}_s(t, r) \quad \text{at} \quad t \to +\infty \textit{ and } \lambda n \gg 1 \qquad (34)$$

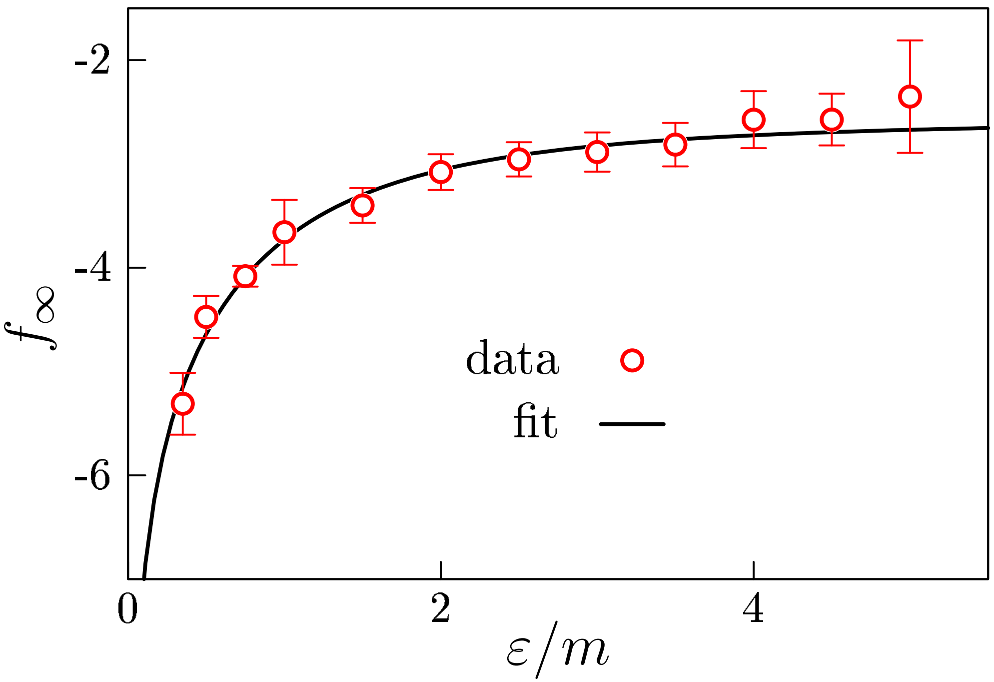


Some rescale of ϕ + dilatation?

Ultrarelativistic limit

Again

$$F \to \lambda n f_\infty(\varepsilon) + g_\infty(\varepsilon) \quad \text{or} \quad \mathcal{P}_n(E) \to \mathrm{e}^{n f_\infty(\varepsilon) + g_\infty(\varepsilon)/\lambda} \tag{35}$$



$$f_\infty \to -2.57 \pm 0.06 \text{ as } \varepsilon/m = \frac{E}{nm} - 1 \to +\infty$$

Corresponds to some solution φ_s in massless $\lambda\varphi^4$?

Conclusions

- Full loop corrections to multiparticle amplitudes in $\lambda\phi^4$ may be obtained with tree-level + semiclassics
  Check for two loops?
- In the limit λn exponent behave as in QM and semiclassical solution ϕ_s has scaling features
  Some rescale of variables + perturbation theory?
- Ultrarelativistic limit $\varepsilon/m \to \infty$ exists
  Find corresponding saddle-point solution in massless theory?