$\{\beta\}$-expansion for Adler function, Bjorken Sum Rule, the diagrams, and perturbation series optimization

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based on

SVM [JHEP 04(2017)169], K.Chetyrkin [NPB985(2022)115988],

P. Baikov&SVM [JHEP 09(2022)185, 03(2023)053]

with the participation of **K. Chetyrkin, P. Baikov, and D. Kotlorz**

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What are Adler D-function, Bjorken Sum Rule C^{Bjp}, the CBK relation

There are renorm-group invariant single scale Q^2 quantities D, C^{Bjp}:

Adler function

$$d_R D(a_s) = D_A = -12\pi^2 Q^2 \frac{d}{dQ^2}\Pi(Q^2); \quad Q^2 = -q^2$$

Bjorken polarized Sum Rule

$$\frac{1}{6}\left|\frac{g_A}{g_V}\right| C^{\text{Bjp}}(a_s) + \text{high twist} = S^{\text{Bjp}}_{\text{NS}}(Q^2) = \int_0^1 \left[g_1^{lp}(x,Q^2) - g_1^{ln}(x,Q^2)\right] dx$$

Crewther relation

–a plausible **conjecture** **[Crewther 1972,1997]** inspired by conformal symmetry

$$D_{ns}(a_s)\cdot C^{\text{Bjp}}(a_s) = \mathbb{1} + \beta(a_s) K(a_s),\ \text{where } K(a_s) - \text{polynom in } a_s = \frac{\alpha_s}{4\pi}$$

Crucial 3-loop analysis **[Broadhurst,Kataev,1993]** in $\overline{\textbf{MS}}$-scheme - **CBK** relation

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[P.Baikov,K.Chetyrkin,J.Kühn, PRL2010]- confirmation in $O(a_s^4)$ **5 loops**.

OUTLINE

1. **Intro**: What is the $\{\beta\}$**-expansion** for RG-invariants
   and what does it express?

2. How to apply the $\{\beta\}$**-expansion**?
   To understand and to control the corresponding PT series in each expansion order, etc.

3. How to obtain the $\{\beta\}$**-expansion** from **QCDe**
   (extended QCD with different fermion representations of gauge group),
   and alternative – from certain **QCD diagrams**
   result to **Adler** $\boldsymbol{D_{ns}(a_s)}$ and **Bjorken** $C^{\mathbf{Bjp}}(a_s)$.

4. **Crewther-Broadhurst-Kataev (CBK) relation** and
   its corollaries from $\{\beta\}$**-expansion** point of view.

5. On attempt of series optimization of **Bjorken** $C^{\mathbf{Bjp}}(a_s)$, example.

6. Generalization of NNA. A conjecture about β_0-dominance

7. **Conclusions**

STORE

Motivation for the revision of series representation

we consider 1-scale Q^2 RG-INVARIANT quantities at $Q^2 = \mu_R^2$, e.g., D_{ns}

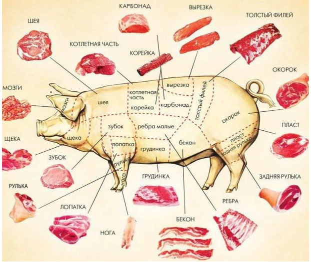
"**Wild**" approach: $\forall d_n$ - numbers, taken **wholly**

$$D(a_s) \sim 1 + a_s d_1 + a_s^2 d_2 + a_s^3 d_3 + \ldots$$

$$d_2 = 31.77 - 1.84 n_f;$$
$$d_3 = 1164.8 - 270.1 n_f - 5.5 n_f^2;$$
$$d_4 = 34765 - 8806.4 n_f + 481.3 n_f^2 - 2.56 n_f^3.$$

Delicate approach: $\forall d_n$ has an intrinsic structure due to a_s-renorm.

$$D(a_s) \sim 1 + \hat{M}(a_s, \{\beta_i\}) \leftarrow \text{2D matrix}$$



$$d_2 = \beta_0 d_2[1] + d_2[0]; \text{ [base notation]}$$
$$d_3 = \beta_0^2 d_3[2] + \beta_1 d_3[0,1] + \beta_0 d_3[1] + d_3[0]$$
$$d_4 = \beta_0^3 d_4[3] + \beta_2 d_3[0,0,1] + \ldots$$

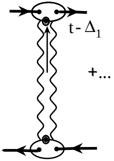
$\rightarrow$ series becomes "thick"

the decomposition is named **$\{\beta\}$-expansion[MS2005-7]**

it shows the **dynamic** knowledge of D exhibited how **a_s-renorm.**

How to apply the $\{\beta\}$-expansion? 1.(If we already have it)

Evident usage of the $\{\beta\}$-**expansion** is the different kinds of **optimization**: one can change the contributions of different origins of a_s-**renorm** playing with the choice of μ_R^2. E.g., well-known **BLM approach [Brodsky et al 1983]**:



(a)

$$d_2 = \underline{\beta_0 d_2[1]} + d_2[0] \to d_2[0] \text{ at } \mu_R^2 \to \mu_{BLM}^2 = \exp(-d_2[1]/d_1)\mu_R^2$$

$$\beta_0 = 11/3C_A - 4/3T_R n_f \;\Big|\; \text{profit at } \underline{\beta_0 d_2[1]} \gg d_2[0]$$

$$d_3 = \underline{\beta_0^2 d_3[2]} + \beta_1 d_3[0,1] + \beta_0 d_3[1] + d_3[0] \to d_3[0]$$

$$d_n = \underline{\beta_0^{n-1} d_n[n-1]} + \ldots + d_n[0] \to d_n[0]$$

generalization BLM $\Rightarrow$ Principle of Maximum Conformality [PMC, Brodsky et al, starting 2013 – up now]:

$$D = 1 + \sum_1^N a_s^n(\mu_R^2) d_n \to D_0 = 1 + \sum_1^N a_s^n(\mu_{PMC}^2) d_n[0]$$

This "conformal limit" **doesn't lead** to **optimized series** in any sense, making the PT convergence even **worse** in $O(a_s^4)$ **[A. Kataev&Molokoedov PRD2023]**

Alternative approaches: S.Brodsky, X.-G.Wu,+. . . (a'la mass mailing of **PMC**)
A.Kataev,+. . .

How to apply the $\{\beta\}$-expansion 2.(If we already have it)

[MS2007, A.Kataev&MS PRD2015] $(d_n, a_s, \mu^2) \xrightarrow{RG} (d'_n, a'_s, \mu'^2) \Rightarrow d'_n = \hat{\boldsymbol{B}}_{nj} d_j$

$$\ln(\mu^2/\mu'^2) = t - t' \equiv \Delta(a') = \Delta_0 + a'\beta_0 \Delta_1 + (a'\beta_0)^2 \Delta_2 + \ldots, \quad \boxed{\Delta_0 = d_2[1]/d_1}$$

↑ **BLM**

$$| \; a^1 \cdot d_1 \rightarrow a'^1 \cdot [d_1];$$

Each $| \; a^2 \cdot d_2 \rightarrow a'^2 \cdot [d'_2 = \beta_0 \, (d_2[1] - \Delta_0) + d_2[0]];$

order $| \; a^3 \cdot d_3 \rightarrow a'^3 \cdot \Big[d'_3 = \beta_0^2 \Big(d_3[2] - 2d_2[1]\Delta_0 + \Delta_0^2 - \Delta_1\Big) + \beta_1 \, (d_3[0,1] - \Delta_0)$

can be $| \quad + \beta_0 \, (d_3[1] - 2d_2[0]\Delta_0) + d_3[0]\Big];$

controlled $| \; a^4 \cdot d_4 \rightarrow a'^4 \cdot \Big[d'_4 = \beta_0^3 (d_4[3] - 3d_3[2]\Delta_0 \ldots - \Delta_2) + \ldots + d_4[0]\Big]$

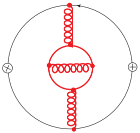
Fitting components $\Delta_0, \; \Delta_1, \; \Delta_2, \ldots$ of new normalization scale μ'^2 to adjust the elements $d'_2, d'_3, d'_4, \ldots$ following to any **optimization procedure**. Higher-order calculations are laborious and require **bloody efforts**, so they should be used with maximum efficiency, **i.e., optimized**.

How to obtain the elements of $\{\beta\}$-expansion? 1. **QCD** diagrams

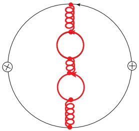
Direct results for the special diagrams **[P.Ball et al NPB1995], C1(IV1-2)** **in QCD** give the elements $d_3[2]$, $d_3[0,1]$ by means of $\boxed{\gamma_g}$ identificat. that admit restore all others:

$$\beta_0 = \frac{11}{3}C_A \boxed{-4T_R n_f};\ \beta_1 = \frac{34}{3}C_A - \frac{20}{3}C_A T_R n_f - \boxed{4T_R n_f C_F}$$

$$\boxed{d_3[0,1] = 3C_F\left(\frac{101}{12} - 8\zeta_3\right)};$$



$$d_3[2] = 3C_F\left(\frac{302}{9} - \frac{76}{3}\zeta_3\right);$$



$$\tilde{d}_3 = d_3 - \beta_0^2 d_3[2];\ \text{roots of } \beta_j(x_j),\ x_j \sim T_R n_f \quad : \quad \beta_0(x_0) = 0;\ \beta_1(x_1) = 0;$$

$$\tilde{d}_3(x_0) - \beta_1(x_0) d_3[0,1] = d_3[0];$$

$$\frac{\tilde{d}_3(x_0) - \tilde{d}_3(x_1)}{\beta_0(x_1)} + \frac{\beta_1(x_0)}{\beta_0(x_1)} d_3[0,1] = d_3[1],$$

How to obtain the elements of $\{\beta\}$-expansion? 2. QCD+$\tilde{g}$

Add 1 degrees of freedom-**d.o.f.** **[K.ChetyrkinPLB1997]**, ***D*** **with MSSM gluinos** $n_{\tilde{g}}$ entering only in intrinsic loops. We have 2 **d.o.f.** $x \sim T_R n_f, y \sim C_A n_{\tilde{g}}$.

$$\exists \text{ a point } (\bar{x},\bar{y}): \quad \beta_0(\bar{x},\bar{y}) = \beta_1(\bar{x},\bar{y}) = 0$$

$$d_3(\bar{x},\bar{y}) = d_3[0] + \mathbf{0}$$

$$d_3[0,1] = \left(d_3(x_0,0) - d_3(\bar{x},\bar{y})\right)/\beta_1(\bar{x}_0,0)$$

The Same results for $d_3[.]$ **as with diagrams** were obtained in **[MS,JHEP2007, Kataev,MS,PRD2015]**.

$$d_3[1] = 3C_F\left[C_A\left(-\frac{3}{4}+\frac{80}{3}\zeta_3-\frac{40}{3}\zeta_5\right) - C_F\left(18+52\zeta_3-80\zeta_5\right)\right];$$

$$d_3[0] = 3C_F\left[C_A^2\left(\frac{523}{36}-72\zeta_3\right)+\frac{71}{3}C_A C_F-\frac{23}{2}C_F^2\right].$$

The key elements d_n and c_n (**Bjorken** $C^{\mathrm{Bjp}}(a_s)$) are related via **CBK**

$$c_2[1]+d_2[1] = c_3[0,1]+d_3[0,1] = c_n[\underbrace{0,0,\ldots,1}_{n-1}] + d_n[\underbrace{0,0,\ldots,1}_{n-1}] = 3C_F\left(\frac{7}{2}-4\zeta_3\right)$$

so, we know also $c_3[0,1]$ and others $c_3[.]$.

How to obtain the elements of $\{\beta\}$-expansion? 3. QCDe

We need in extended QCD with additional **degrees of freedom - d.o.f.**,
e.g., any numbers of fermion multiplets

These **d.o.f.** $\{R\}$ **interact following the universal gauge principle** entering only in intrinsic loops.

$$\{T_R n_f, \frac{C_A}{2} n_{\tilde{g}}, \ldots\} \xrightarrow{general} \{R\} \quad \text{[M.Zoller 2016] for } \beta(a_s, \{R\}),$$

$$D(a_s, \{R\}) \text{ or } C^{\mathrm{Bjp}}(a_s, \{R\}) : \text{QCDe}$$ [K.Chetyrkin NPB985(2022)115988]

d.o.f.:

$\{R\}$- any numbers of different quark representations **[K.Chetyrkin NPB985(2022)]**

$$\mathcal{L}_{QCD} = \ldots + \sum_{r=1}^{N_{\rm rep}} \sum_{q=1}^{n_{f,r}} \left\{ \frac{i}{2} \bar{\psi}_{q,r} \overleftrightarrow{\hat{\partial}} \psi_{q,r} - m_{q,r} \bar{\psi}_{q,r} \psi_{q,r} + g_s \bar{\psi}_{q,r} \hat{A}^a T^{a,r} \psi_{q,r} \right\},$$

$$\mathrm{R} = (q - \text{flavors}, r - \text{Representation})$$

Lie Algebra: $\left[T^{a,r}, T^{b,r}\right] = if^{abc} T^{c,r};\; T^{a,r}_{ik} T^{a,r}_{kj} = \delta_{ij} C_{F,r};\; T_{F,r} \delta^{ab} = \mathbf{Tr}\left(T^{a,r} T^{b,r}\right);$

$$d_R^{a_1 a_2 \ldots a_n} = \frac{1}{n!} \sum_{\text{perm } \pi} \mathrm{Tr}\left\{ T^{a_{\pi(1)},R} T^{a_{\pi(2)},R} \ldots T^{a_{\pi(n)},R} \right\},$$

How to obtain the $\{\beta\}$-**expansion**? 4. Solving a set of equations

The key role plays the **set of zeros** of $\beta_0(\{R\}), \beta_1(\{R\}), \beta_2(\{R\}), \ldots$ and **zeros** of sets of these β_k.
E.g., in $O(a_s^4)$ $\beta_0, \beta_1, \beta_2$ are defined on the axes of variables R_0, R_1, R_2:
($R_0 = T_R n_f, R_1 = C_A n_{\tilde{g}}/2, R_2 = ?,$ d.o.f..)

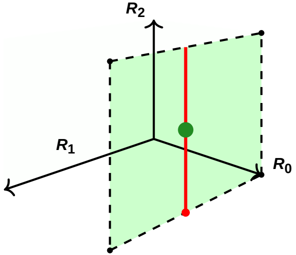
1)∃ 2D surface in 3D $\beta_0(\{\bar{R}_0\}) = 0$, **[QCD]**
Then $d_4(\bar{R}_0) = \beta_2(\{\bar{R}_0\}) d_4[0,0,1] + \beta_1(\{\bar{R}_0\}) d_4[0,1] + d_4[0]$

2)∃ line in 3D $\beta_0(\{\bar{R}_{0,1}\}) = \beta_1(\{\bar{R}_{0,1}\}) = 0$, **[QCD+$\tilde{g}$]**.
Then $d_4(\bar{R}_{0,1}) = \beta_2(\{\bar{R}_{0,1}\}) \underline{d_4[0,0,1]} + d_4[0]$

3) ∃ 3D point $\bar{R}_{0,1,2}$:
$\beta_0(\{\bar{R}_{0,1,2}\}) = \beta_1(\{\bar{R}_{0,1,2}\}) = \beta_2(\{\bar{R}_{0,1,2}\}) = 0$,
Then $D(\bar{R}_{0,1,2}) = (a_s^2 d_2[0], a_s^3 d_3[0], a_s^4 d_4[0])$, step by step

4) ∃ curve in 3D $\beta_0(\{\bar{R}_{0,2}\}) = \beta_2(\{\bar{R}_{0,2}\}) = 0$,
Then $d_4(\bar{R}_{0,2}) = \beta_1(\{\bar{R}_{0,2}\}) \underline{d_4[0,1]} + d_4[0]$

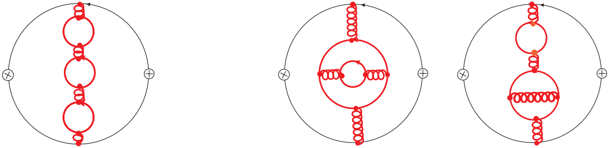
...



Finally this reduces to the solvable set of equations that results to $d_4[.], c_4[.]$.
In $O(a_s^5)$ at 6-loop one has single-valued solution of the similar set of equation [MS2017].

Diagrammatic test for $d_4[.]$ elements of β-expansion

The types of the "abelian" diagrams for the D_A in order a_s^4 with the renormalization of the only one gluon line, see results in **[P.Ball et al NPB1995],C1(V1-2-3)**



$C_F(T_R n_f)^3 \Rightarrow \boldsymbol{\beta_0^3}$, that determines $d_4[3]$

$C_F^2(T_R n_f)^2 \Rightarrow \boldsymbol{\beta_0\beta_2,\ \beta_1\beta_0}$, the sum gives $\beta_2 d_4[0,0,1] + \beta_1\beta_0 d_4[1,1]$

$$\text{diagram's sum} \equiv \hat{P}_{(C_F T_R n_f)^2}\left(\beta_2\, d_4[0,0,1] + \beta_1\beta_0\, d_4[1,1]\right)$$

$$= \left(\frac{44}{9} C_F (T_R n_f)^2\right) d_4[0,0,1] + (-4 C_F T_R n_f)\left(-\frac{4}{3}(T_R n_f)\right) d_4[1,1],$$

Partial results for the diagrams confirm the results for $\boldsymbol{d_4[3], d_4[0,0,1], d_4[1,1]}$ and also for abelian part of $\boldsymbol{d_2[2]}$, obtained from the algebraic procedure based on **QCDe**.

The structure of $\{\beta\}$-expansion

Then one can decompose all **β-terms** explicitly following an **algebraic procedure** **[MS2017]**: all **7** elements of $\boldsymbol{d_4}$ and $\boldsymbol{c_4}$ are **explicitly** obtained here,

$$d_4 = \beta_0^3 d_4[3] + \beta_2 d_4[0,0,1] + \beta_1\beta_0 d_4[1,1] + \beta_0^2 d_4[2] + \beta_1 d_4[0,1] + \beta_0 d_4[1] + d_4[0]$$

$$d_n = \underbrace{\beta_0^{n-1}\, d_n[n-1] + \ldots + \beta_0 d_n[1] + d_n[0]}_{\boldsymbol{N}(n)},$$ **series** D becomes **matrix** $\hat{D}$

$$\hat{D}(a_s, \{\beta_i\}) = \begin{pmatrix} a_s^2 & \ldots & a_s^{n-1} & a_s^n & \ldots \\ & & & d_n[0] & \\ \beta_0 d_2[1] & & \vdots \quad \boldsymbol{N}(n)\left\{ \begin{matrix} \\ \\ \\ \end{matrix}\right. & \beta_0 d_n[1] & \\ & & & \vdots & \\ & & & & \end{pmatrix}$$

$$\boldsymbol{N}(n) = \sum_{l=0}^{(n-1)} p(l) = \{\mathbf{1,2,4,7,12,..,97,..}\} \sim \frac{\sqrt{6n}}{\pi}\cdot(p(n) \leftarrow \text{ partition of number } n) + \ldots$$

$$\uparrow \qquad \uparrow \qquad \uparrow$$

$$\text{PT order}\, n = \{\mathbf{1,..\quad ,4,..\quad ,10,..}\}$$

Hardy-Ramanujan asymptotic for partition of numbers $p(n) \sim \dfrac{1}{4n\sqrt{3}} \exp\left(\pi\sqrt{2n/3}\right)$

Important! We need new d.o.f. Only to perform the decomposition, after that we return from QCDe to the standard QCD, $\{\boldsymbol{R}\} \to \boldsymbol{T}_{\mathrm{R}}\boldsymbol{n_f}$.

The marked traces of **the general gauge principle** saved as **$\{\beta\}$-expansion**.

CBK relation and its corollaries

The elements of $\{\beta\}$**-expansion** for $\boldsymbol{D_{ns}}$, $\boldsymbol{C}^{\text{Bjp}}$ were independently obtained. They provide the *appropriate "bricks"* to analyse **CBK relation**, as well as the way **to check them**.

First time it was applied to **CBK** **[A.Kataev&MS TMP2012]**

$$\boldsymbol{D_{ns}}(\boldsymbol{a_s}) \cdot \boldsymbol{C}^{\text{Bjp}}(\boldsymbol{a_s}) = \mathbb{1} + \beta(\boldsymbol{a_s}) \times \sum_{n=1} a_s^{n-1} K_n \quad (\clubsuit)$$

Structure of K_n

$$K_1 = K_1[1],\; K_2 = K_2[1] + \beta_0 K_2[2], \quad K_3 = K_3[1] + \beta_0 K_3[2] + \beta_0^2 K_3[3] + \beta_1 K_3[1,1],$$
$$K_4 = K_4[1] + \ldots$$

- Products of $\boldsymbol{d_k}[.]$, $\boldsymbol{c_j}[.]$ elements are already presented in the LHS of ($\clubsuit$)
- While the structure of the RHS ($\clubsuit$) orders combinations of the elements that leads to the equations for them.
- So, we can **Confirm** the values of obtained $\boldsymbol{d_k}[.]$, $\boldsymbol{c_j}[.]$ elements satisfied these equations (**CBK relation**) and **Predict** the elements in next orders.

Another way to obtain $\boldsymbol{d_k}[.]$, $\boldsymbol{c_j}[.]$ just inspired by (**CBK relation**)($\clubsuit$) appeared first in **[Cvetic&Kataev PRD2016]**, further in **[Kataev&Molokoedov JHEP2022]**, the results do not agree with ours.

CBK relation, its corollaries for $\mathbb{1}$ term.

1. The $\boldsymbol{D_0} \cdot C_0^{\rm Bjp} = \mathbb{1}$ –"conformal" part of the relation, here $d_n \to d_n[0] \in \boldsymbol{D_0}$,

$$\boldsymbol{c_k}\,[\boldsymbol{0}] + \boldsymbol{d_k}\,[\boldsymbol{0}] = (-)^k \det[\boldsymbol{D_0^{(k)}}] \equiv (-)^k \begin{vmatrix} d_1 & 1 & 0 & & \ldots & 0 \\ d_2 & d_1 & 1 & & \ldots & 0 \\ d_3 & d_2 & d_1 & & \ldots & 0 \\ \ldots & & & & \ldots & 0 \\ d_{k-1} & \ldots & \ldots & & d_1 & 1 \\ \boldsymbol{0} & d_{k-1} & d_{k-2} & \ldots & d_2 & d_1 \end{vmatrix},$$

$\boldsymbol{c_k}\,[\boldsymbol{0}] + \boldsymbol{d_k}\,[\boldsymbol{0}] = \text{Polynom}(\ldots d_{k-1})$, $k = 2, 3, 4$ – **Confirmations!**

$$\begin{aligned} d_4[0] + c_4[0] &= 2d_1 d_3[0] - 3d_1^2 d_2[0] + d_2[0]^2 + d_1^4 \\ &= 3\mathrm{C_F^2}\left[132\mathrm{C_F C_A} - \frac{111}{4}\mathrm{C_F^2} + \left(\frac{175}{2} - 432\zeta_3\right)\mathrm{C_A^2}\right], \end{aligned}$$

$\boldsymbol{c_5}\,[\boldsymbol{0}] + \boldsymbol{d_5}\,[\boldsymbol{0}] = \text{Polynom}(d_4)$, $\quad k = 5$ – **Prediction**

$$\begin{aligned} d_5[0] + c_5[0] &= 2d_1 d_4[0] + 2d_3[0]d_2[0] - 3d_3[0]d_1^2 + 4d_2[0]d_1^3 - 3d_2^2[0]d_1 - d_1^5 \\ &= d_1 \Bigg[C_A^2 C_F^2\, 27\,(43 + 128\zeta_3) + C_F^4 \left(\frac{2485}{2} + 192\zeta_3\right) - C_A C_F^3\,(3097 + 864\zeta_3 \\ &\quad + C_A^3 C_F \left(\frac{206233}{72} + 7969\zeta_3 - 14220\zeta_5\right) + 2\delta d_4 \Bigg]. \end{aligned}$$

2. The factorization of the $\beta(a_s)$, taken wholly, sets the chain of conditions

$$\begin{aligned} K_1[1] &= d_2[1] + \overset{\beta_0}{\downarrow} c_2[1] = d_3[0,1] + \overset{\beta_1}{\downarrow} c_3[0,1] = \underline{c_4[0,0,1] + \overset{\beta_2}{\downarrow} d_4[0,0,1]} = 3C_F\left(\frac{7}{2} - 4\zeta_3\right) \\ &= d_n[\underbrace{0,0,\ldots,1}_{n-1}] + c_n[\underbrace{0,0,\ldots,1}_{n-1}] \end{aligned}$$

Confirmations/**Prediction**

$$\begin{aligned} K_2[1] &= c_3[1] + d_3[1] + d_1(c_2[1] - d_2[1]) = \\ &= \underline{c_4[0,1] + d_4[0,1]} + d_1(c_3[0,1] - d_3[0,1]) \quad \textbf{Confirmation} \\ &= C_F^2\left(-\frac{397}{6} - 136\zeta_3 + 240\zeta_5\right) + C_F C_A\left(\frac{47}{3} - 16\zeta_3\right) \\ &= \underline{c_5[0,0,1] + d_5[0,0,1]} + d_1(c_4[0,0,1] - d_4[0,0,1]) \quad \textbf{Prediction} \\ &= c_n[\underbrace{0,\ldots,1}_{n-2}] + d_n[\underbrace{0,\ldots,1}_{n-2}] + d_1(c_{n-1}[\underbrace{0,\ldots,1}_{n-2}] - d_{n-1}[\underbrace{0,\ldots,1}_{n-2}]). \end{aligned}$$

$$\begin{aligned} K_3[1] &= c_4[1] + d_4[1] + d_1\left(c_3[1] - d_3[1]\right) + d_2[0]c_2[1] + d_2[1]c_2[0] \quad \textbf{Prediction} \\ &= \underline{c_5[0,1] + d_5[0,1]} + d_1\left(c_4[0,1] - d_4[0,1]\right) + d_2[0]c_3[0,1] + c_2[0]d_3[0,1] = \ldots \\ &= c_{n+1}[\underbrace{0,\ldots,1}_{n-2}] + d_{n+1}[\underbrace{0,\ldots,1}_{n-2}] + d_1\left(c_n[\underbrace{0,\ldots,1}_{n-2}] - d_n[\underbrace{0,\ldots,1}_{n-2}]\right) + \\ &\quad d_2[0]\, c_{n-1}[\underbrace{0,\ldots,1}_{n-2}] + c_2[0]\, d_{n-1}[\underbrace{0,\ldots,1}_{n-2}] \end{aligned}$$

General requirements $\boldsymbol{A}, \boldsymbol{B}$ to a series $S(a_s) \stackrel{RG}{\to} S'(a'_s)$ **don't need $\{\beta\}$-expan**.

$$t - t' \equiv \boldsymbol{\Delta_0} + a'_s\beta_0 \boldsymbol{\Delta_1} + (a'_s\beta_0)^2 \boldsymbol{\Delta_2} \Rightarrow$$

$$(\boldsymbol{A}) \quad |\boldsymbol{\Delta_0}| \geqslant a'_s\beta_0 |\boldsymbol{\Delta_1}| \geqslant (a'_s\beta_0)^2 |\boldsymbol{\Delta_2}| \geqslant \ldots$$

$$S(a_s) - 1 = c_1(a_s + (a_s)^2 c_2 + \ldots) \stackrel{RG}{\longrightarrow} c_1(a'_s + (a'_s)^2 c'_2 + \ldots) \Rightarrow$$

$$(\boldsymbol{B}) \quad 1 \geqslant a'_s|c'_2| \geqslant (a'_s)^2 |c'_3| \geqslant (a'_s)^3 |c'_4| \geqslant \ldots$$

The set of inequalities $\boldsymbol{A} \wedge \boldsymbol{B}$ ($2\boldsymbol{D}$ series) is universal for any quantity.

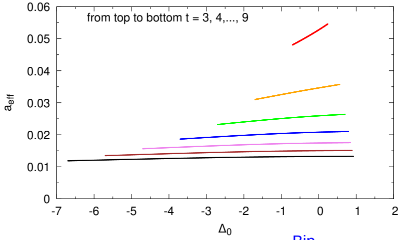
For **Bjorken** $\boldsymbol{C}^{\text{Bjp}}(a_s)$ the constrains form admissible domains for every $t = \ln(\frac{Q^2}{\Lambda^2_{qcd}})$

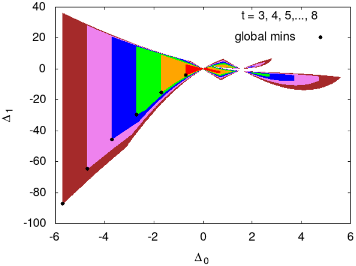
$t = \mathbf{3},\ \mathbf{4},\ \mathbf{5}, \ldots,\ 8,\ 9$:

$Q^2 = \mathbf{2}, \mathbf{5.5}, \mathbf{15} \ldots, 301, 819$ GeV^2

(Left, 1D) $\boldsymbol{\Delta_0}$, $\boldsymbol{\Delta_{1.2}} = \mathbf{0}$;

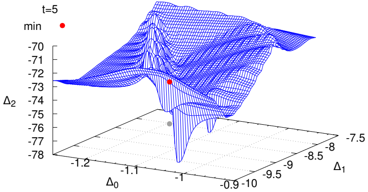
(Right, 2D) $\boldsymbol{\Delta_0}, \boldsymbol{\Delta_1}$, $\boldsymbol{\Delta_2} = \mathbf{0}$

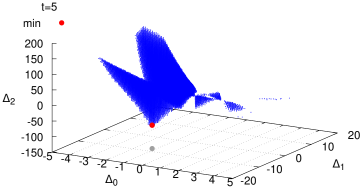




$$\boldsymbol{C}^{\text{Bjp}}(\boldsymbol{\Delta_0}, \boldsymbol{\Delta_1}, \boldsymbol{\Delta_2}) = 1 + \boldsymbol{a}_{\text{eff}} \cdot c_1$$

Series optimization. A model instance





3D

around **min** of $\boldsymbol{a}_{\rm eff}$, $(-1.2, -8, -75)$
admiss. **domain** - above the surface
$t = 5 \to Q^2 \approx \mathbf{15}$ GeV^2

global admissible fuzzy **domain**

“A weak” optimization problem for **Bjorken** $\boldsymbol{C}^{\rm Bjp}(a_s)$

One can interpret “Optimization” **as the min of** $\boldsymbol{a}_{\rm eff}$ obtained **directly, in numerical calculations** in **[D.Kotlorz&MS PRD2019]**

$$\boldsymbol{C}^{\rm Bjp} = 1 + \boldsymbol{a}'_{\rm eff} \cdot c_1, \quad \boldsymbol{a}'_{\rm eff} = a'_s \left(1 + a'_s c'_s + (a'_s)^2 c'_3 + (a'_s)^3 c'_4 + \ldots\right)$$

- 1D-optimization may be effective at **all** $\boldsymbol{t}$ with **1D-profit** $= \mathbf{9 - 10\%}$; **global min** of $\boldsymbol{a}_{\rm eff}$ – on the left edges.
- 2D-optimization may be effective at **all** $\boldsymbol{t}$ with **2D-profit** $= \mathbf{12 - 14\%}$; **global min** of $\boldsymbol{a}_{\rm eff}$ – on the left/down edges.
- 3D-optimization —- at **all** $\boldsymbol{t}$ with **2D-profit** $<$ **3D-profit** $= \mathbf{14 - 17\%}$;

Contribution of $(a_s)^4 c_4$ to the whole $\boldsymbol{a}_{\rm eff}$ is of $\mathbf{7 - 8\%}$

Generalized NNA, illustration on a β_0-dominance

Observation: $b_i \stackrel{def}{=} \beta_i/\beta_0^{i+1} = O(1)$ at $n_f = 0 \div 5$, $i = 1 \div 4$, $\beta_0 \simeq 10$

$$d_3 = \beta_0^2 \left[\overbrace{\Big(\underbrace{d_3[2]}_{\mathbf{12.4}} + b_1\, d_3[0,1] \Big)}^{\mathbf{8.6}} + \overbrace{\frac{1}{\beta_0} d_3[1]}^{32.6} + \overbrace{\frac{1}{\beta_0^2} d_3[0]}^{-28.3} \right] \approx \beta_0^2\, [\,\mathbf{12.9}\,]$$

$$c_3 = \beta_0^2 \left[\overbrace{\Big(\underbrace{c_3[2]}_{\mathbf{-25.6}} + b_1\, c_3[0,1] \Big)}^{\mathbf{-25.9}} + \overbrace{\frac{1}{\beta_0} c_3[1]}^{-17.8} + \overbrace{\frac{1}{\beta_0^2} c_3[0]}^{27.7} \right] \approx \beta_0^2\, [\,\mathbf{-16}\,]$$

$$d_4 = \beta_0^3 \left[\overbrace{\Big(\underbrace{d_4[3]}_{\mathbf{8.7}} + b_1\, d_4[1,1] + b_2\, d_4[0,0,1] \Big)}^{\mathbf{24}} + \overbrace{\frac{1}{\beta_0}(d_4[2] + b_1 d_4[0,1])}^{34.3} + \overbrace{\frac{1}{\beta_0^2} d_4[1]}^{-6.8} + \overbrace{\frac{1}{\beta_0^3} d_4[0]}^{-34.35} \right]$$

$$\approx \beta_0^3\, [\,\mathbf{17.2}\,]$$

$$c_4 = \beta_0^3 \left[\overbrace{\Big(\underbrace{c_4[3]}_{\mathbf{-89.6}} + b_1\, c_4[1,1] + b_2\, c_4[0,0,1] \Big)}^{\mathbf{-112}} + \overbrace{\frac{1}{\beta_0}(c_4[2] + b_1 c_4[0,1])}^{-95.5} + \overbrace{\frac{1}{\beta_0^2} c_4[1]}^{136.5} + \overbrace{\frac{1}{\beta_0^3} c_4[0]}^{9.4} \right]$$

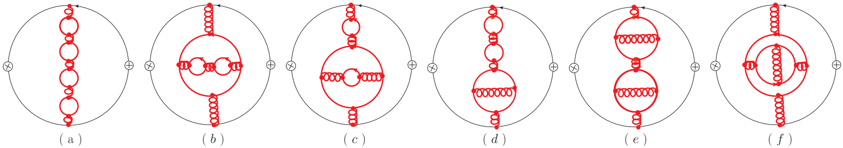
$$\approx \beta_0^3\, [\,\mathbf{-61.7}\,]$$

Conjecture of β_0-dominance. 1

One groups terms of order $(1/\beta_0)^0$ that **presumably dominate** in the result of d_5,
indexes of leading group $\boldsymbol{d_n[j_1,\ldots,j_{n-1}]} \to \boldsymbol{1j_1 + 2j_2 + \ldots + (n-1)j_{n-1} = n-1}$,
number of group elements – $\boldsymbol{p(n-1)}$

$$d_5 = \beta_0^4\Bigg[\Big(\overbrace{d_5[4] + b_1 d_5[2,1] + b_1^2 d_5[0,2] + b_2 d_5[1,0,1] + b_3 d_5[0,0,0,1]}^{p(5-1)}\Big) + \frac{1}{\beta_0}\Big(d_5[3] + b_1 d_5[1,1] + b_2 d_5[0,0,1]\Big) + \frac{1}{\beta_0^2}\Big(d_5[2] + b_1 d_5[0,1]\Big) + \frac{1}{\beta_0^3} d_5[1] + \frac{1}{\beta_0^4} d_5[0]\Bigg] .$$

Types of “abelian” QCD diagrams that **Independently check** the gNNA in order a_s^5.
(a): $C_\mathrm{F}(T_R n_f)^4 \Rightarrow \boldsymbol{\beta_0^4}$;
(b),(c),(d): $C_\mathrm{F}(T_R n_f)^3 \Rightarrow \boldsymbol{\beta_3, \beta_0\beta_2, \beta_0^2\beta_1}$; (e),(f): $C_\mathrm{F}^2(T_R n_f)^2 \Rightarrow \boldsymbol{\beta_1\beta_1, \beta_3}$



and corresponds to renormalization of only one gluon line.

CONCLUSIONS

1. The **$\{\beta\}$-expansion** for PT series is invented and analyzed for the **Renormalization Group invariant** quantities.
   This allows to perform different optimizations of the PT series.

2. The elements of **$\{\beta\}$-expansion** can be determined within **QCDe** following to an algebraic procedure, or
   considering the special types of diagrams in **pure QCD**.

3. The **Crewther-Broadhurst-Kataev relation** is **reproduced** in order of $O(a_s^4)$. The interesting relations between the elements of Adler $\boldsymbol{D_{ns}}$, and Bjorken polarized SR $\boldsymbol{C}^{\mathrm{Bjp}}$ are established in any orders of a_s.

4. Conjecture of a generalization of the Naive Non-Abelianization (gNNA).

STORE, CBK relation, the structure of K – term

The universal form of the second term appears due to the cancellation of a_s^1- terms

$$K_{n\geqslant 3}[1] \;=\; c_{n+1}[1]+d_{n+1}[1]+d_1\left(c_n[1]-d_n[1]\right)+\sum_{k=2}^{n-2}\left(d_k[0]c_{n+1-k}[1]+c_k[0]d_{n+1-k}[1]\right).$$

Partial results for K-term in order $O(a_s^4)$

$$K_1[1] \;=\; d_2[1]+c_2[1]=3C_{\rm F}\left(\frac{7}{2}-4\zeta_3\right)$$

$$\begin{aligned} K_2[1] &= c_3[1]+d_3[1]+d_1\left(c_2[1]-d_2[1]\right) \\ &= C_{\rm F}^2\left(-\frac{397}{6}-136\zeta_3+240\zeta_5\right)+C_{\rm F}C_{\rm A}\left(\frac{47}{3}-16\zeta_3\right) \\ K_2[2] &= c_3[2]+d_3[2]=3C_{\rm F}\left(\frac{163}{6}-\frac{76}{3}\zeta_3\right) \end{aligned}$$

$$\begin{aligned} K_3[1] &= c_4[1]+d_4[1]+d_1\left(c_3[1]-d_3[1]\right)+d_2[0]c_2[1]+d_2[1]c_2[0] \\ K_3[2] &= c_4[2]+d_4[2]+d_1\left(c_3[2]-d_3[2]\right)+d_2[1]c_2[1], \\ K_3[3] &= c_4[3]+d_4[3], \\ K_3[1,1] &= c_4[1,1]+d_4[1,1] \end{aligned}$$

STORE, explicit form of D-elements. 1

$$d_1 = 3\mathrm{C_F};\; d_2[1] = d_1\left(\frac{11}{2} - 4\zeta_3\right);\; d_2[0] = d_1\left(\frac{\mathrm{C_A}}{3} - \frac{\mathrm{C_F}}{2}\right);$$

$$d_4[3] = C_F\left(\frac{6131}{9} - 406\zeta_3 - 180\zeta_5\right);$$

$$d_4[1,1] = C_F\left(385 - \frac{1940}{3}\zeta_3 + 144\zeta_3^2 + 220\zeta_5\right);$$

$$d_4[2] = -C_F\Big[C_F\left(\frac{6733}{8} + 1920\zeta_3 - 3000\zeta_5\right) + C_A\left(\frac{20929}{144} - \frac{12151}{6}\zeta_3 + 792\zeta_3^2 + 1050\zeta_5\right)\Big];$$

$$d_4[0,0,1] = C_F\left(\frac{355}{6} + 136\zeta_3 - 240\zeta_5\right);$$

$$d_4[1] = C_F\Big[-C_F^2\left(\frac{447}{2} - 42\zeta_3 - 4920\zeta_5 + 5040\zeta_7\right) + C_A C_F\left(\frac{3301}{4} - 678\zeta_3 - 2280\zeta_5 + 2520\zeta_7\right) + C_A^2\left(\frac{16373}{36} - \frac{17513}{3}\zeta_3 + 2592\zeta_3^2 + 3030\zeta_5 - 420\zeta_7\right)\Big],$$

$$d_4[0,1] = -C_F\left[C_A\left(\frac{139}{12} + \frac{1054}{3}\zeta_3 - 460\zeta_5\right) + C_F\left(\frac{251}{4} + 144\zeta_3 - 240\zeta_5\right)\right],$$

STORE, explicit form of $D_0, \ C_0$ elements. 2

$$
\begin{aligned}
d_4[0] &= \tilde{d}_4[0] + \delta d_4 \\
&= C_F^4\left(\frac{4157}{8} + 96\zeta_3\right) - C_A C_F^3\left(\frac{2409}{2} + 432\zeta_3\right) + C_A^2 C_F^2\left(\frac{3105}{4} + 648\zeta_3\right) + \\
&\quad C_A^3 C_F\left(\frac{68047}{48} + \frac{8113}{2}\zeta_3 - 7110\zeta_5\right) + \delta d_4\,, \\
\delta d_4 &= -\frac{16}{dR}\left(d_F^{abcd} n_f d_F^{abcd}(13 + 16\zeta_3 - 40\zeta_5) + d_F^{abcd} d_A^{abcd}(-3 + 4\zeta_3 + 20\zeta_5)\right)
\end{aligned}
$$

$$
\begin{aligned}
c_3[0] &= c_1\left(\left(\frac{523}{36} - 72\zeta_3\right)C_A^2 + \frac{65}{3}C_F C_A + \frac{C_F^2}{2}\right) \\
c_4[0] &= \tilde{c}_4[0] - \delta d_4 \\
&= -C_F^4\left(\frac{4823}{8} + 96\zeta_3\right) + C_A C_F^3\left(\frac{3201}{2} + 432\zeta_3\right) - C_A^2 C_F^2\left(\frac{2055}{4} + 1944\zeta_3\right) - \\
&\quad C_A^3 C_F\left(\frac{68047}{48} + \frac{8113}{2}\zeta_3 - 7110\zeta_5\right) - \delta d_4\,;
\end{aligned}
$$

STORE, explicit form of β-function elements. 3

$$\beta_0 = \frac{11}{3}C_A - \frac{4}{3}\mathbf{nT}; \qquad (1)$$

$$\beta_1 = \frac{34}{3}C_A^2 - 4\left[\mathbf{nTC1} + \frac{5}{3}C_A(\mathbf{nT})\right]; \qquad (2)$$

$$\beta_2 = \frac{2857}{54}C_A^3 + 2(\mathbf{nTC2}) - \frac{205}{9}C_A(\mathbf{nTC1}) - \frac{1415}{27}C_A^2(\mathbf{nT}) + \mathbf{nT}\left[\frac{44}{9}(\mathbf{nTC1}) + \frac{158}{27}C_A(\mathbf{nT})\right]; \qquad (3)$$

$$\begin{aligned}\beta_3 = {} & \left(\frac{150653}{486} - \frac{44}{9}\zeta_3\right)C_A^4 - \left(\frac{80}{9} - \frac{704}{3}\zeta_3\right) + d_{\mathrm{AA}} \\ & \left[46(\mathbf{nTC3}) - \left(\frac{4204}{27} - \frac{352}{9}\zeta_3\right)C_A(\mathbf{nTC2}) + \left(\frac{7073}{243} - \frac{656}{9}\zeta_3\right)C_A^2(\mathbf{nTC1}) \right. \\ & \left. - \left(\frac{39143}{81} - \frac{136}{3}\zeta_3\right)C_A^3(\mathbf{nT})\right] + \left(\frac{512}{9} - \frac{1664}{3}\zeta_3\right)\sum_i n_{f,i}\, d_{\mathrm{FA},i} + \\ & \left[\left(\frac{184}{3} - 64\zeta_3\right)(\mathbf{nTC1})^2 - \left(\frac{304}{27} + \frac{128}{9}\zeta_3\right)(\mathbf{nT})(\mathbf{nTC2}) \right. \\ & \left. + \left(\frac{17152}{243} + \frac{448}{9}\zeta_3\right)C_A(\mathbf{nT})(\mathbf{nTC1}) + \left(\frac{7930}{81} + \frac{224}{9}\zeta_3\right)C_A^2(\mathbf{nT})^2\right] - \\ & \left(\frac{704}{9} - \frac{512}{3}\zeta_3\right)\sum_{i,j} n_{f,i}n_{f,j}\, d_{\mathrm{FF},ij} + (\mathbf{nT})^2\left[\frac{1232}{243}(\mathbf{nTC1}) + \frac{424}{243}C_A(\mathbf{nT})\right]. \qquad (4)\end{aligned}$$

$$\mathbf{nT} = \sum_i n_{f,i}T_{F,i}, \quad \mathbf{nTC}\boldsymbol{k} = \sum_i n_{f,i}T_{F,i}C_{F,i}^k, \quad \mathbf{nd}^{\boldsymbol{abcd}} = \sum_i n_{f,i}d_{F,i}^{abcd},$$