Supercurvatures in $\mathcal{N} = 2$ supergravity and higher spin theories

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Pereslavl-Zalessky, May 21, 2024

XXII international seminar on high-energy physics QUARKS-2024

Based on work with Evgeny Ivanov, arXiv:24xx.xxxxx

Introduction: supergravity

- Supergravity is the gauge theory of supersymmetry.
- Local supersymmetry is a unique symmetry principle to bind together the gravitational field (spin 2) and matter fields of spin $s < 2$.
- Improved UV behavior.

Higher-curvature invariants:

- Counterterms [Utiyama, DeWitt, 1962].
- Inclusion of quadratic curvature terms in action gives renormalizable theory [Stelle, 1977].
- $R + R^2$ Starobinsky theory [Starobinsky, 1980].
- Pure R^2 gravity is ghost-free [Alvarez-Gaume, Kehagias, Kounnas, Lust, Riotto, 2016].
- Superconformal anomalies.
- String effective action [Fradkin, Tseytlin 1985], [Callan, Friedan, Martinec, Perry, 1985].

- Supergravity theory with higher curvature terms [Cecotti, 1987].

Introduction: supergravity

- Superspace approach to extended ($\mathcal{N} \geq 2$) SUGRA:
  - Supergravity in superspace [Howe 1982]
  - $\mathcal{N} = 2$ supergravity in harmonic superspace $\Rightarrow$ prepotentials [Galperin, Ivanov, Ogievetsky, Sokatchev 1984-1987]
  - $\mathcal{N} = 2$ linearized supergravity [Zupnik 1998], [Buchbinder, Ivanov, Zaigraev 2021]
- Higher-derivative $\mathcal{N} = 2$ invariants:
  - Square of the Weyl tensor [Bergshoeff, de Roo, de Wit 1981]
  - $\mathcal{N} = 2$ supersymmetrization of R^4 invariant [Moura 2003], [de Wit, Katmadas, Zalk 2011]
  - $\mathcal{N} = 2$ supersymmetrization of $R^2_{mn} - \frac{1}{3}R^2$ [Butter, de Wit, Kuzenko, Lodato 2013]
  - $\mathcal{N} = 2$ supersymmetrization of R^2 [Kuzenko, Novak 2015]
- Higher-derivative invariants in harmonic superspace:
  - R^4, D^4R^4, D^6R^4 in linearized $\mathcal{N} = 8$ SUGRA [Drummond, Heslop,Howe, Kerstan 2003]
  - $L = \mathcal{N} - 1$ superinvariants in an on-shell *harmonic superspace* [Bossard, Howe, Stelle, Vanhove 2011]

? Supergravity invariants in harmonic superspace

? Differential supergeometry in harmonic superspace

? Manifestly $\mathcal{N} = 2$ covariant quantization of extended supergravity

? Interaction of supergravity and higher spins

Introduction: higher spins

- Free massless higher-spin theories [Fronsdal 1978], [Fang, Fronsdal 1978]
- It is shown that a gauge theory of self-interacting massless spin-3 particles which is analogous to Yang-Mills or the theory of gravity does not exist. A way out may be the existence of an interacting infinite family of massless particles of various spins. [Berends, Burgers, van Dam 1985]
- Explicit Construction of Conserved Currents for Massless Fields of Arbitrary Spin [Berends, Burgers, van Dam 1986]
- Trilinear interaction terms for the interaction between a massless spin-s_1 field and two spin-s_2 fields ($s_1 \geq 2s_2$):

$$\mathcal{L}_{int} \sim \phi_{s_1} \times j_{s_2}$$

- Vasiliev equations are formally consistent gauge invariant nonlinear equations [Vasiliev 1990]
- The classification of all cubic vertices (s_1, s_2, s_3) [Metsaev, 2005]

$$s_1 + s_2 + s_3 - 2min\{s_1, s_2, s_3\} \leq N(\partial_x) \leq s_1 + s_2 + s_3.$$

? Higher-spin gauge invariant $\mathcal{N} = 2$ supercurvatures

? $\mathcal{N} = 2$ gauge invariant higher-spin supercurrents

? $\mathcal{N} = 2$ interaction vertices

? $\mathcal{N} = 2$ generalizations of Fradkin-Tseytlin actions

Supersymmetric higher-spin theories

- The component approach to the description of 4D, $\mathcal{N}=1$ supersymmetric free massless higher spin models was initiated in [Courtright 1979]; [Vasiliev 1980].
- The complete off-shell Lagrangian formulation of 4D free higher spin $\mathcal{N}=1$ models (including those on the AdS background) has been given in terms of $\mathcal{N}=1$ superfields in a series of works by S. Kuzenko with collaborators [Kuzenko et al, 1993, 1994].
- $4D, \mathcal{N}=1$ unconstrained higher-spin prepotentials [Gates, Koutrolikos 2013].
- Conserved $\mathcal{N}=1$ higher spin supercurrents and cubic interactions [Buchbinder, Gates, Koutrolikos 2018], [Gates, Koutrolikos 2019].
- An off-shell manifestly $\mathcal{N}=2$ supersymmetric unconstrained formulation of $4D, \mathcal{N}=2$ superextension of the Fronsdal theory for integer spins in the harmonic superspace approach [Buchbinder, Ivanov, Zaigraev 2021].
- Cubic interactions of $\mathcal{N}=2$ higher spins with hypermultiplet [Buchbinder, Ivanov, Zaigraev 2022].
- Superconformal $\mathcal{N}=2$ higher spins and superconformal hypermultiplet higher-spin interactions [Buchbinder, Ivanov, Zaigraev 2024] (E.A. Ivanov talk).

Supersymmetry

Supersymmetry is "unique" extension of Poincaré group [Haag–Łopuszański–Sohnius 1975]:

$$\left\{Q^i_\alpha, \bar{Q}_{\dot\alpha j}\right\} = 2\delta^i_j \sigma^m_{\alpha\dot\alpha} P_m, \quad \{Q^i_\alpha, Q^j_\beta\} = \epsilon_{\alpha\beta} Z^{[ij]}, \quad \{\bar{Q}_{\dot\alpha i}, \bar{Q}_{\dot\beta j}\} = \epsilon_{\dot\alpha\dot\beta} \bar{Z}_{[ij]}, \qquad i = 1, 2 \ldots \mathcal{N}.$$

- In field theories with linearly realized supersymmetry fields are unified in supermultiplets. Supersymmetry mix bosonic and fermionic fields:

$$\delta B \sim \epsilon F, \qquad \delta F \sim \epsilon \partial B.$$

- Form of supersymmetry transformation is model dependent if SUSY realized on physical d.o.f. In interacting theory supersymmetry transformations are non-linear in fields.
- Auxiliary fields are needed for three purposes:
  1. Supersymmetry transformations are model-independent and manifestly linear.
  2. There are equal number of fermionic and bosonic d.o.f. off-shell.
  3. Super-Poincaré algebra is closed off-shell (otherwise, constraints must be imposed during quantization).
- The problem of searching for auxiliary fields is not-trivial and solved only for some $\mathcal{N} = 1, 2, 3$ theories. May be some theories do not admit off-shell formulation, e.g. $\mathcal{N} = 4$ SYM, $\mathcal{N} = 8$ SUGRA.

- The most natural and useful way describe off-shell supersymmetric theories is superspace. Superspace is natural generalization of Minkowski space, which is necessary for construction of manifestly Poincare-invariant theories.
- Minkowski space-time can be realized as coset space:

$$\mathbb{M}^4 := \frac{ISO(1,3)}{SO(1,3)} = \frac{\{L_{nm}, P_n\}}{\{L_{nm}\}} = (x^m).$$

Coset construction can be easily generalized to supersymmetry:

- Real superspace:

$$\mathbb{R}^{4|4\mathcal{N}} := \frac{\mathcal{SP}(1,3|\mathcal{N}) \times SU(\mathcal{N})}{SO(1,3) \times SU(\mathcal{N})} = \frac{\{L_{nm}, su(\mathcal{N}), P_n, Q^i_\alpha, \bar{Q}_{\dot{\alpha} i}\}}{\{L_{nm}, su(\mathcal{N})\}} = \left(x^m, \theta^\alpha_i, \bar{\theta}^{\dot{\alpha} i}\right).$$

- Chiral superspace:

$$\mathbb{C}^{4|2\mathcal{N}} := \frac{\{L_{nm}, su(\mathcal{N}), P_n, Q^i_\alpha, \bar{Q}_{\dot{\alpha} i}\}}{\{L_{nm}, su(\mathcal{N}), \bar{Q}_{\dot{\alpha} i}\}} = (x^m, \theta^\alpha_i).$$

- Superspaces $\mathbb{R}^{4|4}$ and $\mathbb{C}^{4|2}$ are inevitable for description of $\mathcal{N} = 1$ theories: chiral multiplet, vector multiplet, supergravity multiplet, higher-spin off-shell multiplets have formulation in terms of unconstrained superfields.

Harmonic superspace

- Harmonic superspace [Galperin, Ivanov, Kalitsyn, Ogievetsky, Sokatchev 1984] is efficient approach of dealing with supersymmetric theories with 8 real SUSY generators in a manifestly covariant manner.
- In harmonic superspace there are auxiliary coordinates – harmonics $u_i^{\pm}$:

$$\mathbb{HR}^{4+2|8} := \frac{\{P_m, L_{mn}, Q^i_\alpha, su(2)\}}{\{L_{mn}, u(1)\}} = \mathbb{R}^{4|8} \times S^2 = \{x^m, \theta^+_\alpha, \bar{\theta}^+_{\dot{\alpha}}, \theta^-_\alpha, \bar{\theta}^-_{\dot{\alpha}}, u^{\pm i}\}$$

- Superfields in HSS: $Q^{(n)} = Q^{(n)}(x^a, \theta^i_\alpha, u^{\pm i})$ have infinitely many components.
- Using harmonics one can convert $su(2)$ indices to $u(1)$:

$$Q^i_\alpha \quad \to \quad Q^+_\alpha = Q^i_\alpha u^+_i, \quad Q^-_\alpha = Q^i_\alpha u^-_i .$$

- Harmonic superspace have new invariant subspace containing only half of the original Grassmann variables. Analytic superspace:

$$\mathbb{HA}^{4+2|4} := \frac{\{L_{mn}, P_m, Q^{\pm}_\alpha, \bar{Q}^{\pm}_{\dot{\alpha}}, su(2)\}}{\{L_{mn}, Q^+_\alpha, \bar{Q}^+_{\dot{\alpha}}, u(1)\}} = (x^m, \theta^+_\alpha, \bar{\theta}^+_{\dot{\alpha}}, u^{\pm i}) := \zeta.$$

Hypermultiplet in harmonic superspace

- Hypermultiplet is the fundamental $\mathcal{N}=2$ matter multiplet.
- On-shell hypermultiplet contain doublet of complex scalars $f^i(x)$ and a pair of singlet spinors $\psi_\alpha(x), \kappa_\alpha(x)$.
- A finite set of auxiliary fields does not exist for hypermultiplet ("no-go theorems" in extended SUSY).
- **Hypermultiplet** in HSS is described by an unconstrained analytic superfield $q^+(\zeta)$. It contains a doublet of complex scalars f^i and a pair of singlet spinors $\psi_\alpha, \kappa_\alpha$, as well as an infinite set of auxiliary fields which comes from the harmonic S^2 expansions:

$$q^+(\zeta) = f^i u_i^+ + \theta^{+\alpha}\psi_\alpha + \bar{\theta}^+_{\dot{\alpha}}\bar{\kappa}^{\dot{\alpha}} + \text{auxiliary fields}.$$

- The free hypermultiplet action has the form:

$$S_{free} = -\int d\zeta^{(-4)}\tilde{q}^+\mathcal{D}^{++}q^+ = -\frac{1}{2}\int d\zeta^{(-4)}\, q^{+a}\mathcal{D}^{++}q^+_a.$$

- Here we used harmonic derivative:

$$\mathcal{D}^{++} = \underbrace{\partial^{++}}_{u^{+i}\frac{\partial}{\partial u^{-i}}} - 4i\theta^{+\beta}\bar{\theta}^{+\dot{\beta}}\underbrace{\partial_{\beta\dot{\beta}}}_{\frac{1}{2}\sigma^m_{\beta\dot{\beta}}\partial_m} + \theta^{+\beta}\partial^+_\beta + \bar{\theta}^{+\dot{\beta}}\partial^+_{\dot{\beta}} + \left[(\theta^+)^2 - (\bar{\theta}^+)^2\right]\partial_5.$$

- Equation of motion $\mathcal{D}^{++}q^+ = 0$ allow to exclude all auxiliary fields and lead to free Klein-Gordon and Weyl equations.

$\mathcal{N} = 2$ supergravity

- Gauge supergroup of $\mathcal{N} = 2$ Einstein supergravity is supertranslations in harmonic superspace, maintaining analyticity:

$$\delta_\lambda x^{\alpha\dot\alpha} = \lambda^{\alpha\dot\alpha}(\zeta), \quad \delta_\lambda \theta^{+\hat\alpha} = \lambda^{+\hat\alpha}(\zeta), \quad \delta_\lambda \theta^{-\hat\alpha} = \lambda^{-\hat\alpha}(\zeta, \theta^-), \quad \delta_\lambda u_i^\pm = 0, \quad \delta_\lambda x^5 = \lambda^5(\zeta).$$

$$\delta_\lambda z^M = [\hat\Lambda, z^M], \qquad \hat\Lambda := \lambda^M \partial_M.$$

- Harmonic derivative $\mathcal{D}^{++}$ is not covariant under supertranslations:

$$\delta_\lambda \mathcal{D}^{++} = -\mathcal{D}^{++}\lambda^M \partial_M - 4i\lambda^{+\rho}\bar\theta^{+\dot\rho}\partial_{\rho\dot\rho} - 4i\theta^{+\rho}\bar\lambda^{+\dot\rho}\partial_{\rho\dot\rho} + \lambda^{+\hat\rho}\partial^+_{\hat\rho} + 2(\lambda^{\hat +}\theta^{\hat +})\partial_5.$$

- Analytic gauge prepotentials:

$$\mathcal{D}^{++} \quad \rightarrow \quad \mathfrak{D}^{++} = \mathcal{D}^{++} + h^{++\alpha\dot\alpha}\partial_{\alpha\dot\alpha} + h^{++\hat\alpha+}\partial^-_{\hat\alpha} + h^{++\hat\alpha-}\partial^+_{\hat\alpha} + h^{++5}\partial_5,$$

$$\begin{cases} \delta_\lambda h^{++\alpha\dot\alpha} = \mathfrak{D}^{++}\lambda^{\alpha\dot\alpha} + 4i\lambda^{+\alpha}\bar\theta^{+\dot\alpha} + 4i\theta^{+\alpha}\bar\lambda^{+\dot\alpha} - \hat\Lambda h^{++\alpha\dot\alpha}, \\ \delta_\lambda h^{++\hat\alpha+} = \mathfrak{D}^{++}\lambda^{+\hat\alpha} - \hat\Lambda h^{++\hat\alpha+}, \\ \delta_\lambda h^{++\hat\alpha-} = \mathfrak{D}^{++}\lambda^{-\hat\alpha} - \lambda^{+\hat\alpha} - \hat\Lambda h^{++\hat\alpha-}, \\ \delta_\lambda h^{++5} = \mathfrak{D}^{++}\lambda^5 - 2\lambda^{+\hat\alpha}\theta^+_{\hat\alpha} - \hat\Lambda h^{++5}. \end{cases} \quad \Rightarrow \quad \delta_\lambda \mathfrak{D}^{++} = 0.$$

$\mathcal{N} = 2$ supergravity

- Wess-Zumino gauge:

$$h^{++\alpha\dot{\alpha}}_{WZ} = -4i\theta^{+\beta}\bar{\theta}^{+\dot{\beta}}\Phi^{\alpha\dot{\alpha}}_{\beta\dot{\beta}} + 16(\bar{\theta}^{+})^2\theta^{+\beta}\psi^{\alpha\dot{\alpha}i}_{\beta}u^{-}_{i} - 16(\theta^{+})^2\bar{\theta}^{+\dot{\beta}}\bar{\psi}^{\alpha\dot{\alpha}i}_{\dot{\beta}}u^{-}_{i} + (\theta^{+})^4 V^{\alpha\dot{\alpha}(ij)}u^{-}_{i}u^{-}_{j},$$

$$h^{++5}_{WZ} = -4i\theta^{+\beta}\bar{\theta}^{+\dot{\beta}}C_{\beta\dot{\beta}} + 8(\bar{\theta}^{+})^2\theta^{+\beta}\rho^{i}_{\beta}u^{-}_{i} - 8(\theta^{+})^2\bar{\theta}^{+\dot{\beta}}\bar{\rho}^{i}_{\dot{\beta}}u^{-}_{i} + (\theta^{+})^4 S^{(ij)}u^{-}_{i}u^{-}_{j},$$

$$h^{++\alpha+}_{WZ} = (\bar{\theta}^{+})^2\theta^{+}_{\beta}T^{(\alpha\beta)} + (\bar{\theta}^{+})^2\theta^{+\alpha}T + (\theta^{+})^2\bar{\theta}^{+}_{\dot{\beta}}P^{\alpha\dot{\beta}} + (\theta^{+})^4\chi^{\alpha i}u^{-}_{i},$$

$$h^{++\dot{\alpha}+}_{WZ} = (\theta^{+})^2\bar{\theta}^{+}_{\dot{\beta}}\bar{T}^{(\dot{\alpha}\dot{\beta})} + (\theta^{+})^2\bar{\theta}^{+\dot{\alpha}}\bar{T} - (\bar{\theta}^{+})^2\theta^{+}_{\beta}\bar{P}^{\beta\dot{\alpha}} + (\theta^{+})^4\bar{\chi}^{\dot{\alpha}i}u^{-}_{i}.$$

- $\mathcal{N} = 2$ off-shell "minimal" supergravity multiplet [Fradkin, Vasiliev 1979], [de Wit, van Holten 1979]:

$$\text{Physical fields :} \qquad \Phi^{\alpha\dot{\alpha}\beta\dot{\beta}} = \Phi^{(\alpha\beta)(\dot{\alpha}\dot{\beta})} + \epsilon^{\alpha\beta}\epsilon^{\dot{\alpha}\dot{\beta}}\Phi, \psi^{\alpha\dot{\alpha}i}_{\beta}, C_{\beta\dot{\beta}},$$

$$\text{Auxilliary fields :} \qquad V^{\alpha\dot{\alpha}(ij)}, \rho^{i}_{\beta}, S^{(ij)}, T, T^{(\alpha\beta)}, P^{\alpha\dot{\beta}}, \chi^{\alpha i}.$$

- Linearized gauge freedom:

$$\delta_\lambda \underbrace{\Phi^{\alpha\dot{\alpha}}_{\beta\dot{\beta}}}_{graviton} = \partial_{\beta\dot{\beta}}a^{\alpha\dot{\alpha}} - l_{(\beta}^{\ \ \alpha)}\delta^{\dot{\alpha}}_{\dot{\beta}} - \delta^{\alpha}_{\beta}l_{(\dot{\beta}}^{\ \ \dot{\alpha})}, \quad \delta_\lambda \underbrace{\psi^{\alpha\dot{\alpha}i}_{\beta}}_{gravitino} = -\partial^{\dot{\alpha}}_{\beta}\epsilon^{\alpha i}, \quad \delta_\lambda \underbrace{C_{\alpha\dot{\alpha}}}_{graviphoton} = \partial_{\alpha\dot{\alpha}}c.$$

- Totally there are $\mathbf{40_B + 40_F}$ off-shell d.o.f.

Covariant superfields

- Covariant harmonic derivative:

$$\mathfrak{D}^{++} = \mathcal{D}^{++} + \hat{\mathcal{H}}^{++}, \qquad \hat{\mathcal{H}}^{++} := h^{++M}\partial_M.$$

$$\delta_\epsilon \mathfrak{D}^{++} = 0, \qquad \delta_\epsilon \mathcal{D}^{++} = 0, \qquad \delta_\epsilon \hat{\mathcal{H}}^{++} = 0.$$

- Spinor derivatives, covariant with respect to $\mathcal{N} = 2$ supersymmetry $\delta_\epsilon \mathcal{D}^{\pm}_{\hat{\alpha}} = 0$ are:

$$\begin{aligned}
&\mathcal{D}^{+}_{\hat{\alpha}} := \partial^{+}_{\hat{\alpha}}, \\
&\mathcal{D}^{-}_{\alpha} := -\partial^{-}_{\alpha} + 4i\bar{\theta}^{-\dot{\alpha}}\partial_{\alpha\dot{\alpha}} - 2i\theta^{-}_{\alpha}\partial_5, \\
&\bar{\mathcal{D}}^{-}_{\dot{\alpha}} := -\partial^{-}_{\dot{\alpha}} - 4i\theta^{-\alpha}\partial_{\alpha\dot{\alpha}} + 2i\bar{\theta}^{-}_{\dot{\alpha}}\partial_5.
\end{aligned}$$

- Covariant superfields:

$$\hat{\mathcal{H}}^{++} = G^{++\alpha\dot{\alpha}}\partial_{\alpha\dot{\alpha}} + G^{++\alpha+}\mathcal{D}^{-}_{\alpha} + G^{++\dot{\alpha}+}\bar{\mathcal{D}}^{-}_{\dot{\alpha}} + G^{++5}\partial_5, \qquad \delta_\epsilon G^{++M} = 0.$$

- Gauge transformations:

$$\delta_\lambda \mathcal{H}^{++} = [\mathcal{D}^{++}, \hat{\Lambda}], \qquad \hat{\Lambda} := \lambda^M \partial_M = \Lambda^M \mathcal{D}_M$$

- Linearized gauge transformations of covariant superfields have form:

$$\delta_\lambda G^{++\alpha\dot{\alpha}} = \mathcal{D}^{++}\Lambda^{\alpha\dot{\alpha}}, \quad \delta_\lambda G^{++5} = \mathcal{D}^{++}\Lambda^{5}, \quad \delta_\lambda G^{++\hat{\alpha}+} = \mathcal{D}^{++}\Lambda^{+\hat{\alpha}}.$$

Zero curvature equation

- Flat harmonic derivatives:

$$\mathcal{D}^{++} = \partial^{++} - 4i\theta^{+\rho}\bar{\theta}^{+\dot{\rho}}\partial_{\rho\dot{\rho}} + \theta^{+\hat{\rho}}\partial^{+}_{\hat{\rho}} + (\theta^{\hat{+}})^2\partial_5,$$
$$\mathcal{D}^{--} = \partial^{--} - 4i\theta^{-\rho}\bar{\theta}^{-\dot{\rho}}\partial_{\rho\dot{\rho}} + \theta^{-\hat{\rho}}\partial^{-}_{\hat{\rho}} + (\theta^{\hat{-}})^2\partial_5,$$
$$\mathcal{D}^{0} = \partial^{0} + \theta^{+\hat{\rho}}\partial^{-}_{\hat{\rho}} - \theta^{-\hat{\rho}}\partial^{+}_{\hat{\rho}}.$$

$$[\mathcal{D}^{++}, \mathcal{D}^{--}] = \mathcal{D}^{0}.$$

- Curved harmonic derivatives:

$$\mathcal{D}^{++} \to \mathfrak{D}^{++} = \mathcal{D}^{++} + \hat{\mathcal{H}}^{++}, \qquad \mathcal{D}^{--} \to \mathfrak{D}^{--} := \mathcal{D}^{--} + \hat{\mathcal{H}}^{--}$$
$$\hat{\mathcal{H}}^{--} := G^{--\alpha\dot{\alpha}}\partial_{\alpha\dot{\alpha}} + G^{--\hat{\alpha}+}\mathcal{D}^{-}_{\hat{\alpha}} + G^{--\hat{\alpha}-}\mathcal{D}^{+}_{\hat{\alpha}} + G^{--5}\partial_5, \qquad \delta_\epsilon G^{--M} = 0.$$
$$[\mathfrak{D}^{++}, \mathfrak{D}^{--}] = \mathcal{D}^{0} \qquad \Rightarrow \qquad [\mathcal{D}^{++}, \hat{\mathcal{H}}^{--}] = [\mathcal{D}^{--}, \hat{\mathcal{H}}^{++}].$$

- Gauge freedom:

$$\delta_\lambda \hat{\mathcal{H}}^{++} = [\mathcal{D}^{++}, \hat{\Lambda}], \qquad \delta_\lambda \hat{\mathcal{H}}^{--} = [\mathcal{D}^{--}, \hat{\Lambda}].$$
$$\delta_\lambda G^{--\alpha\dot{\alpha}} = \mathcal{D}^{--}\Lambda^{\alpha\dot{\alpha}}, \quad \delta_\lambda G^{--5} = D^{--}\Lambda^{5},$$
$$\delta_\lambda G^{--\hat{\alpha}+} = \mathcal{D}^{--}\Lambda^{+\hat{\alpha}} + \Lambda^{-\hat{\alpha}}, \quad \delta_\lambda G^{--\hat{\alpha}-} = \mathcal{D}^{--}\Lambda^{-\hat{\alpha}}.$$

- Linearized $\mathcal{N}=2$ supergravity action:

$$S_{lin} = -\int d^4x d^8\theta du \left[G^{++\alpha\dot{\alpha}} G^{--}_{\alpha\dot{\alpha}} + 4 G^{++5} G^{--5} \right].$$

- Components in gravity sector:

$$G^{++\alpha\dot{\alpha}}_{(\Phi)} = -4i\theta^{+\beta}\bar{\theta}^{+\dot{\beta}}\Phi^{\alpha\dot{\alpha}}_{\beta\dot{\beta}} + 4(\theta^+)^2\bar{\theta}^{+\dot{\beta}}\bar{\theta}^{-\dot{\alpha}}B^{\alpha}_{\dot{\beta}} - 4(\bar{\theta}^+)^2\theta^{+\beta}\theta^{-\alpha}B^{\dot{\alpha}}_{\beta},$$

$$G^{--\alpha\dot{\alpha}}_{(\Phi)} = \dots - 8i(\theta^-)^4\theta^+_\rho\bar{\theta}^+_{\dot{\rho}} \underbrace{\left(\mathcal{R}^{(\alpha\rho)(\dot{\alpha}\dot{\rho})} - \frac{1}{8}\epsilon^{\alpha\rho}\epsilon^{\dot{\alpha}\dot{\rho}}\mathcal{R}\right)}_{\text{Einstein tensor}}.$$

$$G^{++5}_{(\Phi)} = 2(\theta^+)^2\bar{\theta}^{+\dot{\beta}}\theta^{-\beta}B_{\beta\dot{\beta}} + 2(\bar{\theta}^+)^2\theta^{+\beta}\bar{\theta}^{-\dot{\beta}}B_{\beta\dot{\beta}},$$

$$G^{--5}_{(\Phi)} = \dots - \frac{i}{2}(\theta^-)^4(\theta^+)^2\mathcal{R} + \frac{i}{2}(\theta^-)^4(\bar{\theta}^+)^2\mathcal{R}.$$

- We use notation:

$$B^{\alpha\dot{\alpha}} = -2\left(\frac{1}{2}\partial_{\beta\dot{\beta}}\Phi^{(\alpha\beta)(\dot{\alpha}\dot{\beta})} - \frac{3}{2}\partial^{\alpha\dot{\alpha}}\Phi\right).$$

- Ricci curvature:

$$\mathcal{R}_{(\alpha\beta)(\dot{\alpha}\dot{\beta})} := 2\partial_{(\alpha(\dot{\alpha}}\partial^{\rho\dot{\rho}}\Phi_{(\beta)\rho)(\dot{\beta})\dot{\rho})} - \frac{1}{2}\Box\Phi_{(\alpha\beta)(\dot{\alpha}\dot{\beta})} - 2\partial_{(\alpha(\dot{\alpha}}\partial_{\beta)\dot{\beta})}\Phi.$$

- Scalar curvature:

$$\mathcal{R} := 4\partial^{\alpha\dot{\alpha}}\partial^{\beta\dot{\beta}}\Phi_{(\alpha\beta)(\dot{\alpha}\dot{\beta})} - 6\Box\Phi.$$

$\mathcal{N}=2$ supergravity equations of motion

- Superfield equation of motion:

$$(\bar{\mathcal{D}}^+)^2\mathcal{D}^{+\alpha}G^{--}_{\alpha\dot{\alpha}} - 4(\mathcal{D}^+)^2\bar{\mathcal{D}}^+_{\dot{\alpha}}G^{--5} \approx 0.$$

- In component superfield EOM lead to algebraic equations for auxiliary fields:

$$T_{(\alpha\beta)} = -4i\mathcal{F}_{(\alpha\beta)}, \quad \bar{T}_{(\dot{\alpha}\dot{\beta})} = 4i\bar{\mathcal{F}}_{(\dot{\alpha}\dot{\beta})},$$
$$T = \bar{T} = 0, \quad P_{\alpha\dot{\alpha}} = \bar{P}_{\alpha\dot{\alpha}} = 0, \quad V^{\alpha\dot{\alpha}(ij)} = 0, \quad S^{(ij)} = 0,$$
$$\rho^i_\alpha = 0, \quad \chi^i_\alpha = 0, \qquad \bar{\rho}^i_{\dot{\alpha}} = 0, \quad \bar{\chi}^i_{\dot{\alpha}} = 0.$$

- In special gauge, EOM of physical fields are:

$$\Box\Phi^{(\alpha\beta)(\dot{\alpha}\dot{\beta})} = 0, \quad \Phi = 0, \quad \Box C_{\alpha\dot{\alpha}} = 0,$$

$$\partial^\alpha_{\dot{\alpha}}\psi^i_{(\alpha\beta)\dot{\beta}} = 0, \qquad \psi^{\dot{\alpha}i} = 0.$$

- This imply for prepotentials (in special gauge):

$$\Box h^{\pm\pm M} = 0.$$

$\mathcal{N} = 2$ supercurvatures

Gauge-invariant $\mathcal{N} = 2$ supercurvatures have a surprisingly simple structure:

- Einstein supertensor (analytic):

$$\mathcal{F}^{++\alpha\dot{\alpha}} = (\mathcal{D}^+)^4 G^{--\alpha\dot{\alpha}} = -8i\theta^+_\rho \bar{\theta}^+_{\dot{\rho}} \left(\mathcal{R}^{(\alpha\rho)(\dot{\alpha}\dot{\rho})} - \frac{1}{8}\epsilon^{\alpha\rho}\epsilon^{\dot{\alpha}\dot{\rho}}\mathcal{R} \right) + \dots,$$

- Scalar supercurvature (analytic):

$$\mathcal{F}^{++5} = (\mathcal{D}^+)^4 G^{--5} = \frac{i}{2}(\theta^+)^2\mathcal{R} - \frac{i}{2}(\bar{\theta}^+)\mathcal{R} + \dots.,$$

- $\mathcal{N} = 2$ supercurvatures are *supersymmetry* and *gauge invariant*.
- On superfield EOM of linearized $\mathcal{N} = 2$ supergravity:

$$(\bar{\mathcal{D}}^+)^2\mathcal{D}^{+\alpha} G^{--}_{\alpha\dot{\alpha}} - 4(\mathcal{D}^+)^2\bar{\mathcal{D}}^+_{\dot{\alpha}} G^{--5} \approx 0 \qquad \Rightarrow \qquad \mathcal{F}^{++\alpha\dot{\alpha}} \approx 0, \qquad \mathcal{F}^{++5} \approx 0.$$

$\mathcal{N} = 2$ Weyl tensor

- Gauge invariant Weyl supercurvature:

$$\mathcal{W}_{(\alpha\beta)} = (\bar{\mathcal{D}}^+)^2 \left(\mathcal{D}^+_{(\alpha} G^{---}_{\beta)} + \mathcal{D}^-_{(\alpha} G^{--+}_{\beta)} - \partial^{\dot\rho}_{(\alpha} G^{--}_{\beta)\dot\rho} \right) = 32\theta^{-(\gamma}\theta^{+\delta)} \mathcal{R}_{(\alpha\beta\gamma\delta)} + \dots .$$

- Linearized (self-dual) Weyl tensor:

$$\mathcal{R}_{(\alpha\beta\gamma\delta)} := \partial^{\dot\alpha}_{(\alpha} \partial^{\dot\beta}_{\beta} \Phi_{\gamma\delta)(\dot\alpha\dot\beta)}.$$

- Weyl supercurvature is *chiral* and *harmonic independent*:

$$\bar{\mathcal{D}}^{\pm} \mathcal{W}_{(\alpha\beta)} = 0, \qquad \mathcal{D}^{\pm\pm} \mathcal{W}_{(\alpha\beta)} = 0.$$

- On mass-shell:

$$\mathcal{W}_{(\alpha\beta)} \not\approx 0.$$

- Weyl supercurvature satisfy conservation equation on the equations of motion:

$$(\mathcal{D}^+)^2 \mathcal{W}_{(\alpha\beta)} \approx 0.$$

Linearized R^2 invariants

- R^2 invariant have dimension:

$$[d^4x] = -4, \quad [R] = 2 \qquad \Rightarrow \qquad [d^4x\, R^2] = 0.$$

- Superspace integration measure:

$$\underbrace{[d^4xd^8\theta du]}_{\text{Full harmonic superspace}} = 0, \qquad \underbrace{[d^4xd^4\theta^+du]}_{\text{Analytic superspace}} = -2, \qquad \underbrace{[d^4xd^4\theta]}_{\mathcal{N}=2 \text{ chiral superspace}} = -2.$$

- Dimensions of supercurvatures:

$$[\mathcal{F}^{++\alpha\dot{\alpha}}] = 1, \qquad [\mathcal{F}^{++5}] = 1, \qquad [\mathcal{W}_{(\alpha\beta)}] = 1.$$

- R^2 invariants are given by integrals of supercurvatures over analytic/chiral superspaces:

$$I_1 := \int d\zeta^{(-4)} \mathcal{F}^{++\alpha\dot{\alpha}} \mathcal{F}^{++}_{\alpha\dot{\alpha}} \sim \int d^4x \left(\mathcal{R}^{(\alpha\beta)(\dot{\alpha}\dot{\beta})} \mathcal{R}_{(\alpha\beta)(\dot{\alpha}\dot{\beta})} - \frac{1}{16}\mathcal{R}^2 \right),$$

$$I_2 := \int d\zeta^{(-4)} \mathcal{F}^{++5} \mathcal{F}^{++5} \sim \int d^4x\, \mathcal{R}^2,$$

$$I_3 = \int d^4xd^4\theta\, \mathcal{W}^{(\alpha\beta)} \mathcal{W}_{(\alpha\beta)} \sim \int d^4x\, \mathcal{R}^{(\alpha\beta\gamma\delta)} \mathcal{R}_{(\alpha\beta\gamma\delta)}.$$

Other higher-derivative $\mathcal{N}=2$ supergravity invariants

- R^4 *invariant*

$$\int d^4xd^8\theta du\ \left(\mathcal{D}^{--}\mathcal{F}^{++5}\right)^4 \sim \int d^4xR^4,$$

- R^{4+n} *invariant*

$$\int d^4xd^8\theta du\ \left(\mathcal{D}^{--}\mathcal{F}^{++5}\right)^4\left[(\mathcal{D}^{-})^2\mathcal{F}^{++5}\right]^n \sim \int d^4xR^{4+n},$$

- G^4 *invariant*

$$\int d^4xd^8\theta du\ \left(\mathcal{D}^{--}\mathcal{F}^{++\alpha_1\dot{\alpha}_1}\right)\left(\mathcal{D}^{--}\mathcal{F}^{++\alpha_2\dot{\alpha}_2}\right)\left(\mathcal{D}^{--}\mathcal{F}^{++}_{\alpha_1\dot{\alpha}_2}\right)\left(\mathcal{D}^{--}\mathcal{F}^{++}_{\alpha_2\dot{\alpha}_1}\right)$$
$$\sim \int d^4x\ \mathcal{G}^{\alpha_1\beta_1\dot{\alpha}_1\dot{\beta}_1}\mathcal{G}^{\alpha_2\beta_2\dot{\alpha}_2\dot{\beta}_2}\mathcal{G}_{\beta_2\alpha_1\dot{\beta}_1\dot{\alpha}_2}\mathcal{G}_{\beta_1\alpha_2\dot{\beta}_2\dot{\alpha}_1}.$$

- *Bel-Robinson tensor squared*

$$\int d^4xd^8\theta du\ \mathcal{W}^{(\alpha\beta)}\mathcal{W}_{(\alpha\beta)}\bar{\mathcal{W}}^{(\dot{\alpha}\dot{\beta})}\bar{\mathcal{W}}_{(\dot{\alpha}\dot{\beta})} \sim \int d^4x\ \mathcal{R}^{(\alpha\beta\gamma\delta)}\mathcal{R}_{(\alpha\beta\gamma\delta)}\bar{\mathcal{R}}^{(\dot{\alpha}\dot{\beta}\dot{\gamma}\dot{\delta})}\bar{\mathcal{R}}_{(\dot{\alpha}\dot{\beta}\dot{\gamma}\dot{\delta})}.$$

- R^3 *invariants and two-loop finiteness of supergravity*

$$\mathcal{R}_{\alpha\beta}{}^{\gamma\delta}\mathcal{R}_{\gamma\delta}{}^{\rho\kappa}\mathcal{R}_{\rho\kappa}{}^{\alpha\beta}$$ Impossible to supersymmetrize!

Conserved gauge-invariant supercurrents: conservation equations

- Analytic prepotentials of $\mathcal{N}=2$ higher spins:

$$h^{++\alpha(s-1)\dot\alpha(s-1)}, \quad h^{++\alpha(s-1)\dot\alpha(s-2)}, \quad h^{++\alpha(s-2)\dot\alpha(s-1)}, \quad h^{++\alpha(s-2)\dot\alpha(s-2)}.$$

- Supersymmetry invariant analytic differential operator:

$$\hat{\mathcal{H}}^{++}_{(s)} = h^{++\alpha(s-2)\dot\alpha(s-2)M}\partial_M \partial^{s-2}_{\alpha(s-2)\dot\alpha(s-2)}, \qquad \partial_M := (\partial_{\alpha\dot\alpha}, \partial^-_{\alpha}, \partial^-_{\dot\alpha}, \partial_5),$$

$$\delta_\lambda \hat{\mathcal{H}}^{++}_{(s)} = [\mathcal{D}^{++}, \hat\Lambda_{(s)}], \qquad \hat\Lambda_{(s)} := \lambda^{\alpha(s-2)\dot\alpha(s-2)M}\partial_M \partial^{s-2}_{\alpha(s-2)\dot\alpha(s-2)}.$$

- Alternative representation of prepotentials is given by Mezincescu non-analytic prepotentials:

$$\hat{\mathcal{H}}^{++}_{(s)} = (\mathcal{D}^+)^4 \left[\Psi^{-\alpha(s-2)\dot\alpha(s-2)\hat\beta}\mathcal{D}^-_{\hat\beta}\right]\partial^{s-2}_{\alpha(s-2)\dot\alpha(s-2)}.$$

- Mezincescu prepotentials are defined up to gauge freedom and pre-gauge freedom:

$$\delta_{\lambda,b}\Psi^-_{\alpha(s-1)\dot\alpha(s-2)} = \mathcal{D}^{++}K^{(-3)}_{\alpha(s-1)\dot\alpha(s-2)} + \mathcal{D}^+_{(\alpha}B^{--}_{\alpha(s-2))\dot\alpha(s-2)} + \mathcal{D}^{+\beta}B^{--}_{(\alpha(s-1)\beta)\dot\alpha(s-2)} + \bar{\mathcal{D}}^{+\dot\beta}B^{--}_{\alpha(s-1)(\dot\alpha(s-2)\dot\beta)}.$$

- This motivate to consider cubic couplings:

$$S_{int} = \int d^4x d^8\theta du\, \Psi^{-\alpha(s-1)\dot\alpha(s-2)}J^+_{\alpha(s-1)\dot\alpha(s-2)} + c.c. \quad \Rightarrow \quad \begin{cases} \mathcal{D}^{++}J^+_{\alpha(s-1)\dot\alpha(s-2)} \approx 0, \\ D^+_\beta J^+_{\alpha(s-1)\dot\alpha(s-2)} \approx 0, \\ \bar{\mathcal{D}}^+_{(\dot\beta}J^+_{\alpha(s-1)\dot\alpha(s-2))} + c.c. \approx 0. \end{cases}$$

Supercurrents from supercurvatures

Spin 4 conserved gauge-invariant supercurrent:

$$J^+_{(\alpha_1\alpha_2\alpha_3)(\dot\alpha_1\dot\alpha_2)} = \mathcal{D}^+_{(\alpha_1}\mathcal{W}_{\alpha_2\alpha_3)}\bar{\mathcal{W}}_{(\dot\alpha_1\dot\alpha_2)}.$$

- Harmonic equation:

$$\mathcal{D}^{++}J^+_{(\alpha_1\alpha_2\alpha_3)(\dot\alpha_1\dot\alpha_2)} = 0 \quad \Leftarrow \quad \mathcal{D}^{++}\mathcal{W}_{(\alpha\beta)} = 0.$$

- “Chiral” equation:

$$\mathcal{D}^+_\beta J^+_{(\alpha_1\alpha_2\alpha_3)(\dot\alpha_1\dot\alpha_2)} \approx 0 \quad \Leftarrow \quad \mathcal{D}^+_\beta\bar{\mathcal{W}}_{(\dot\alpha\dot\beta)} = 0, \quad (\mathcal{D}^+)^2\mathcal{W}_{(\alpha\beta)} \approx 0.$$

- “Symmetrized” condition:

$$\bar{\mathcal{D}}^+_{(\dot\beta}J^+_{(\alpha_1\alpha_2\alpha_3)(\dot\alpha_1\dot\alpha_2))} + c.c. = 0.$$

Spin 2s conserved gauge-invariant supercurrent:

$$J^+_{\alpha(s-1)\dot\alpha(s-2)} = \mathcal{D}^+_{(\alpha}\partial^{s-2}_{\alpha\dot\alpha}\mathcal{W}_{\alpha(2))}\partial^{s-2}_{\alpha\dot\alpha}\bar{\mathcal{W}}_{\dot\alpha(2))}.$$

Outlook

- $\mathcal{N}=2$ *higher spin supercurvatures*
- *Non-linear* $\mathcal{N}=2$ *supergravity invariants*
- *Differential geometry of curved* $\mathcal{N}=2$ *harmonic superspace*
- *Superfield* $\mathcal{N}=2$ *Gauss-Bonnet theorem in harmonic superspace*
- $\mathcal{N}=2$ *conformal supergravity invariants*
- $6D$, $\mathcal{N}=(1,0)$ *supergravity invariants*
- $\mathcal{N}=2$ *AdS supergravity*
- $\mathcal{N}=2$ *massive supergravity*
- $\mathcal{N}=3$ *supergravity in harmonic superspace*
- $\mathcal{N}\geq 4$*??*
- $\mathcal{N}=2$ *Ogievetsky-Socatchev multiplet*

Thank you for your attention!