Local quench within the Keldysh technique

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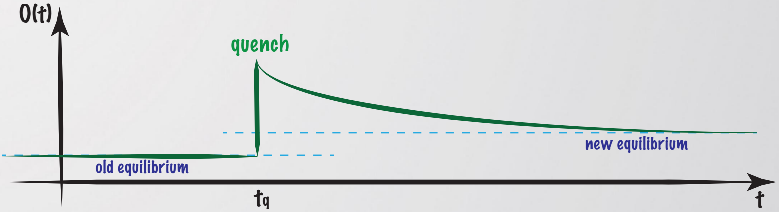
Plan of the talk

- Motivation
- Keldysh technique: semiclassical approximation
- Quench: general formalism and examples
- Connection with CFT calculations
- Conclusions

Motivation: what is quench

Quench is a controllable way to create a nonequlibrium state from known equilibrium one.



- **GLOBAL QUENCH** - the process of sudden changes of the parameters of the entire system.
- **LOCAL QUENCH** - the system is perturbed in the vicinity of some point $\mathbf{x}_q$ by the action of the operator $\hat{Q}(\mathbf{x}_q)$

Motivation: already known results (CFT)

- State after local quench

$$|\psi(t_q+0)\rangle = \mathcal{N} e^{-\epsilon \hat{H}} \hat{Q}(x_q)|0\rangle.$$

- Average value of observable

$$\langle \hat{O} \rangle_t = \langle \psi(t)|\hat{O}|\psi(t)\rangle$$

- Using analytical continuation from the euclidean time ($\hat{O}(\tau) = e^{\tau \hat{H}} \hat{O} e^{-\tau \hat{H}}$)

$$\langle \hat{O} \rangle_t = \left. \frac{\langle 0|\hat{Q}^\dagger(\epsilon, x_q)\hat{O}(\tau)\hat{Q}(-\epsilon, x_q)|0\rangle}{\langle 0|\hat{Q}^\dagger(\epsilon, x_q)\hat{Q}(-\epsilon, x_q)|0\rangle} \right|_{\tau \to it}$$

- If $\hat{Q}$ is a primary operator with dimensions $(h, \bar{h})$ and $\hat{O}$ is an energy density
  P. Caputa, M. Nozaki, T. Takayanagi, Prog. Theor. Exp. Phys. 2014, 093B06 (2014)

$$\delta\varepsilon(\tau, x) = \frac{2h\epsilon^2}{\pi(x_q - x - i\epsilon - i\tau)^2(x_q - x + i\epsilon - i\tau)^2} + \frac{2\bar{h}\epsilon^2}{\pi(x_q - x + i\epsilon + i\tau)^2(x_q - x - i\epsilon + i\tau)^2}.$$

Motivation: problem under consideration

- Consider a local perturbation of the system at space point x_q at time t_q (a local quench) with the operator:

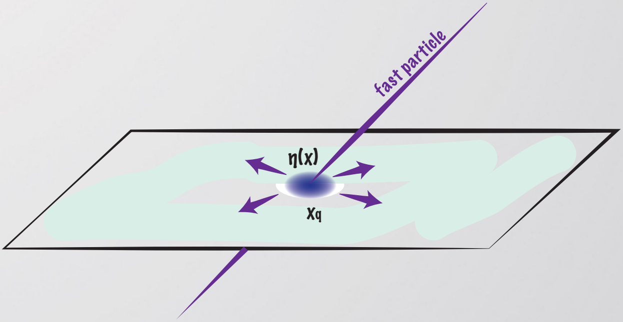
$$\hat{Q}(\mathrm{x_q}) = e^{-i\frac{\alpha}{\hbar}V(\hat{\varphi}_s(\mathrm{x_q}))},$$
$$\hat{\rho}(t_0) \to \hat{\rho}_Q(t_q, \mathrm{x_q}) = \hat{Q}(\mathrm{x_q})\hat{\rho}(t_q)\hat{Q}^\dagger(\mathrm{x_q})$$

- $V(\hat{\varphi}(x_q))$ - some potential (for this talk $\hat{\varphi}^n(x_q)$).
- Field operator $\hat{\varphi}_s(\mathrm{x_q}) = \int dx\eta(\mathrm{x}-\mathrm{x_q})\hat{\varphi}(\mathrm{x})$ is "smeared"in the vicinity of point x_q in order to deal with the problem of products of field operators in coinciding points.
- "Smearing"function $\eta(x - x_q)$ is a smooth function that is non-zero only in a small vicinity of the point x_q.
- Dimensional parameter α describes the magnitude of the perturbation.

Motivation: problem under consideration

Local quench can be described as the additional term of the Hamiltonian $\delta\hat{H}(t) = \alpha\delta(t - t_q)V(\hat{\varphi}_s(\mathrm{x_q}))$.

Quench with $\alpha V(\hat{\varphi}_s(\mathrm{x_q})) = g\hat{\varphi}_s^4(\mathrm{x_q})$ corresponds to the instantaneous appearance of interaction in the system at point $\mathrm{x_q}$.



Keldysh technique: semiclassical approximation

General form of an average of observable O at time t in Keldysh technique:

$$\langle O[\hat{\varphi}(\mathrm{x})]\rangle_t = \mathrm{tr}(\hat{O}(\hat{\varphi})\rho(t)) =$$
$$\int \mathfrak{D}\Pi(\mathrm{x})\mathfrak{D}\Phi(\mathrm{x})\ \mathcal{W}[\Phi(\mathrm{x}),\Pi(\mathrm{x})] \int_{i.c.} \mathcal{D}\varphi_{cl}(t,\mathrm{x}) \int \mathcal{D}\varphi_q(t,\mathrm{x}) O[\varphi_{cl}(t,\mathrm{x})] e^{\frac{i}{\hbar}S_K[\varphi_{cl},\varphi_q]}.$$

here "classical" φ_{cl} and "quantum" φ_q fields:

$$\varphi_{cl}(x) = \frac{1}{2}\left(\varphi_F(x) + \varphi_B(x)\right), \quad \hbar\varphi_q(x) = \varphi_F(x) - \varphi_B(x).$$

An integral with $i.c.$ means integration with initial conditions $\varphi_{cl}(t_0,\mathrm{x}) = \Phi(\mathrm{x})$, $\partial_t\varphi_{cl}(t_0,\mathrm{x}) = \Pi(\mathrm{x})$.

Keldysh technique: semiclassical approximation

Wigner functional is expressed through the initial density matrix of the system; thereby, it defines the properties of this system at the initial time t_0:

$$W[\Phi(x),\Pi(x)] = \int \mathfrak{D}\,\beta(x) e^{i\int d^{d-1}x\beta(x)\Pi(x)} \langle \Phi(x) + \frac{\hbar}{2}\beta(x)|\hat{\rho}(t_0)|\Phi(x) - \frac{\hbar}{2}\beta(x)\rangle.$$

For the scalar theory:

$$S = \frac{1}{2}\int d^d x \left(\partial_\mu\varphi(x)\partial^\mu\varphi(x) - m^2\varphi^2(x) - \frac{g}{2}\varphi^4(x)\right).$$

Keldysh action is:

$$S_K[\varphi_{cl},\varphi_q] = -\hbar\int_{t_0}^{\infty} dt \int d^{d-1}x \left(\varphi_q A[\varphi_{cl}] + \frac{g\hbar^2}{4}\varphi_{cl}\varphi_q^3\right),$$

Here $A[\varphi_{cl}] = (\partial_\mu\partial^\mu + m^2)\varphi_{cl} + g\varphi_{cl}^3$ is EoM. It selects fields on the classical trajectories.

Keldysh technique: semiclassical approximation

Semiclassical expansion:

$$e^{-i\frac{g\hbar^2}{4}\int\limits_{t_0}^{\infty} dt \int d^{d-1}x\, \varphi_{cl}\varphi_q^3} = \underbrace{1}_{LO} - \underbrace{i\frac{g\hbar^2}{4}\int\limits_{t_0}^{\infty} dt \int d^{d-1}x\ \varphi_{cl}\varphi_q^3}_{NLO} + \cdots$$

LO $\rightarrow$ Classical Statistical Approximation, Classical method. After integration over fields φ_q и φ_c:

$$\langle O[\hat{\varphi}(\mathrm{x})]\rangle_t = \int \mathfrak{D}\Phi(\mathrm{x})\mathfrak{D}\Pi(\mathrm{x})W[\Phi(\mathrm{x}),\Pi(\mathrm{x})]O[\phi_c(t,\mathrm{x})],$$

where ϕ_c is the solution of the classical equation of motion: $(\partial_\mu\partial^\mu + m^2)\phi_c + g\phi_c^3 = 0$ with the initial values: $\phi_c(t_0,\mathrm{x}) = \Phi(\mathrm{x})$, $\quad \partial_t\phi_c(t_0,\mathrm{x}) = \Pi(\mathrm{x})$.

The method: Find classical trajectory and average over all possible initial conditions with the weight given by the Wigner functional.

Keldysh technique: semiclassical approximation

Introduce notation for averaging over initial conditions as

$$\int \mathfrak{D}\Phi(x)\mathfrak{D}\Pi(x)W[\Phi(x),\Pi(x)](\dots) \equiv \langle \dots \rangle_{i.c.},$$

so the average for the Classical Approximation can be rewritten as:

$$\langle O[\hat{\varphi}(x)]\rangle_t = \langle O[\phi_c(t,x)]\rangle_{i.c.}. \tag{1}$$

The semiclassical expansion in the Keldysh technique is constructed using the parameter $\hbar^2 g$:

$$S_K[\varphi_{cl},\varphi_q] = -\hbar \int_{t_0}^{\infty} dt \int d^{d-1}x \left(\varphi_q A[\varphi_{cl}] + \frac{g\hbar^2}{4}\varphi_{cl}\varphi_q^3\right),$$

therefore for a noninteracting system g=0,

the classical approximation gives an exact answer!

Quench

Density matrix after quench: $\hat{\rho}(t_0) \to \hat{\rho}_Q(t_q, \mathrm{x_q}) = \hat{Q}(\mathrm{x_q})\hat{\rho}(t_q)\hat{Q}^\dagger(\mathrm{x_q})$
Then the Wigner functional after local quench:

$$W_Q[\Phi(\mathrm{x}), \Pi(\mathrm{x})] = \int \mathfrak{D}\,\beta(\mathrm{x}) e^{i\int d^{d-\mathbf{1}}\mathrm{x}\,\beta(\mathrm{x})\Pi(\mathrm{x})} \langle \Phi(\mathrm{x}) + \frac{\hbar}{2}\beta(\mathrm{x}) | \hat{Q}(\mathrm{x_q})\hat{\rho}(t_0)\hat{Q}^\dagger(\mathrm{x_q}) | \Phi(\mathrm{x}) - \frac{\hbar}{2}\beta(\mathrm{x})\rangle.$$

Note, that

$$\beta(y) e^{i\int d^{d-\mathbf{1}}\mathrm{x}\,\beta(\mathrm{x})\Pi(x)} = -i\frac{\delta}{\delta\Pi(y)} e^{i\int d^{d-\mathbf{1}}\mathrm{x}\,\beta(\mathrm{x})\Pi(\mathrm{x})}$$

So,

$$W_Q[\Phi(\mathrm{x}), \Pi(\mathrm{x})] = Q\Big(\Phi_s, \frac{\delta}{\delta\Pi_s}\Big) W[\Phi(\mathrm{x}), \Pi(\mathrm{x})],$$

where

$$\mathrm{Q}\Big(\Phi_\mathrm{s}, \frac{\delta}{\delta\Pi_\mathrm{s}}\Big) = e^{-i\frac{\alpha}{\hbar}\left(V\left(\Phi_s - i\frac{\hbar}{\mathbf{2}}\frac{\delta}{\delta\Pi_s}\right) - V\left(\Phi_s + i\frac{\hbar}{\mathbf{2}}\frac{\delta}{\delta\Pi_s}\right)\right)},$$

$$\Phi_s = \int d^{d-\mathbf{1}}\mathrm{x}\,\eta(\mathrm{x} - \mathrm{x_q})\Phi(\mathrm{x}),$$

$$\frac{\delta}{\delta\Pi_s} = \int d^{d-\mathbf{1}}\mathrm{x}\,\eta(\mathrm{x} - \mathrm{x_q})\frac{\delta}{\delta\Pi(\mathrm{x})}.$$

Quench

Then, after functional integration by parts

$$\langle \hat{O} \rangle_t^Q = \int \mathfrak{D}\Phi(\mathrm{x})\mathfrak{D}\Pi(\mathrm{x}) W[\Phi(\mathrm{x}), \Pi(\mathrm{x})]\, \mathrm{Q}\left(\Phi_{\mathrm{s}}, -\frac{\delta}{\delta \Pi_{\mathrm{s}}}\right) O[\phi_c(t, \Phi(\mathrm{x}), \Pi(\mathrm{x}))].$$

In order to find the average of the operator after quench, it is necessary to perform integration over the initial conditions with the original Wigner functional, but for a modified observable.

Quench: example

- Quench: Local sudden change of mass

$$\hat{Q}(x_q) = e^{-i\frac{\alpha}{\hbar}\hat{\varphi}_s^2(x_q)}$$

$$\mathrm{Q}\left(\Phi_s, -\frac{\delta}{\delta\Pi_s}\right) = e^{2\alpha\Phi_s\cdot\frac{\delta}{\delta\Pi_s}}$$

- Observable: Energy density

$$\varepsilon(t,x) = \frac{1}{2}(\partial_t\varphi)^2 + \frac{1}{2}(\partial_x\varphi)^2 + \frac{1}{2}m^2\varphi^2.$$

- Solution of EoM $\Phi(x) = \phi_c(0,x), \Pi(x) = \partial_t\phi_c(0,x)$:

$$\phi_c(t,x) = -\int dy\Big(\partial_t G_R(t,x-y)\Phi(y) + G_R(t,x-y)\Pi(y)\Big).$$

Retarded Green function is defined from the retarded solution of equation:

$$(\partial_t^2 - \partial_x^2 + m^2)G_R(t,x-x') = -\delta(t)\delta(x-x'),$$

$$G_R(t,x-x') = -\theta(t)\int\frac{dp}{2\pi}\frac{\sin(\omega_p t)}{\omega_p}e^{-ip(x-x')}, \quad \omega_p = \sqrt{p^2+m^2}.$$

Quench: example

- Averaging over initial conditions:

Retarded Green function does not depend on the initial conditions, so the average of the classical solutions ϕ_c is performed with the Keldysh Green function:

$$iG_K(t-t',x-x') = \langle \phi_c(t,x)\phi_c(t',x')\rangle_{i.c.} = \frac{1}{2} tr\big(\hat{\rho}(t_0)\{\hat{\varphi}(t,x),\hat{\varphi}(t',x')\}\big).$$

Define the "smeared" Keldysh Green Function $G_K^s(t,x)$ and the constant $\langle \Phi_s^2\rangle_{i.c}$:

$$\langle \phi_c(t,x)\Phi_s\rangle_{i.c} = iG_K^s(t,x) \equiv$$

$$\int dy\ \eta(y-x_q) iG_K(t,x-y),$$

$$\langle \Phi_s^2\rangle_{i.c} \equiv \int dydz\ \eta(y-x_q)\eta(z-x_q) iG_K(0,y-z).$$

Quench: example

Energy density after quench:

$$\langle\hat{\varepsilon}\rangle_t^Q = \langle\hat{\varepsilon}\rangle_t - 2i\alpha\Big(m^2 G_K^s(t,x)G_R^s(t,x) + \partial_t G_K^s(t,x)\partial_t G_R^s(t,x) + \partial_x G_K^s(t,x)\partial_x G_R^s(t,x)\Big)$$
$$+2\alpha^2\langle\Phi_s^2\rangle_{i.c}\Big(m^2\big(G_R^s(t,x)\big)^2 + \big(\partial_t G_R^s(t,x)\big)^2 + \big(\partial_x G_R^s(t,x)\big)^2\Big).$$

Note: Energy density is a real, the imaginary unity is included in the definition of the Keldysh Green function.

The Keldysh Green function is singular at coinciding points. However, the constant $\langle\Phi_s^2\rangle_{i.c}$ is regularised with the help of the "smearing" function $\eta(x - x_q)$. This function was introduced in the quench definition exactly to eliminate such a divergence. Its physical meaning is that the energy is released not exactly at the point x_q, but in a certain vicinity specified by the “smearing” function. Therefore, the final answer depends on this function and diverges if it approaches the delta-function.

Quench: example

Energy density after quench:

$$\langle\hat{\varepsilon}\rangle_t^Q = \langle\hat{\varepsilon}\rangle_t - 2i\alpha\Big(m^2 G_K^s(t,x)G_R^s(t,x) + \partial_t G_K^s(t,x)\partial_t G_R^s(t,x) + \partial_x G_K^s(t,x)\partial_x G_R^s(t,x)\Big)$$
$$+2\alpha^2\langle\Phi_s^2\rangle_{i.c}\Big(m^2\big(G_R^s(t,x)\big)^2 + \big(\partial_t G_R^s(t,x)\big)^2 + \big(\partial_x G_R^s(t,x)\big)^2\Big).$$

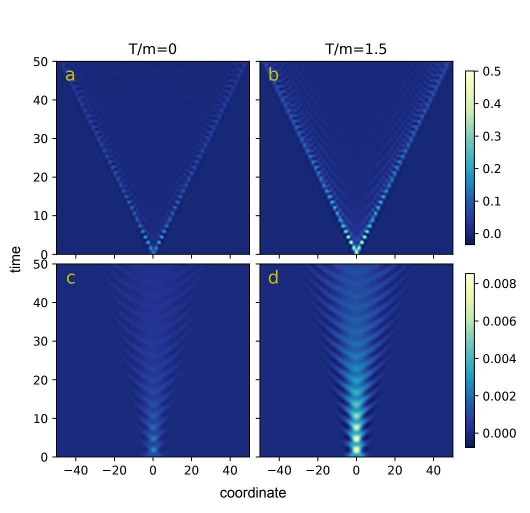
- α^0 Energy density before quench
- α^1 Linear response of the system to a local disturbance. Describes the redistribution of energy between different parts of the system and does not contribute to the total energy absorbed by the system.
- α^2 Shows the energy absorbed by the system after quench. If Keldysh Green function is described by a single-particle distribution function f_p, for the free theory:

$$iG_K(t, x-x') = \hbar\int\frac{dp}{2\pi}\frac{\cos(\omega_p t)}{2\omega_p}(2f_p+1)e^{-ip(x-x')}.$$

then the total energy that the system received after the quench:

$$\delta E = \int dx\big(\langle\hat{\varepsilon}\rangle_t^Q - \langle\hat{\varepsilon}\rangle_t\big) = 2\alpha^2\langle\Phi_s^2\rangle_{i.c}\int dy\,\eta^2(y) \sim \frac{1}{\epsilon}\log\left(\frac{\min\left(\Lambda,\epsilon^{-1}\right)}{m}\right)$$

Quench: energy density



The thermal state with temperatures $T = 0$ (a,c) and $T = 1.5m$ (b,d) . The "smearing" function is Gaussian with a width of $\epsilon m = 0.25$ (a,b) and $\epsilon m = 2$ (c,d).
Maximum momentum of particles created during quench $p_{\max} \sim \frac{1}{\epsilon}$, maximum group velocity of particles :

$$v_{\max} = \frac{\partial \omega_p}{\partial p} \sim \frac{p_{\max}}{\sqrt{p_{\max}^2 + m^2}} \sim \frac{1}{\sqrt{1 + m^2\epsilon^2}}$$

Front propagation:
$\epsilon \ll m^{-1} \rightarrow v_{\max} \sim 1$
$\epsilon \gg m^{-1} \rightarrow v_{\max} \sim \frac{1}{m\epsilon}$

Connection with CFT

Consider the vertex operator $\hat{Q}(x) = \hat{\mathcal{V}}_\alpha(x) =: e^{i\alpha\hat{\varphi}(x)}:$ with conformal dimensions $h = \bar{h} = \alpha^2/(8\pi)$

For Keldysh technique

- Potential $V(\varphi) = -\varphi$
- Vacuum initial state $T = 0$ ($f_p = 0$)
- "Smearing" function ($|\psi_0\rangle = \mathcal{N} e^{-\epsilon\hat{H}} : e^{i\alpha\hat{\varphi}(x_q)} : |0\rangle = e^{i\alpha\hat{\varphi}_s(x_q)}|0\rangle$)

  $$\eta(x) = \int \frac{dp}{2\pi} e^{ipx - \epsilon\omega_p} = \frac{m\epsilon}{\pi\sqrt{x^2+\epsilon^2}} K_1\left(m\sqrt{x^2+\epsilon^2}\right),$$

  where $K_\nu(z)$ is the MacDonald function, and ϵ – is the small parameter (the width of the "smearing" function).
- Energy density (very simple, no $G_K(t,x)$)

$$\langle\hat{\varepsilon}\rangle_t^Q = \langle\hat{\varepsilon}\rangle_t + \frac{1}{2}\alpha^2\left(m^2\left(G_R^s(t,x)\right)^2 + \left(\partial_t G_R^s(t,x)\right)^2 + \left(\partial_x G_R^s(t,x)\right)^2\right).$$

Connection with CFT

- Smeared retarded Green function

$$G_R^s(t,x) = \int dy\ \eta(y-x_q) G_R(t, x-y) = -\theta(t) \int \frac{dp}{2\pi} \frac{\sin(\omega_p t)}{\omega_p} e^{-ip(x-x_q)-\epsilon\omega_p}$$

$$= \frac{i}{2\pi}\theta(t) \left(K_0\left(m\sqrt{(x-x_q)^2+(\epsilon-it)^2} \right) - K_0\left(m\sqrt{(x-x_q)^2+(\epsilon+it)^2} \right) \right).$$

- For $m \to 0$

$$G_R^s(t,x) = \frac{i}{4\pi} \log\left(\frac{(x-x_q)^2+(\epsilon+it)^2}{(x-x_q)^2+(\epsilon-it)^2} \right)$$

- Energy density after the local quench :

$$\langle\hat{\varepsilon}\rangle_t^Q = \langle\hat{\varepsilon}\rangle_t + \frac{\alpha^2}{4\pi^2}\left(\frac{\epsilon^2}{((x-x_q-t)^2+\epsilon^2)^2} + \frac{\epsilon^2}{((x-x_q+t)^2+\epsilon^2)^2} \right).$$

Motivation: already known results (CFT)

- State after local quench

$$|\psi(t_q+0)\rangle = \mathcal{N} e^{-\epsilon \hat{H}} \hat{Q}(x_q)|0\rangle.$$

- Average value of observable

$$\langle \hat{O} \rangle_t = \langle \psi(t)|\hat{O}|\psi(t)\rangle$$

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$$\langle \hat{O} \rangle_t = \left. \frac{\langle 0|\hat{Q}^\dagger(\epsilon, x_q)\hat{O}(\tau)\hat{Q}(-\epsilon, x_q)|0\rangle}{\langle 0|\hat{Q}^\dagger(\epsilon, x_q)\hat{Q}(-\epsilon, x_q)|0\rangle} \right|_{\tau \to it}$$

- If $\hat{Q}$ is a primary operator with dimensions $(h, \bar{h})$ and $\hat{O}$ is an energy density $(h = \bar{h} = \alpha^2/(8\pi))$

$$\delta\varepsilon(\tau, x) = \frac{2h\epsilon^2}{\pi(x_q - x - i\epsilon - i\tau)^2(x_q - x + i\epsilon - i\tau)^2} + \frac{2\bar{h}\epsilon^2}{\pi(x_q - x + i\epsilon + i\tau)^2(x_q - x - i\epsilon + i\tau)^2}.$$

Conclusions

- We propose a new approach for the description of a local perturbation (quench) in scalar field theory with the help of the Keldysh technique. This approach does not use the analytical continuation procedure, which in some cases may be ambiguous. Moreover, the method presented in the work allows to consider systems with an arbitrary initial state.
- For the quench $\hat{Q}(x_q) = e^{-i\frac{\alpha}{\hbar}\hat{\varphi}_s^{\mathbf{2}}(x_q)}$, the evolution of the energy density was calculated for both the vacuum initial state and the state with an arbitrary initial distribution function f_p. Two regimes of propagation of the disturbance front are described, depending on the size of the local disturbance region (the width of the "smearing" function ϵ).
- The approach to the description of the dynamics of a system after an instantaneous local perturbation obtained in this work can be generalised to the case of nonzero interaction, at least for the semiclassical approximation within the Keldysh technique. This is a topic for further investigation.

More details: А.А.Р., А.Г. Семенов, Письма в ЖЭТФ 118, 921 (2023)