

# Nonequilibrium Schwinger-Keldysh formalism for mixed states

based on [arXiv:2309.03687](https://arxiv.org/abs/2309.03687)

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- Consider general nonequilibrium field theory

$$S[\phi] = \frac{1}{2} \int dt \left( \dot{\phi}^T A \dot{\phi} + \dot{\phi}^T B \phi + \phi^T B^T \dot{\phi} + \phi^T C \phi \right)$$

$\phi^I$  — fields,  $I = (\mathbf{x}, i)$  — multi-indices,  $A, B, C$  — time-dependent operator coefficients ( $A = A_{IJ}(t), \dots$ ).

# Problem statement I

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- Field equations read

$$F = -\frac{d}{dt} A \frac{d}{dt} - \frac{d}{dt} B + B^T \frac{d}{dt} + C,$$

and imply Klein-Gordon type inner product

$$(\phi_1, \phi_2) = i\phi_1^\dagger (W\phi_2) - i(W\phi_1)^\dagger \phi_2, \quad W = A \frac{d}{dt} + B$$

## Problem statement II

- **Goal** is to calculate in-in Green's correlation function generating functional

$$Z[J_1, J_2] = \text{tr} \left[ \hat{U}_{J_1}(T, 0) \hat{\rho} \hat{U}_{-J_2}^\dagger(T, 0) \right].$$

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where the Hamiltonian was modified by source term  $-J^T(t)\phi(t)$ .

- Density matrix is general Gaussian density matrix, defined in coordinate space

$$\langle \varphi_+ | \hat{\rho} | \varphi_- \rangle = \text{const} \times \exp \left\{ -\frac{1}{2} \varphi^T \Omega \varphi + \mathbf{j}^T \varphi \right\},$$

where

$$\varphi = \begin{bmatrix} \varphi_+ \\ \varphi_- \end{bmatrix}, \quad \mathbf{j} = \begin{bmatrix} j_+ \\ j_- \end{bmatrix}, \quad \Omega = \begin{bmatrix} R & S \\ S^* & R^* \end{bmatrix}$$

- Result of functional Gaussian integration reads

$$Z[\mathbf{J}] = \text{const} \times \exp \left\{ -\frac{i}{2} \int_0^T dt dt' \mathbf{J}^T(t) \mathbf{G}(t, t') \mathbf{J}(t) - \int_0^T dt \mathbf{J}^T(t) \mathbf{G}(t, 0) \mathbf{j} + \frac{i}{2} \mathbf{j}^T \mathbf{G}(0, 0) \mathbf{j} \right\},$$

where the source reads

$$\mathbf{J} = \begin{bmatrix} J_1 \\ J_2 \end{bmatrix}$$

- Green's function has block-matrix form

$$\mathbf{G}(t, t') = \begin{bmatrix} G_{\text{T}}(t, t') & G_{<}(t, t') \\ G_{>}(t, t') & G_{\bar{\text{T}}}(t, t') \end{bmatrix},$$

## Formal result II

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- and satisfy inhomogeneous equation

$$\mathbf{F}\mathbf{G}(t, t') = \mathbf{I} \delta(t - t'),$$

supplemented by boundary condition

$$(i\mathbf{W} + \mathbf{\Omega})\mathbf{G}(t, t')|_{t=0} = 0, \quad \begin{bmatrix} I & I \\ I & -I \end{bmatrix} \mathbf{W}\mathbf{G}(t, t')|_{t=T} = 0, \\ \begin{bmatrix} I & -I \\ I & -I \end{bmatrix} \mathbf{G}(t, t')|_{t=T} = 0.$$

where

$$\mathbf{F} = \begin{bmatrix} F & 0 \\ 0 & -F \end{bmatrix}, \quad \mathbf{W} = \begin{bmatrix} W & 0 \\ 0 & -W \end{bmatrix}$$



# Basis functions I

- Explicit form of Green's functions can be expressed through basis functions

$$\begin{aligned} \mathbf{F}\mathbf{v}_{\pm}(t) &= 0, \\ (i\mathbf{W} + \mathbf{\Omega})\mathbf{v}_{-}(t)|_{t=0} &= 0, & \begin{bmatrix} I & I \end{bmatrix} \mathbf{W}\mathbf{v}_{+}(t)|_{t=T} &= 0, \\ & & \begin{bmatrix} I & -I \end{bmatrix} \mathbf{v}_{+}(t)|_{t=T} &= 0. \end{aligned}$$

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- We will construct it using the simpler ones

$$\mathbf{F}\mathbf{v}(t) = 0, \quad (i\mathbf{W} - \boldsymbol{\omega})\mathbf{v}(t)|_{t=0} = 0, \quad \boldsymbol{\omega} = \begin{bmatrix} \omega & 0 \\ 0 & \omega^* \end{bmatrix}.$$

having block-diagonal form

$$\mathbf{v} = \begin{bmatrix} v & 0 \\ 0 & v^* \end{bmatrix}, \quad \mathbf{v}^* = \begin{bmatrix} v^* & 0 \\ 0 & v \end{bmatrix}.$$

## Basis functions II

- Desired basis functions are obtained through Bogolyubov transformation

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- Answer for  $\mathbf{G}(t, t')$  has following final form

$$\begin{aligned} i\mathbf{G}(t, t') &= i\mathbf{G}_0(t, t') + \mathbf{v}_+(t) \boldsymbol{\nu} \mathbf{v}_+^T(t'), \\ i\mathbf{G}_0(t, t') &= \mathbf{v}_+(t) \mathbf{v}_+^\dagger(t') \theta(t - t') + \mathbf{v}^*(t) \mathbf{v}_+^T(t') \theta(t' - t) \\ \boldsymbol{\nu} &= \left[ \mathbf{I} + \mathbf{X} - \sqrt{2\omega_{\text{re}}} \mathbf{X} (\omega + \Omega)^{-1} \mathbf{X} \sqrt{2\omega_{\text{re}}} \right]^{-1} - \mathbf{X}. \end{aligned}$$

which has no “good” block structure.

- How to choose  $\omega$ ?

- Decompose Heisenberg field operator

$$\hat{\phi}(t) = v(t)\hat{a} + v^*(t)\hat{a}^\dagger$$

define non-anomalous and anomalous averages

$$\nu = \text{tr}[\hat{\rho}\hat{a}^\dagger\hat{a}], \quad \kappa = \text{tr}[\hat{\rho}\hat{a}\hat{a}],$$

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$$\omega = R^{1/2}\sqrt{I - \sigma^2}R^{1/2}, \quad \sigma \equiv R^{-1/2}SR^{-1/2}.$$

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- Block-matrix components  $\mathbf{G}(t, t')$  have simple form, in particular

$$iG_{>}(t, t') = v(t) (\nu + I) v^\dagger(t') + v^*(t) \nu v^T(t').$$

# Euclidean density matrix I

- Consider the particular type of the density matrix, defined by the Euclidean action

$$\rho_E(\varphi_+, \varphi_-; J_E) = \frac{1}{Z} \int_{\phi(\tau_{\pm})=\varphi_{\pm}} D\phi \exp \left\{ -S_E[\phi] - \int_0^\beta d\tau J_E(\tau)\phi(\tau) \right\},$$

where  $S_E$  is Euclidean action

$$iS[\phi(t)]|_{t=-i\tau} = -S_E[\phi_E(\tau)].$$

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- Operator coefficients of the Euclidean action are defined by the initial one as

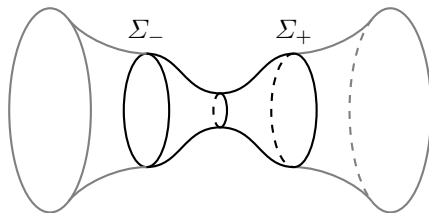
$$A_E(\tau) = A(-i\tau), \quad B_E(\tau) = -iB(-i\tau), \quad C_E(\tau) = -C(-i\tau),$$

and hermiticity implies

$$A_E(\beta - \tau) = A_E^*(\tau), \quad B_E(\beta - \tau) = -B_E^*(\tau), \quad C_E(\beta - \tau) = C_E^*(\tau)$$

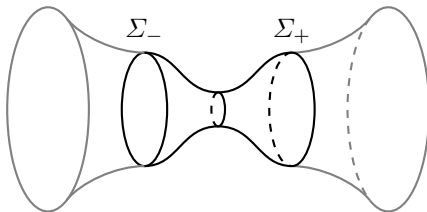
# Euclidean density matrix II

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- Density matrix is Gaussian, which parameters read

$$\Omega = \begin{bmatrix} -\vec{W}_E G_D(\beta, \beta) \overleftarrow{W}_E & \vec{W}_E G_D(\beta, 0) \overleftarrow{W}_E \\ \vec{W}_E G_D(0, \beta) \overleftarrow{W}_E & -\vec{W}_E G_D(0, 0) \overleftarrow{W}_E \end{bmatrix}$$

- Special choice of  $\omega$  implies following properties of basis functions

$$v(t - i\beta) = v(t) \frac{\nu + I}{\nu}, \quad v^*(t - i\beta) = v^*(t) \frac{\nu}{\nu + I}.$$

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- that lead do famous Kubo-Martin-Schwinger condition

$$G_{>}(t - i\beta, t') = G_{<}(t, t')$$

despite non-stationary nature of system!

- Generating functional of in-in Green's functions for general non-equilibrium system and initial state was calculated
- Special choice of basis functions, allowing particle interpretation was made
- For Euclidean initial state analytic structure of basis functions was examined and KMS condition was derived



Thank you for your attention!