Eisenhart lift of Koopman-von Neumann Mechanics

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Outline of the talk

The talk is based on the paper Bikram Keshari Parida, Abhijit Sen, Shailesh Dhasmana, Zurab K. Silagadze, Eisenhart lift of Koopman-von Neumann Mechanics, https://arxiv.org/abs/2207.05073 Published in Journal of Geometry and Physics 185 (2023), 104732.

Koopman-von Neumann mechanics

Through the Liouville equation

$$\frac{\partial \rho(q,p,t)}{\partial t} = \frac{\partial H_{cl}}{\partial q}\frac{\partial \rho}{\partial p} - \frac{\partial H_{cl}}{\partial p}\frac{\partial \rho}{\partial q}.$$

The classical wave function $\psi(q,p,t) = \sqrt{\rho(q,p,t)}$ obeys the same Liouville equation, which can be rewritten in Schrödinger-type form

$$i\frac{\partial \psi(q,p,t)}{\partial t} = \hat{L}\psi, \quad \hat{L} = i\left(\frac{\partial H_{cl}}{\partial q}\frac{\partial}{\partial p} - \frac{\partial H_{cl}}{\partial p}\frac{\partial}{\partial q}\right).$$

- It is possible to develop a formulation of classical mechanics in Hilbert space that completely resembles the quantum formalism, except that, of course, all interference effects are absent. Koopman 1931, von Neumann 1932.

D. Mauro, Topics in Koopman-von Neumann Theory,
https://doi.org/10.48550/arXiv.quant-ph/0301172

- Ordinary axioms of quantum mechanics.
- $|\Psi(t)\rangle = \hat{U}(t)|\Psi(0)\rangle$: unitary representation of a group of time shifts. According to Stone's theorem, there must exist a Hermitian generating operator with $i\frac{d|\Psi\rangle}{dt} = \hat{L}|\Psi\rangle$.
- Ehrenfest's theorem $\frac{d}{dt}\langle\hat{q}\rangle = \langle\frac{\hat{p}}{m}\rangle$, $\frac{d}{dt}\langle\hat{p}\rangle = -\langle\frac{d}{dq}U(\hat{q})\rangle$ requires

$$i[\hat{L},\hat{q}] = \frac{\hat{p}}{m}, \quad i[\hat{L},\hat{p}] = -\frac{d}{dq}U(\hat{q}).$$

- $[\hat{q},\hat{p}] = i\hbar$ -> quantum mechanics: $\hbar\hat{L} = \hat{H} = \frac{\hat{p}^2}{2m} + U(\hat{q})$.
- $[\hat{q},\hat{p}] = 0$ -> we cannot construct $\hat{L}$ from only dynamic variables $\hat{q},\hat{p}$. To correct the situation, we introduce two additional Hermitian operators $\hat{\lambda}_q, \hat{\lambda}_p$, satisfying the conditions $[\hat{q},\hat{\lambda}_q] = i$, $[\hat{p},\hat{\lambda}_p] = i$. Then $\hat{L} = \frac{\hat{p}}{m}\hat{\lambda}_x - \frac{dU(\hat{q})}{dq}\hat{\lambda}_p$.

F. Wilczek, Notes on Koopman von Neumann Mechanics, and a Step Beyond. https://frankwilczek.com/2015/koopmanVonNeumann02.pdf

Via Wigner function

$$W(q,p)=\frac{1}{\sqrt{2\pi\hbar}}\int e^{\frac{i}{\hbar}py}\Psi^*(q+y/2,t)\Psi(q-y/2,t)dy.$$

$\hbar\to k\hbar,\ y=k\hbar\lambda_p,\ u=q-\frac{k\hbar\lambda_p}{2},\ v=q+\frac{k\hbar\lambda_p}{2}$:

$$W(q,p)=\sqrt{\frac{k\hbar}{2\pi}}\int e^{ip\lambda_p}\rho(u,v,t)d\lambda_p,\quad ;\rho(u,v,t)=\Psi^*(v)\Psi(u).$$

$$ik\hbar\frac{\partial\rho}{\partial t}=\left[\hat{H}_u-\hat{H}_v\right]\rho,\ \ \hat{H}_u=\frac{(k\hbar)^2}{2m}\frac{\partial^2}{\partial u^2}+U(u).$$

This is reminiscent of the chiral decomposition method.
Generalized pseudo-differential Bopp operators:

$$\hat{u}=\hat{q}-\frac{k\hbar\hat{\lambda}_p}{2},\ \hat{v}=\hat{q}+\frac{k\hbar\hat{\lambda}_p}{2},\ \hat{p}_u=\hat{p}+\frac{k\hbar\hat{\lambda}_q}{2},\ \hat{p}_v=\hat{p}-\frac{k\hbar\hat{\lambda}_q}{2}.$$

$$[\hat{u},\hat{p}_u]=ik\hbar,\ \ [\hat{v},\hat{p}_v]=-ik\hbar\ \ k\to 0 \text{ means } [\hat{q},\hat{p}]=0.$$

Chiral decomposition method

The difference of Hamiltonians of two uncoupled one-dimensinal oscillators yield an interesting non-commutative system in the plane:

P. D. Alvarez, J. Gomis, K. Kamimura, M. S. Plyushchay, Anisotropic harmonic oscillator, non-commutative Landau problem and exotic Newton-Hooke symmetry, Phys. Lett. **B 659**, 906-912 (2008). https://arxiv.org/abs/0711.2644

P. D. Alvarez, J. Gomis, K. Kamimura, and M. S. Plyushchay, (2+1)D Exotic Newton-Hooke Symmetry, Duality and Projective Phase, Annals Phys. **322** (2007) 1556-1586. https://arxiv.org/abs/hep-th/0702014

P.-M. Zhang, P. A. Horvathy, Chiral Decomposition in the Non-Commutative Landau Problem, Annals Phys. 327 (2012) 1730–1743. https://arxiv.org/abs/1112.0409.

$k \to 0$ limit and Koopman-von Neumann equation

$$\hat{H}_u - \hat{H}_v = \frac{k\hat{p}\hat{P}}{m} + U\left(\hat{q} + \frac{k\hat{Q}}{2}\right) - U\left(\hat{q} - \frac{k\hat{Q}}{2}\right), \quad \hat{\lambda}_q = \frac{\hat{P}}{\hbar}, \ \hat{\lambda}_p = -\frac{\hat{Q}}{\hbar}.$$

$$i\hbar\frac{\partial \Psi_{KvN}}{\partial t} = \left[\frac{\hat{p}\hat{P}}{m} + \frac{1}{k}U\left(\hat{q} + \frac{k\hat{Q}}{2}\right) - \frac{1}{k}U\left(\hat{q} - \frac{k\hat{Q}}{2}\right)\right]\Psi_{KvN},$$

where

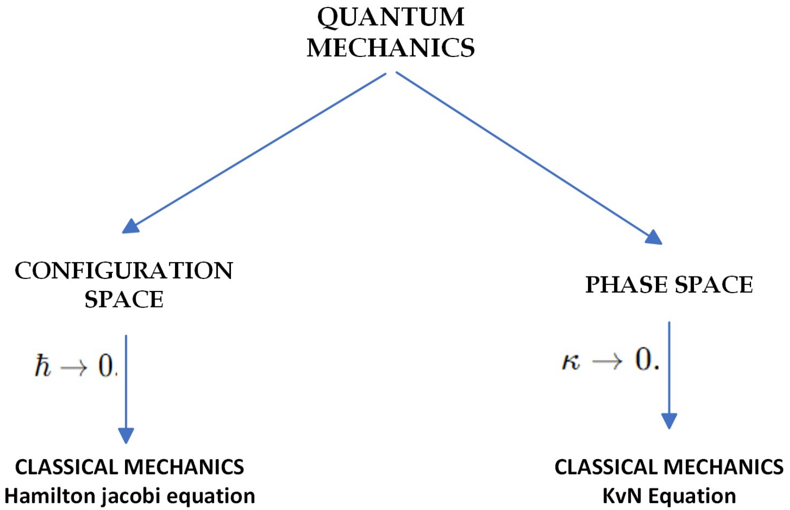
$$\Psi_{KvN}(q, Q, t) \sim \rho(u, v, t).$$

We have a well defined $k \to 0$ limit:

$$i\hbar\frac{\partial \Psi_{KvN}}{\partial t} = \left[\frac{\hat{p}\hat{P}}{m} + \frac{\partial U(q)}{\partial q}Q\right]\Psi_{KvN} = \hat{H}_{KvN}\Psi_{KvN}.$$

D.I. Bondar *et al.*, Operational dynamic modeling transcending quantum and classical mechanics, Phys. Rev. Lett. 109 (2012) 190403.
https://arxiv.org/abs/1105.4014

Comparison of $\hbar \to 0$ and $k \to 0$ limits



Sudarshan's interpretation

If we introduce $\hat{Q}$ and $\hat{P}$ operators as follows

$$\hat{Q} = i\hbar\frac{\partial}{\partial p}, \quad \hat{P} = -i\hbar\frac{\partial}{\partial q},$$

then the Liouville-Schrödinger equation takes the form

$$i\hbar\frac{\partial\psi(q,p,t)}{\partial t} = \hat{H}\psi, \quad \hat{H} = \frac{\partial H_{cl}}{\partial q}\hat{Q} + \frac{\partial H_{cl}}{\partial p}\hat{P},$$

and it can be interpreted as the Schrödinger equation in the (q,p)-representation (with diagonal operators q and p) of a genuine quantum system with two pairs of canonical variables (q,P) and (Q,p).

E. C. G. Sudarshan, Interaction between classical and quantum systems and the measurement of quantum observables, Pramana 6(3) (1976), 117.
https://link.springer.com/article/10.1007/BF02847120

Quantum Mechanics Free Subsystems (QMFS)

Let us assume that the Hamiltonian of the quantum system is equal to

$$\hat{H} = f(q,p,t)\hat{P} + g(q,p,t)\hat{Q} + h(q,p,t),$$

where $f(q,p,t)$, $g(q,p,t)$, $h(q,p,t)$ are arbitrary functions, and q, P and Q, p represent are two pairs of quantum mechanical conjugate variables that obey canonical commutation relations. Then the Heisenberg equations of motion for the commuting variables q, p

$$\frac{dq}{dt} = \frac{\partial H}{\partial P} = f(q,p,t), \quad \frac{dp}{dt} = -\frac{\partial H}{\partial Q} = -g(q,p,t),$$

do not contain "hidden" variables $\hat{Q}, \hat{P}$ and will correspond to classical Hamiltonian dynamics if there exists a classical Hamiltonian function $H_{cl}(q,p,t)$ such that

$$f(q,p,t) = \frac{\partial H_{cl}}{\partial p}, \quad g(q,p,t) = \frac{\partial H_{cl}}{\partial q}.$$

Experimental implementation of QMFS

M. Tsang, C. M. Caves, Evading quantum mechanics: Engineering a classical subsystem within a quantum environment, Phys. Rev. X 2 (2012), 031016. https://arxiv.org/abs/1203.2317 A pair of positive and negative mass oscillators can be used for this purpose. The quantum Hamiltonian in this case has the form

$$H = \frac{p_1^2}{2m} + \frac{1}{2}m\omega^2 q_1^2 - \frac{p_2^2}{2m} - \frac{1}{2}m\omega^2 q_2^2.$$

In terms of new canonical variables

$$q = q_1 + q_2, \quad Q = \frac{1}{2}(q_1 - q_2), \quad p = p_1 - p_2, \quad P = \frac{1}{2}(p_1 + p_2),$$

The Hamiltonian takes the form $H = \frac{pP}{m} + m\omega^2 qQ$, and is a KvN-type Hamiltonian.

Sidney Coleman: "The career of a young theoretical physicist consists of treating the harmonic oscillator in ever-increasing levels of abstraction."

KvN mechanics is realized in QMFS

- Similarity of the Sudarshan interpretation of the KvN mechanics with the idea of QMFS is obvious.
- (q, p) subsystem of KvN mechanics is nothing more than QMFS.
- Resumption of interest in KvN mechanics was caused by the need to create suitable formalism for hybrid classical-quantum systems.
- The identification of quantum-mechanics-free subsystems with Sudarshan's interpretation of KvN mechanics, combined with the fact that such systems were actually implemented experimentally, makes the KvN mechanics, in a sense, engineering science.

Z.K. Silagadze, Evading Quantum Mechanics à la Sudarshan: quantum-mechanics-free subsystem as a realization of Koopman-von Neumann mechanics, https://arxiv.org/abs/2308.08919. Published in Foundations of Physics 53 (2023), 92.

Quantum gravity destroys classicality?

- Modification of quantum mechanics, expected from quantum gravity, can lead to deformation of classical mechanics (O.I Chashchina, A. Sen, Z.K. Silagadze, On deformations of classical mechanics due to Planck-scale physics, Int. J. Mod. Phys. D29 (2020), 2050070 https://arxiv.org/abs/1902.09728).
- This deformation actually destroys the classicality if Sudarshan's views on KvN mechanics are taken seriously.
- You are not required to accept the Sudarshan interpretation in order to develop the KvN mechanics.
- However, we now see that the existence of quantum-mechanics-free subsystems indicates that we should take Sudarshan's interpretation of KvN mechanics seriously.
- Therefore, we expect that, due to the universal nature of gravity, if the effects of quantum gravity do modify quantum mechanics, these effects will destroy the classical dynamics in QMFS.

Eisenhart-Duval lift

Embedding nonrelativistic physics inside a gravitational wave

Eisenhart theorem (1929):
Dynamical trajectories of non-relativistic (NR) mechanics can always be lifted to geodesics of a specific relativistic spacetime with one dimension more. Conversely, to any geodesic of this specific class of spacetimes corresponds a solution of a NR dynamical system.

- The metric is uniquely determined by the form of the NR Lagrangian.
- Relativistic spacetime has a metric with Lorentz signature and carries a covariantly constant null vector (Bargmann structure, Duval, 1985).
- Bargmann space is in fact the space-time of a plane gravitational wave in 5-dimensions.

X. Bekaert, K. Morand, Embedding nonrelativistic physics inside a gravitational wave, Phys. Rev. D 88 (2013), 063008.
https://arxiv.org/abs/1307.6263

The simplest way to explain Eisenhart-Duval lift is to use Hamiltonian approach. The first step is to promote time t to a dynamical variable:

$$H = \frac{1}{2m}\sum_{i,j=1}^{n} h^{ij}(q)p_i p_j + V(q,t) \rightarrow \tilde{H} = p_t + H(q,p,t) = 0.$$

The main idea behind the Eisenhart lift is to introduce a new momentum p_s conjugate to a dummy configuration space variable s to make the Hamiltonian homogeneous in canonical momenta and turn it into a geodesic Hamiltonian (homogeneous quadratic function of momenta):

$$\mathcal{H} = \frac{1}{2m}\sum_{i,j=1}^{n} h^{ij}(q)p_i p_j + \frac{1}{m^2}\, p_s^2\, V(q,t) + \frac{1}{m}\, p_s p_t = \frac{1}{2m}\sum_{A,B=1}^{n+2} g^{AB} p_A p_B.$$

The constraint $\mathcal{H} = 0$ can be interpreted as a mass-shell condition for a massless particle in space-time with a Brinkmann-type metric

$$dS^2 = \sum_{i,j=1}^{n} h_{ij}\, dq^i\, dq^j + 2ds\, dt - 2\frac{V(q,t)}{m}\, dt^2.$$

Eisenhart-Duval metric and Schrödinger equation

The massless Klein-Gordon equation in general metric is given by

$$\Box\phi = \frac{1}{\sqrt{-g}}\,\partial_A\left(\sqrt{-g}\,g^{AB}\partial_B\phi\right) = 0,$$

which, after the field transformation (null-reduction)

$$\phi(q,t,s) = e^{is}\varphi\,(q,t)\,,$$

is reduced to the Schrödinger equation

$$i\frac{\partial\varphi}{\partial t} = -\frac{1}{2m}\nabla^2\varphi + V\varphi.$$

The Schrödinger equation can be considered as a null-reduction (reduction in the s-direction) of the Klein-Gordon equation in the Eisenhart-Duval metric background.

The simplest way to geometrize the KvN mechanics is to begin from the KvN Hamiltonian and consider it as describing classical (not KvN) system:

$$H = \frac{pP}{m} + \frac{\partial V}{\partial q} Q.$$

Homogenizing this Hamiltonian, we get

$$\mathcal{H} = \frac{pP}{m} + \frac{\partial V}{\partial q} Q \frac{p_s^2}{m^2} + \frac{1}{m} p_s p_t,$$

which corresponds to the inverse metric

$$g^{qQ} = g^{Qq} = 1, \;\; g^{st} = g^{ts} = 1, \;\; g^{ss} = \frac{2}{m} \frac{\partial V}{\partial q} Q,$$

all other components being zero. Inverting g^{AB} to calculate the metric tensor g_{AB}, we get the corresponding Eisenhart metric

$$dS^2 = 2dq\, dQ + 2dt\, ds - \frac{2Q}{m} \frac{\partial V(q)}{\partial q} dt^2.$$

KvN equation from null-reduction

Curved space KG for the massless scalar field $\chi(t,s,q,Q)$ for KvN Eisenhart-Duval metric is

$$\frac{\partial^2\chi}{\partial q\partial Q}+\frac{Q}{m}\frac{\partial V}{\partial q}\frac{\partial^2\chi}{\partial s^2}+\frac{\partial^2\chi}{\partial t\partial s}=0,$$

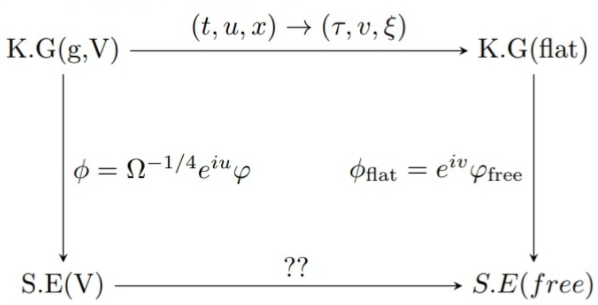
which after the field redefinition

$$\chi(t,s,q,Q)=e^{ims}\psi_{KvN}(t,q,Q),$$

reduces to the equation of the form

$$i\frac{\partial\psi_{KvN}}{\partial t}=\left(Q\frac{\partial V}{\partial q}-\frac{1}{m}\frac{\partial^2}{\partial q\partial Q}\right)\psi_{KvN},$$

which is the KvN equation in the (q,Q)-representation for the classical Hamiltonian $H=\frac{p^2}{2m}+V(q)$.



Conditions for flatness: the Cotton tensor ($d = 3$) or the Weyl tensor ($d > 3$) vanishes.

$$C_{\mu\nu\lambda} = \nabla_\lambda R_{\mu\nu} - \nabla_\nu R_{\mu\lambda} + \frac{1}{4} \left(g_{\mu\lambda} \nabla_\nu R - g_{\mu\nu} \nabla_\lambda R \right).$$

Conformally flat Bargmann spaces

C. Duval, P.A. Horváthy, L. Palla, Conformal Properties of Chern-Simons Vortices in External Fields. Phys. Rev. D50 (1994), 6658-6661.
https://arxiv.org/abs/hep-th/9404047

$$V = \frac{1}{2}A(t)\vec{x}^2 + \vec{B(t)} \cdot \vec{x} + D(t).$$

Uniform gravitational field:

$$t = \tau, \quad x = \xi - \frac{g\tau^2}{2}, \;\; u = v + g\xi\tau - \frac{g^2\tau^3}{3},$$

$$\varphi_{grv}(t,x) = e^{-i\left[g\xi\tau - \frac{g^2\tau^3}{3}\right]}\varphi_{free}(\tau,\xi).$$

S. Dhasmana, A. Sen, Z.K. Silagadze, Equivalence of a harmonic oscillator to a free particle and Eisenhart lift, Annals Phys. 434 (2021) 168623. https://arxiv.org/abs/2106.09523

Phenomena of neutron interference in the presence of a weak gravitational potential: R. Colella, A.W. Overhauser, S.A. Werner, Observation of gravitationally induced quantum interference, Phys. Rev. Lett. 34 (1975), 1472-1474. https://doi.org/10.1103/PhysRevLett.34.1472

Transformation between harmonic oscillator and free particle:

$$x = \frac{\xi + c_2\,\omega_0\tau + c_1}{\sqrt{1+\omega_0^2\tau^2}}, \quad t = \frac{1}{\omega_0}\tan^{-1}(\omega_0\tau), \quad u =$$

$$v + \frac{\omega_0^2\xi^2\tau}{2(1+\omega_0^2\tau^2)} - \frac{(c_2 - c_1\,\omega_0\tau)\,\omega_0}{1+\omega_0^2\tau^2}\,\xi + \frac{(c_1^2 - c_2^2)\omega_0^2\tau - 2c_1c_2\,\omega_0}{2(1+\omega_0^2\tau^2)} + c_3,$$

whereas the wave function transforms as follows:

$$\varphi_{H.O}(t,x) = (1+\omega_0^2\tau^2)^{1/4} \times$$
$$e^{-i\left(\frac{\omega_0^2\xi^2\tau}{2(1+\omega_0^2\tau^2)} - \frac{(c_2-c_1\omega_0\tau)\omega_0}{(1+\omega_0^2\tau^2)}\xi + \frac{(c_1^2-c_2^2)\omega_0^2\tau - 2c_1c_2\omega_0}{2(1+\omega_0^2\tau^2)} + c_3\right)}\varphi_{free}(\tau,\xi)$$

S. Dhasmana, A. Sen, Z.K. Silagadze, Equivalence of a harmonic oscillator to a free particle and Eisenhart lift, Annals Phys. 434 (2021) 168623. https://arxiv.org/abs/2106.09523

The case of Koopmann von Neumann mechanics

For a harmonic oscillator, the coordinate transformation has the form

$$t = \tan^{-1}(\tau), \quad u = \frac{-\eta}{\sqrt{\tau^2+1}}, \quad v = \frac{-\xi}{\sqrt{\tau^2+1}},$$

$$s = \zeta + \frac{1}{2(\tau^2+1)}\left[\left(\eta^2 - \xi^2\right)\tau\right]. \quad q = \frac{v+u}{2}, \quad Q = u - v.$$

Corresponding transformation of the KvN wave function

$$\psi_{HO} = \sqrt{1+\tau^2}\,\exp\left[\frac{im\left[\left(\xi^2-\eta^2\right)\tau\right]}{2(\tau^2+1)}\right]\psi_{free}.$$

For a linear potential, the coordinate transformation has the form

$$t = \tau, \; u = \eta - \frac{1}{2}g\tau^2, \; v = \xi - \frac{1}{2}g\tau^2, \; s = \zeta + (\eta - \xi)g\tau.$$

Corresponding transformation of the KvN wave function

$$\psi_{Linear}(q, Q, t) = e^{-iQg\tau}\psi_{free}\left(q + \frac{1}{2}gt^2, Q, t\right),$$

is unitary and represents Einstein's equivalence principle in KvN mechanics.

A. Sen, B.K. Parida, S. Dhasmana, Z.K. Silagadze, Eisenhart lift of Koopman-von Neumann mechanics, J. Geom. Phys. 185 (2023), 104732 https://arxiv.org/abs/2207.05073

Conclusions

- A general holonomic conservative system in classical dynamics with d degrees of freedom is geometrically described by the Eisenhart-Duval lift in terms of the geodesics of the Lorentzian metric in the $(d+2)$-dimensional space-time.
- For treating time dependent dynamical systems and their symmetries, this geometric perspective is particularly convenient.
- The same geometric perspective provided by the Eisenhart-Duval lift can also be used in quantum theory, since null reduction of the massless KG equation from Eisenhart-Duval space-time leads to the Schrödinger equation.
- The Eisenhart-Duval toolkit can be applied to KvN mechanics as well, much like the quantum case.