# Fermions localized on solitons in flat and curved space-time

**Ya Shnir**

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**J Kunz,, I Perapechka**
**and N Sawado**

Quarks-2024, Pereslavl, 22 May 2024

# Outline

- Warming up: Fermions localized by kinks in 1+1 dim
- Fermions localized by baby Skyrmions in 2+1 dim
- Backreaction of the fermions
- Fermionic zero mode localized on the non-Abelian monopole
- Self-gravitating non-Abelian monopole coupled to fermions
- Self-gravitating Skyrmion coupled to fermions
- Summary and outlook

# Fermions localization on the kink in 1+1 dim

$$L = \frac{1}{2}(\partial_\mu \phi)^2 + i\bar{\psi}\gamma^\mu \partial_\mu \psi + g\phi\bar{\psi}\psi - \frac{1}{2}(\phi^2 - 1)^2$$

*R.Jackiw and C.Rebbi*
*Phys. Rev. D13 3398 (1976)*

- **Field equations:** $i\gamma^\mu \partial_\mu \psi = g\phi\psi\,; \qquad \partial_\mu \partial^\mu \phi = 2\phi(1-\phi^2) - g\bar{\psi}\psi$

**Fixed background (g≪1):**

$$\psi = e^{-i\epsilon t}\begin{pmatrix} v_1 - v_2 \\ v_1 + v_2 \end{pmatrix} \qquad \int dx\, |\bar{\psi}\psi| = 1$$

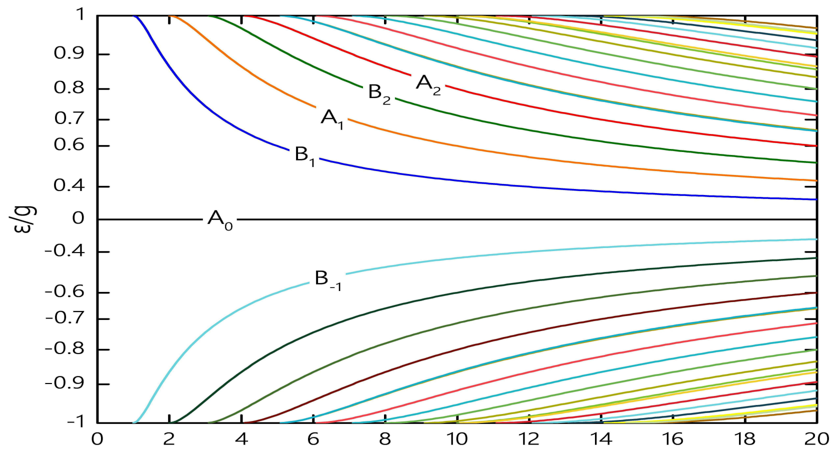
$$\phi_K = \tanh x \qquad |\epsilon| \le g$$

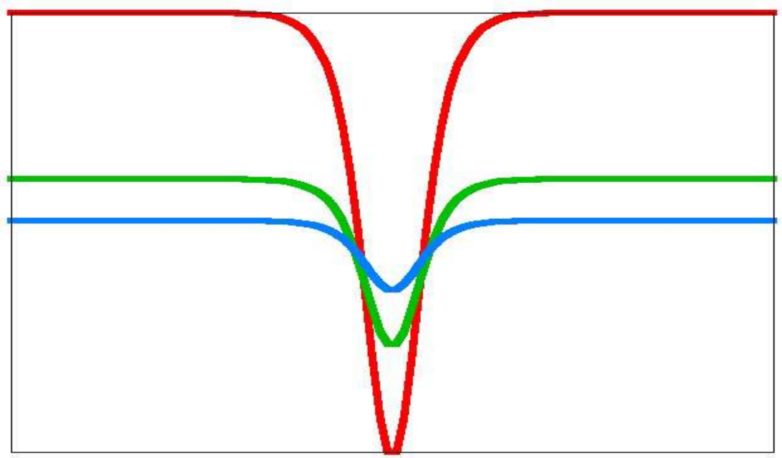
$$(\partial_x + g\tanh x)v_1 = -\epsilon v_2$$
$$(\partial_x - g\tanh x)v_2 = \epsilon v_1$$

$\Rightarrow$

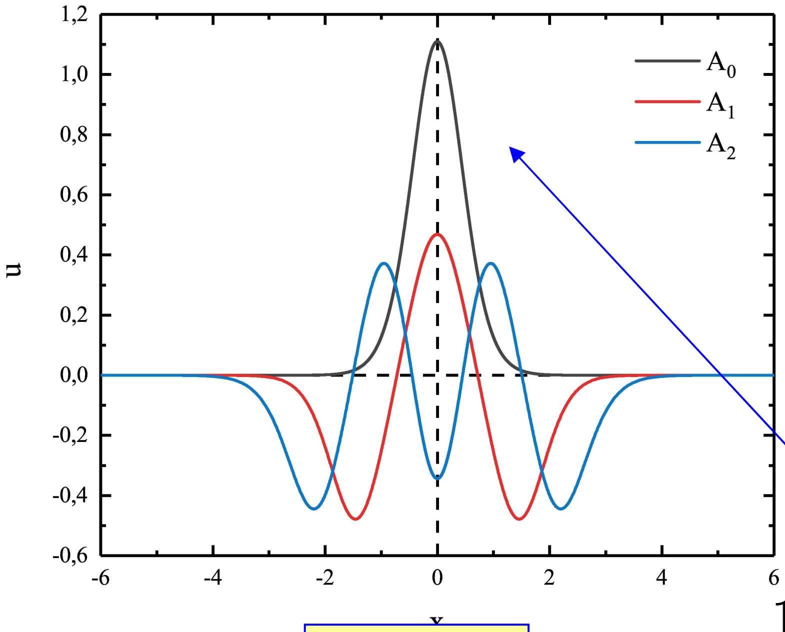
$$\left(-\partial_x^2 + U_\pm(x)\right) v_{1,2} = \epsilon^2 v_{1,2}$$

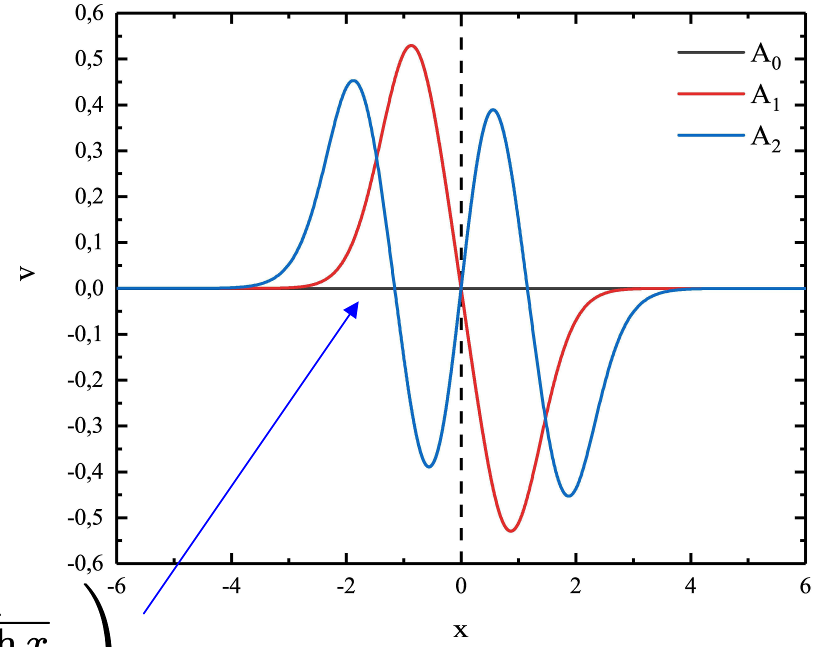
$$U_\pm(x) = g^2 - g(g \pm 1)\mathrm{sech}^2 x$$



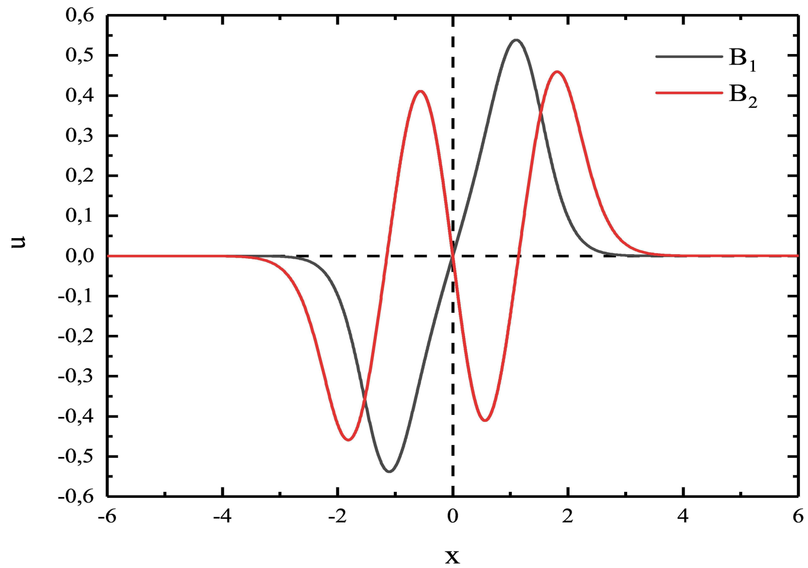
# Fermionic modes

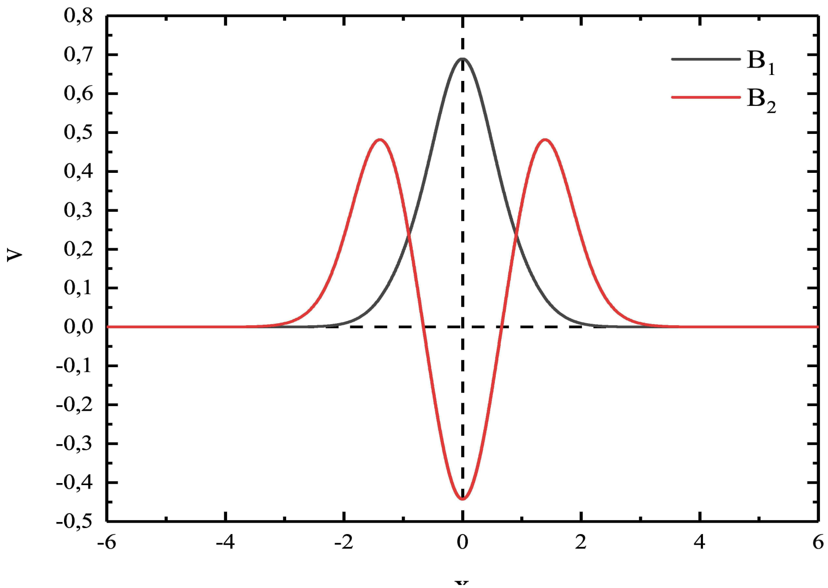


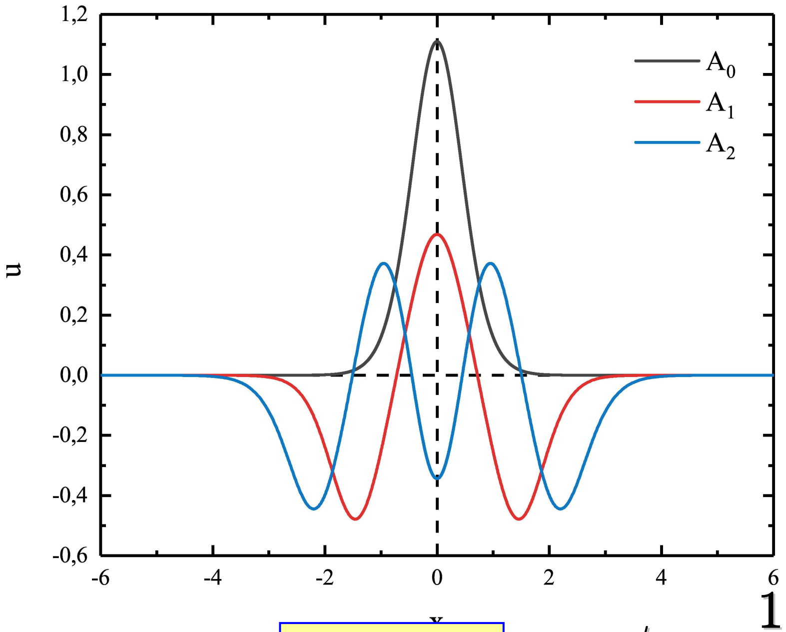

A$_0$ mode

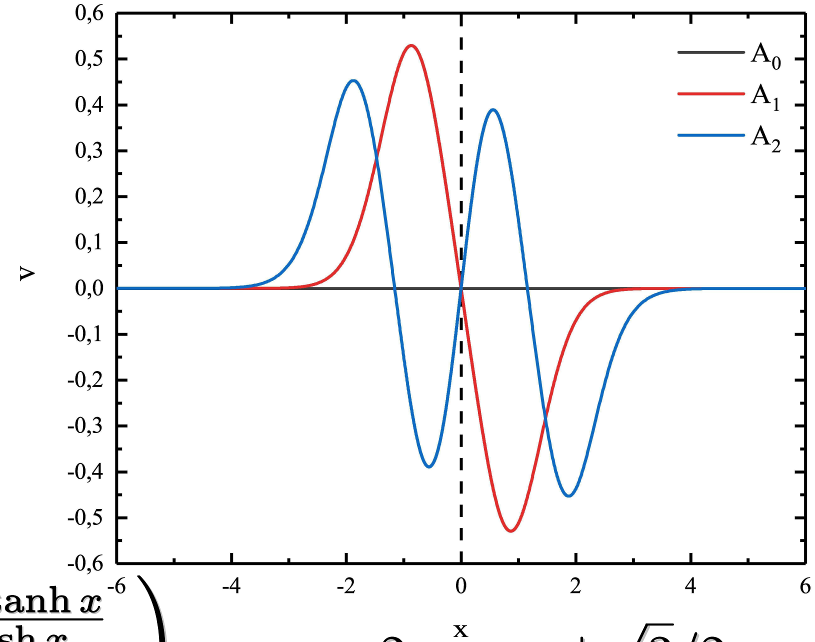
$$\psi_0 = \frac{1}{2}\begin{pmatrix} \frac{1}{\cosh x} \\ 0 \end{pmatrix}$$



## Fermionic modes

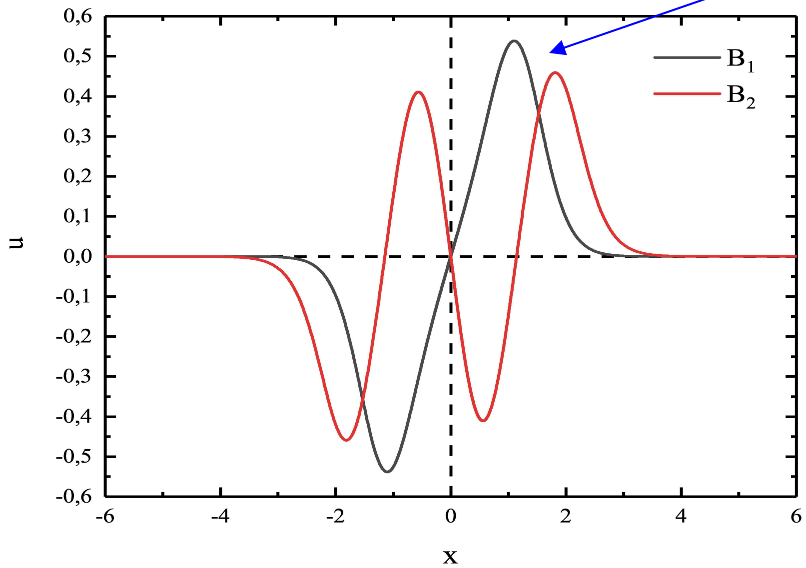


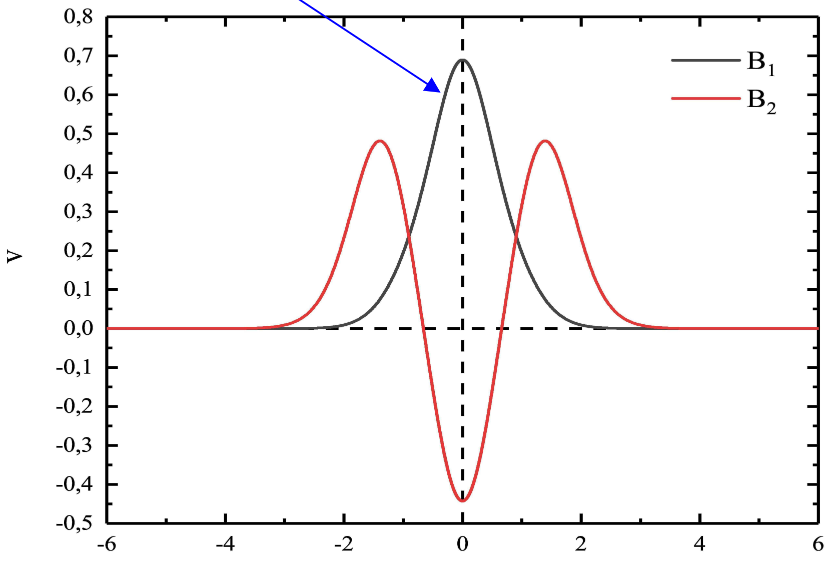




B$_1$ mode

$$\psi_1 = \frac{1}{2}\begin{pmatrix} \frac{\sqrt{3}\tanh x}{\cosh x} \\ \frac{1}{\cosh^2 x} \end{pmatrix}$$

$$g = 2, \quad \varepsilon = \pm\sqrt{3}/2$$



- **N=1 SUSY kink**

$$L = \frac{1}{2}(\partial_\mu\phi)^2 + i\bar{\psi}\gamma^\mu\partial_\mu\psi + g\phi\bar{\psi}\psi - \frac{1}{2}(\phi^2 - 1)^2$$

⇓

$$L_{N=1} = (\partial_\mu\phi)^2 + i\bar{\psi}\gamma^\mu\partial_\mu\psi + F^2 + 2FW - W'\bar{\psi}\psi$$

***F - auxiliary field:*** $F = -W$

⇓

$$L_{N=1} = (\partial_\mu\phi)^2 + i\bar{\psi}\gamma^\mu\partial_\mu\psi - W'\bar{\psi}\psi - W^2$$

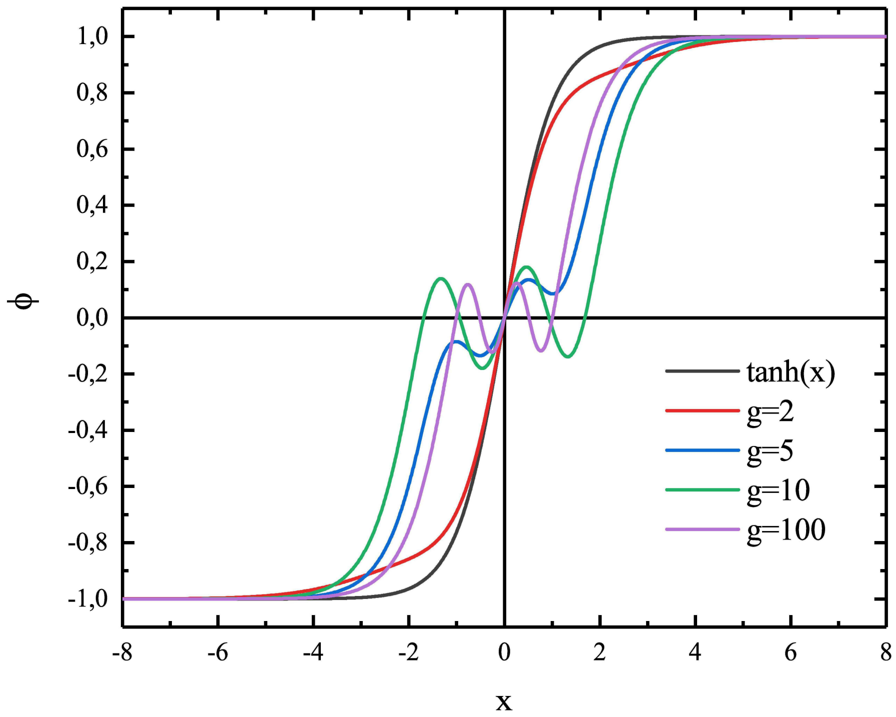
$$W[\phi] = \frac{1}{\sqrt{2}}(\phi^2 - 1)$$

- **SUSY transformations:** $\delta\phi = \eta\psi; \qquad \delta\psi = \eta(\gamma^\mu\partial_\mu\phi - W)$

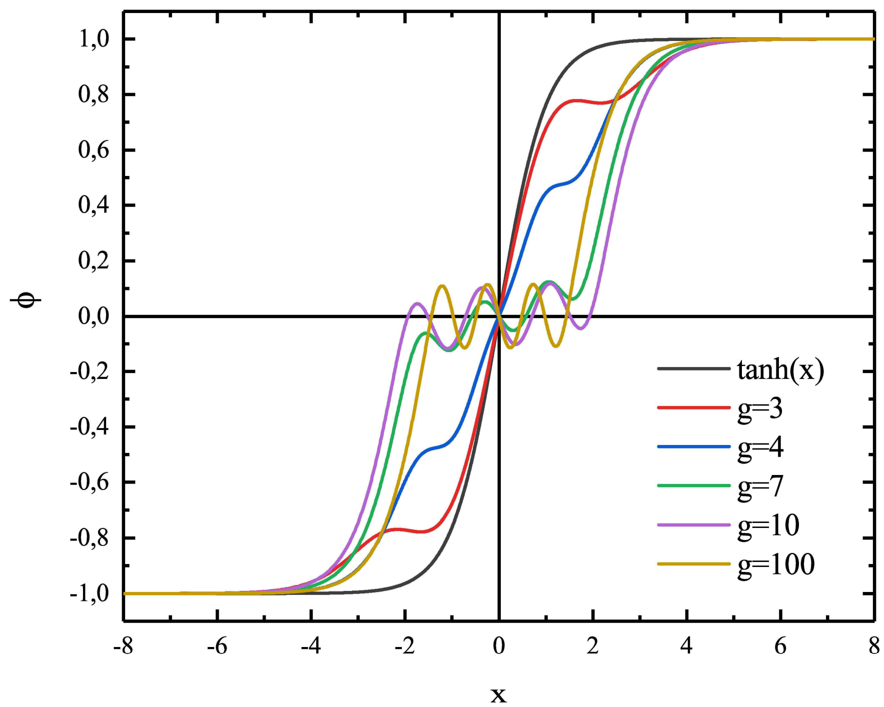
**Fermionic zero mode of the kink:**
**Grassmann-valued deformation of the bosonic field**

$$\psi_0 = \frac{1}{2}\begin{pmatrix} \frac{1}{\cosh x} \\ 0 \end{pmatrix}$$

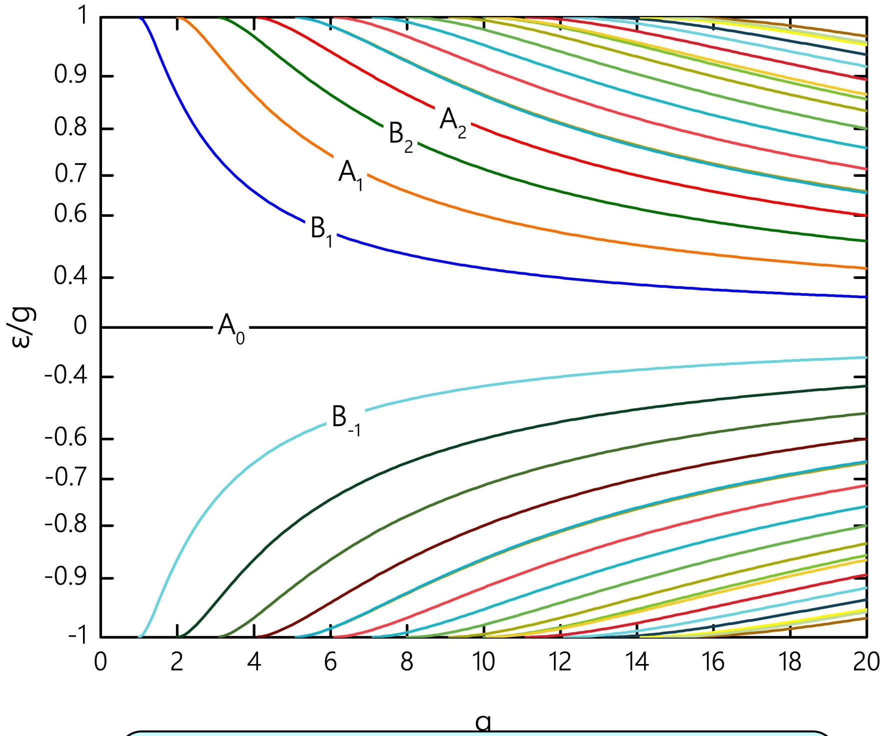
# Backreaction of the fermions

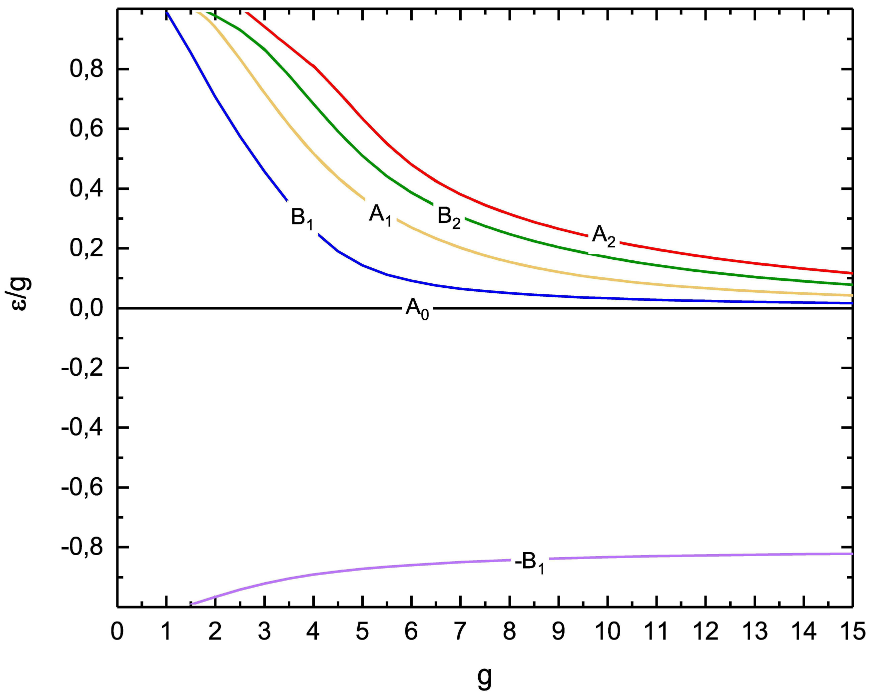



Kink + $A_1$ mode




Kink + $B_1$ mode




without backreaction

with backreaction

**Symmetry on a fixed background:** $x \to x, \quad \phi \to -\phi, \quad uv \to uv, \quad v \to u, \quad u \to v$

Backreaction breaks the symmetry of the spectral flow

# Kinks bounded by fermions

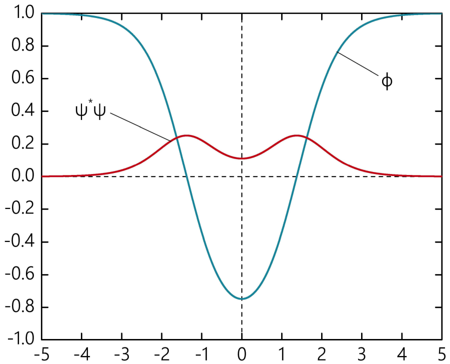
$$L = \frac{1}{2}(\partial_\mu \phi)^2 + i\bar{\psi}\gamma^\mu \partial_\mu \psi + g\phi\bar{\psi}\psi - U(\phi)$$

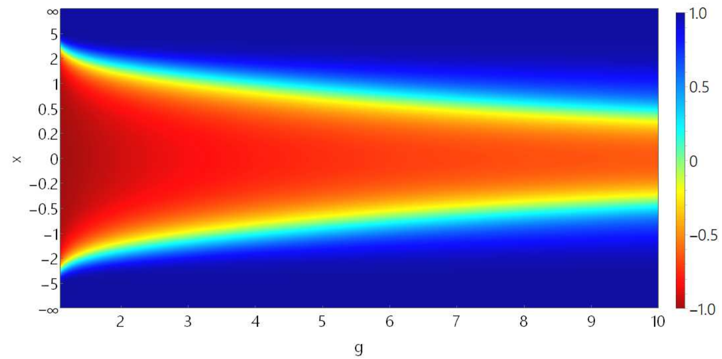
- **SG model:** $U(\phi) = 1 - \cos\phi$
- **$\phi^4$ model:** $U(\phi) = \frac{1}{2}\left(1-\phi^2\right)^2$

Kinks (decoupled limit g=0): $\phi_{SG} = 4\arctan e^x, \quad \phi_{\phi^4} = \tanh x$

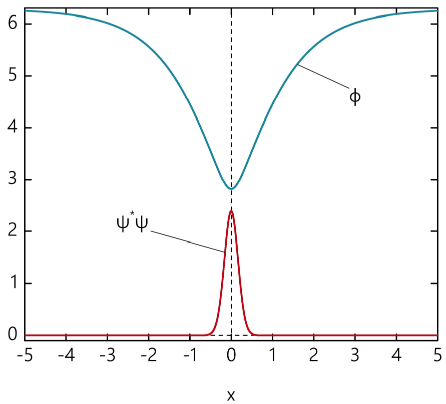
Bounded KK pair

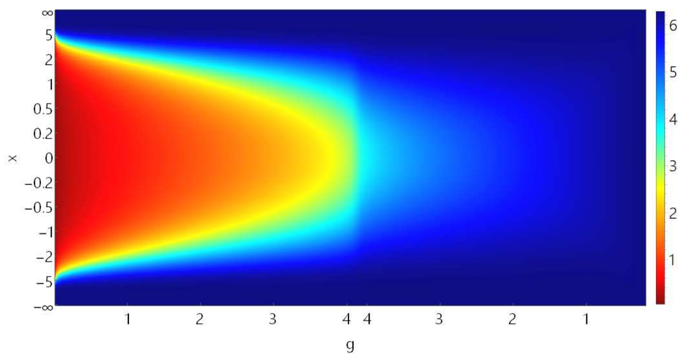
- **$\phi^4$ model:**




- **SG model:**



# Non-Abelian SU(2) monopole: fermionic zero mode

$$L_{YMH} = \frac{1}{2}\mathrm{Tr}\,(F_{\mu\nu}F_{\mu\nu}) - \mathrm{Tr}\,(D_\mu\Phi)^2 + \lambda\mathrm{Tr}\,(\Phi^2 - a^2)^2$$

$$D_\mu = \partial_\mu - igA_\mu^a\frac{\sigma^a}{2}$$

- **'t Hooft–Polyakov monopole:** $\Phi : S^2_\infty \mapsto S^2_{vac}, \quad \Pi_2(S^2) = \mathbb{Z}$ $\qquad \Phi = \phi^a\sigma^a$

$$\phi^a = \frac{r^a}{gr^2}H(r), \quad A_n^a = \varepsilon_{amn}\frac{r^m}{gr^2}[1 - W(r)], \quad A_0^a = 0$$

**+ fermions:**

$$L_{\rm sp} = \frac{i}{2}\left((\hat{\not{D}}\bar\psi)\psi - \bar\psi\hat{\not{D}}\psi\right) - m\bar\psi\psi - \frac{i}{2}h\bar\psi\gamma^5\phi\psi$$

$$\begin{cases} D_\nu F^{a\nu\mu} = -e\epsilon^{abc}\phi^b D^\mu\phi^c - \frac{e}{2}\bar\psi\gamma^\mu\sigma^a\psi\,, \\ D_\mu D^\mu\phi^a + \lambda\phi^a\left(\phi^2 - 1\right) + ih\bar\psi\gamma^5\sigma^a\psi = 0\,, \\ i\hat{\not{D}}\psi - i\frac{h}{2}\gamma^5\sigma^a\phi^a\psi - m\psi = 0 \end{cases}$$

- **spin-isospin fermions:**

$$\psi = e^{-i\omega t}\begin{pmatrix}\chi \\ \eta\end{pmatrix}$$

$$\int d^3x\psi^\dagger\psi = 1$$

$$\chi = \frac{u(r)}{\sqrt{2}}\begin{pmatrix}0 & -1 \\ 1 & 0\end{pmatrix}, \qquad \eta = i\frac{v(r)}{\sqrt{2}}\begin{pmatrix}\sin\theta e^{-i\varphi} & -\cos\theta \\ -\cos\theta & -\sin\theta e^{i\varphi}\end{pmatrix}.$$

**Two dimensionless parameters of the model:**

$$\beta = \frac{M_s}{M_v}, \quad h = \frac{2M_f}{M_v}$$

- **m=0** $$u' + u\left(\frac{1-W}{x} - \frac{h}{2}H\right) = 0, \quad v' + v\left(\frac{1+W}{x} + \frac{h}{2}H\right) = 0$$

- **BPS limit:** $\beta \to 0, \quad \hat{\phi}^a \underset{r\to\infty}{\to} \hat{r}^a$ ⟹ $W = \frac{x}{\sinh x}, \quad H = \coth x - \frac{1}{x}, \quad x = agr$

Generalized angular momentum: $$\vec{J} = \vec{L} + \vec{S} + \vec{T} = \vec{L} + \vec{\sigma} \otimes \mathbb{I} + \mathbb{I} \otimes \vec{\tau}$$

Spherical symmetry: $\vec{S} + \vec{T} = 0$

$$v = 0\,, \quad u \sim e^{-\int dx\left[\frac{1-W(x)}{x} - \frac{h}{2}H(x)\right]}$$

**Fermionic zero mode ($\omega$=0)**

- **BPS limit:** $v = 0\,, \quad u = \frac{1}{\cosh^2(x/2)} \qquad (h = -2)$

# Fermions+GR (Dirac stars)

H Weyl and V Fock (1929)

$$ds^2 = \eta_{ab}(\mathbf{e}^a_\mu dx^\mu)(\mathbf{e}^b_\nu dx^\nu) \qquad \gamma^\alpha = \mathbf{e}^\alpha_\mu \gamma^\mu$$

$$\mathcal{L}_{sp} = -i\frac{1}{2}\left(\gamma^\mu D_\mu \bar{\Psi}\Psi - \bar{\Psi}\gamma^\mu D_\mu \Psi\right) + \mu\bar{\Psi}\Psi$$

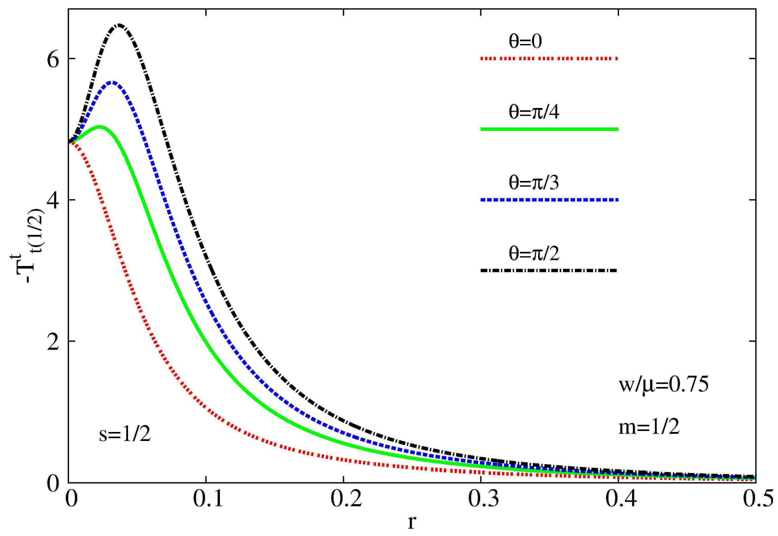
$$D_\mu \Psi = (\partial_\mu - \Gamma_\mu)\Psi$$

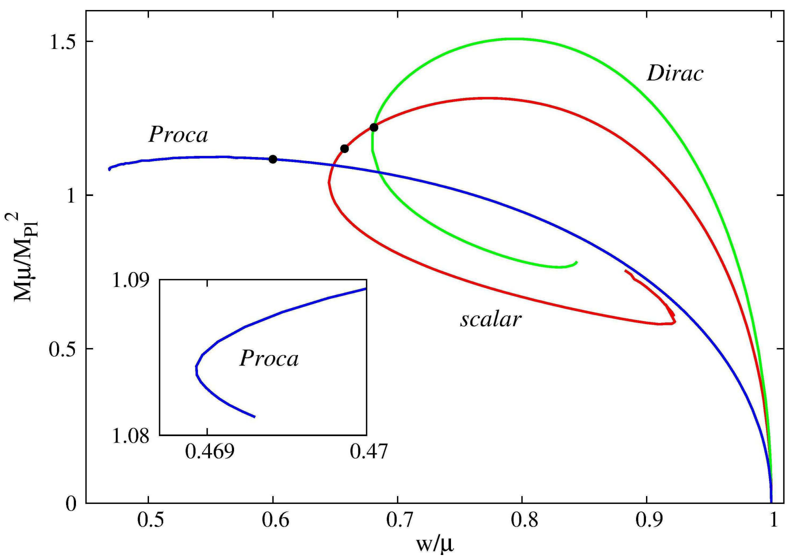
The spinor connection matrices

- **Fermionic current:** $j_\mu = \bar{\Psi}\gamma_\mu\Psi$
  - **Metric tetrad:** $\mathbf{e}^0_\mu dx^\mu = e^{F_0} dt\ , \quad \mathbf{e}^1_\mu dx^\mu = e^{F_1} dr\ ,$

$$\mathbf{e}^2_\mu dx^\mu = e^{F_1} r d\theta\ , \qquad \mathbf{e}^3_\mu dx^\mu = e^{F_2} r \sin\theta \left(d\varphi - \frac{W}{r} dt\right)$$

*(Herdeiro, Perapechka, Radu & Ya S 2019)*



# Localized Fermions+GR

Self-gravitating fermions?

Assumptions

- **only single-particle normalizible state is considered**
- **second quantization of the fields is ignored**
- **gravity is treated purely classically**

REVIEWS OF MODERN PHYSICS VOLUME 29, NUMBER 3 JULY, 1957

## Interaction of Neutrinos and Gravitational Fields

DIETER R. BRILL AND JOHN A. WHEELER

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### 1. INTRODUCTION; GRAVITATION THE ONLY FORCE IN WHICH NEUTRINOS ARE SUBJECT TO SIMPLE ANALYSIS

KNOWLEDGE of neutrinos to date is confined mainly to emission and absorption processes; that is, to the domain of elementary particle transformations. For comparison, imagine that one knew about electrons

handed polarization that are demanded by the recently gained knowledge.[1–3] Section 4 separates out the radial wave equation for the motion of a neutrino in a centrally symmetric gravitational field, and identifies one term in this equation with a spin-orbit coupling. Section 5 compares and contrasts the energy level spectrum in the case of spherical symmetry for (1) an electron in

# Non-Abelian self-gravtating monopole+fermions

$$S = \int d^4x \sqrt{-g}\left[\frac{R}{16\pi G} + \frac{1}{2}\mathrm{Tr}(F_{\mu\nu}F^{\mu\nu}) - \mathrm{Tr}(D_\mu\phi\; D^\mu\phi) + \lambda \mathrm{Tr}\left(\phi^2 - a^2\right)^2 + L_{\mathrm{sp}}\right.$$

$$L_{\mathrm{sp}} = \frac{i}{2}\left((\hat{\not{D}}\bar{\psi})\psi - \bar{\psi}\hat{\not{D}}\psi\right) - \frac{i}{2}h\bar{\psi}\gamma^5\phi\psi, \qquad \hat{D}_\mu\psi = (\partial_\mu - \Gamma_\mu + ieA_\mu)\psi$$

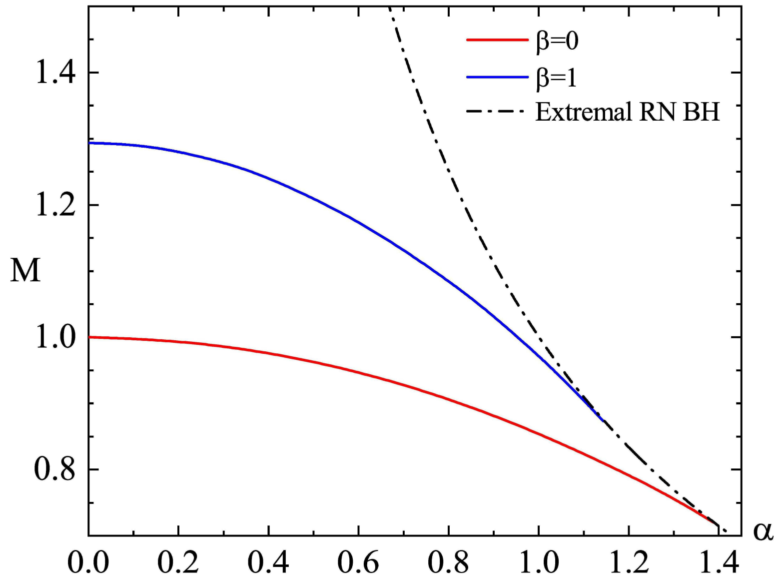
$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 8\pi G\left[(T_{\mu\nu})_{YM} + (T_{\mu\nu})_\phi + (T_{\mu\nu})_s\right]$$

**Three dimensionless parameters of the model:** $(\alpha^2 = 4\pi G a^2)$

$$\alpha = \sqrt{4\pi}\frac{M_v}{gM_{Pl}}, \quad \beta = \frac{M_s}{M_v}, \quad h = \frac{2M_f}{M_v}$$

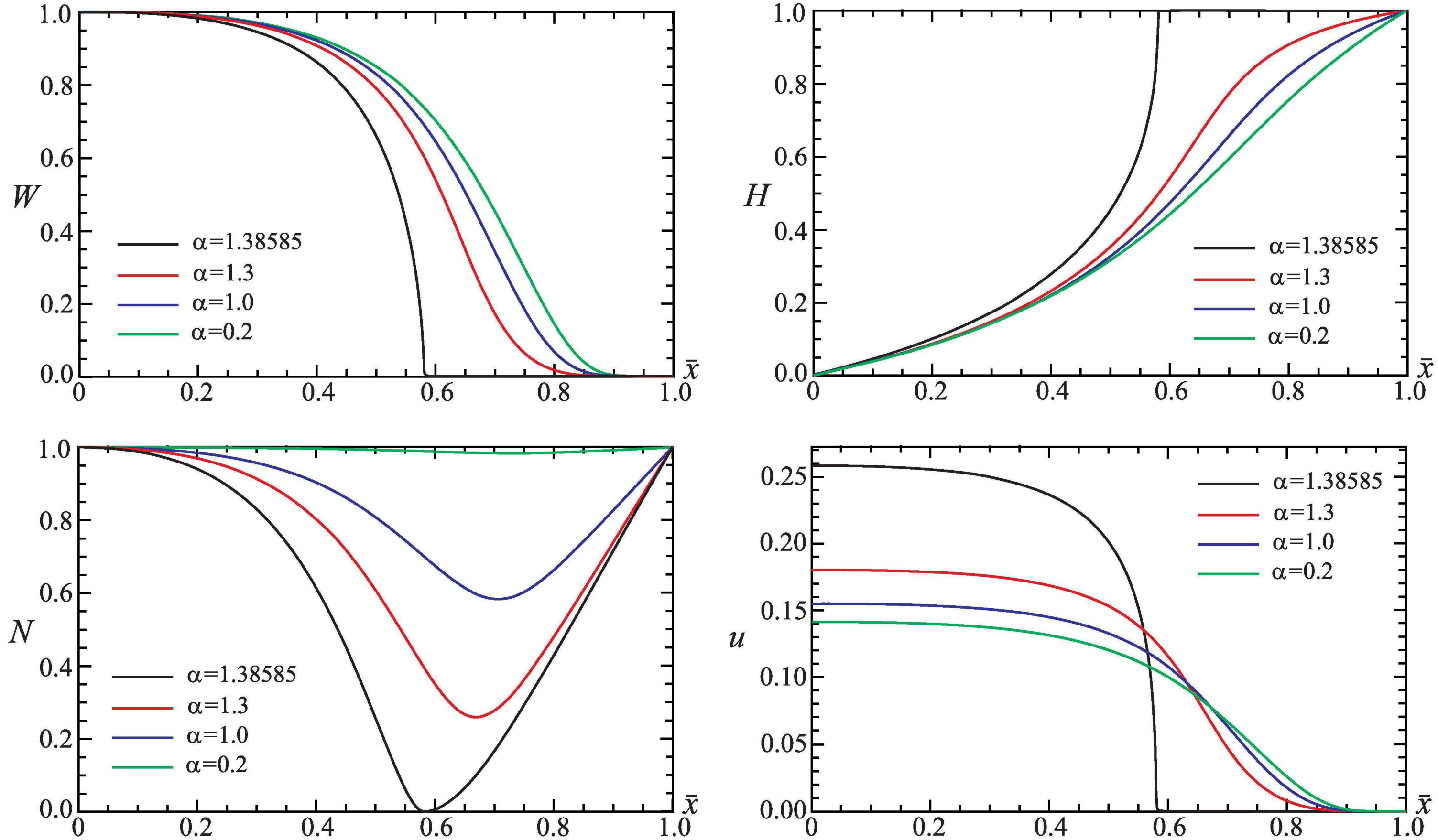
- **decoupled limit *h*=0**
**self - gravitating 't Hooft–Polyakov monopole**

*Breitenlohner, Forgacs, Maison (1992),*
*Lee, Nair, Weinberg (1992)*



Spherical symmetry:

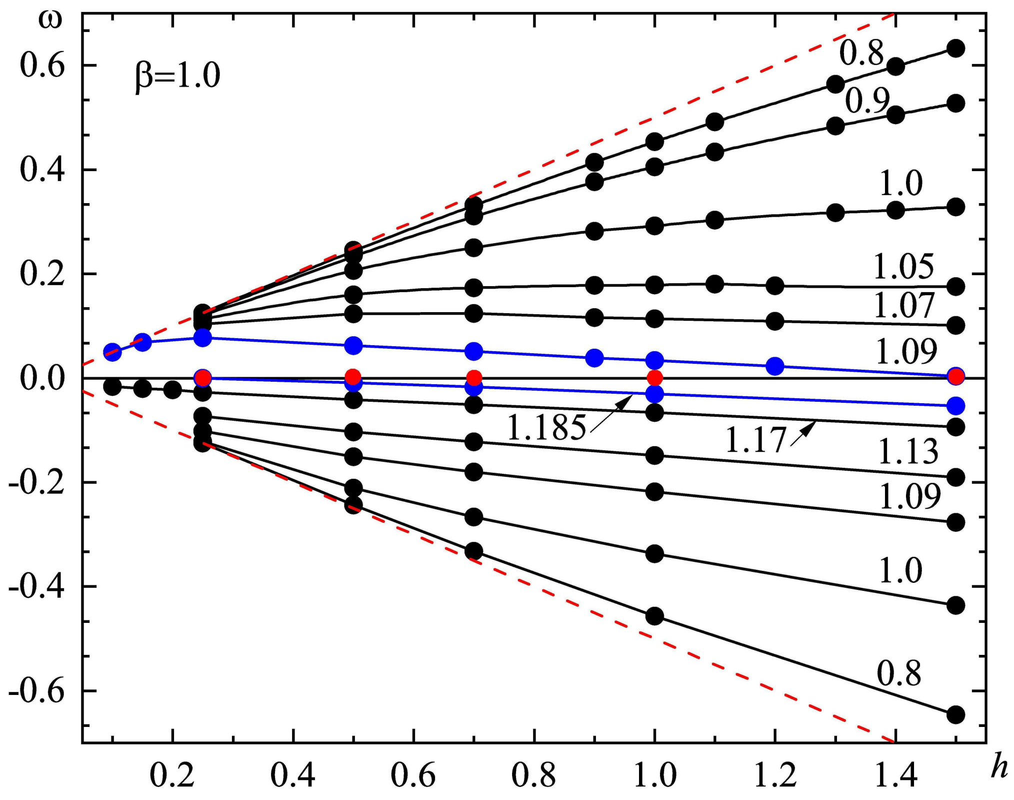
$$ds^2 = -\sigma^2(r)N(r)dt^2 + \frac{1}{N(r)}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)$$



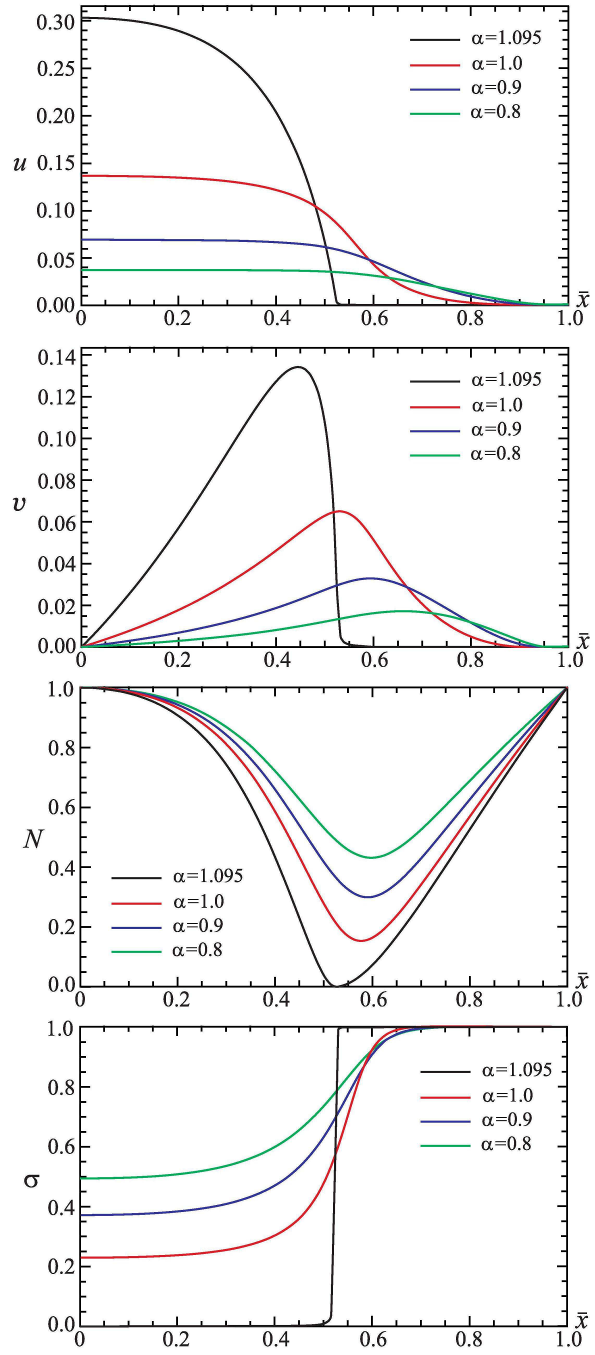

β=0, h=-1, ω=0

No fermion hair for RN BH

## Non-zero modes: $|\omega| < |h/2|$




β=1, h=1




No fermion hair for RN BH
(possible loophole: axially symmetric modes)

# Gravitating Skyrmions

$$S = \int \left\{ \frac{R}{\alpha^2} + \mathcal{L}_{Sk} \right\} \sqrt{-g} d^4x$$

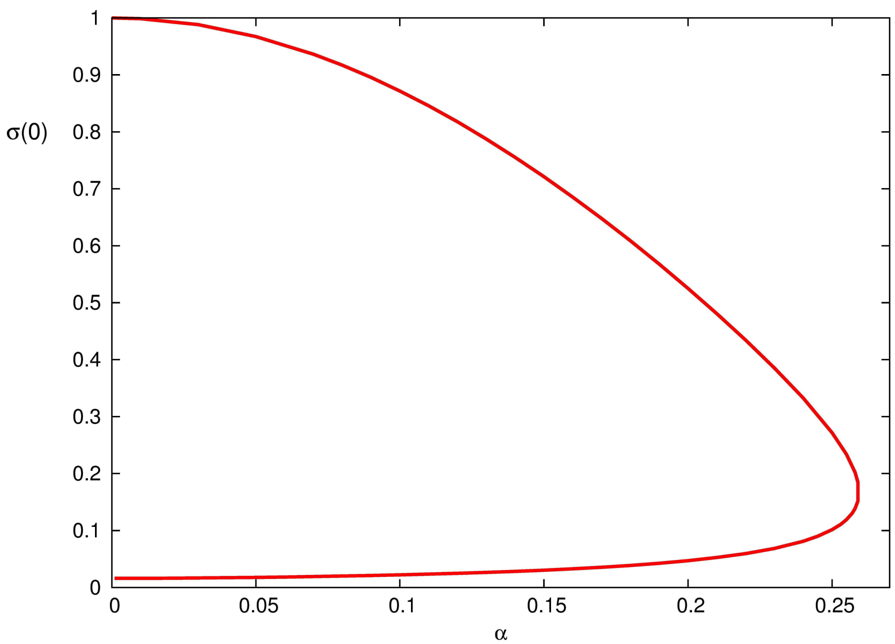
- **The Skyrme field:** $U(\vec{r}, t) \underset{r \to \infty}{\longrightarrow} \mathbb{I}$

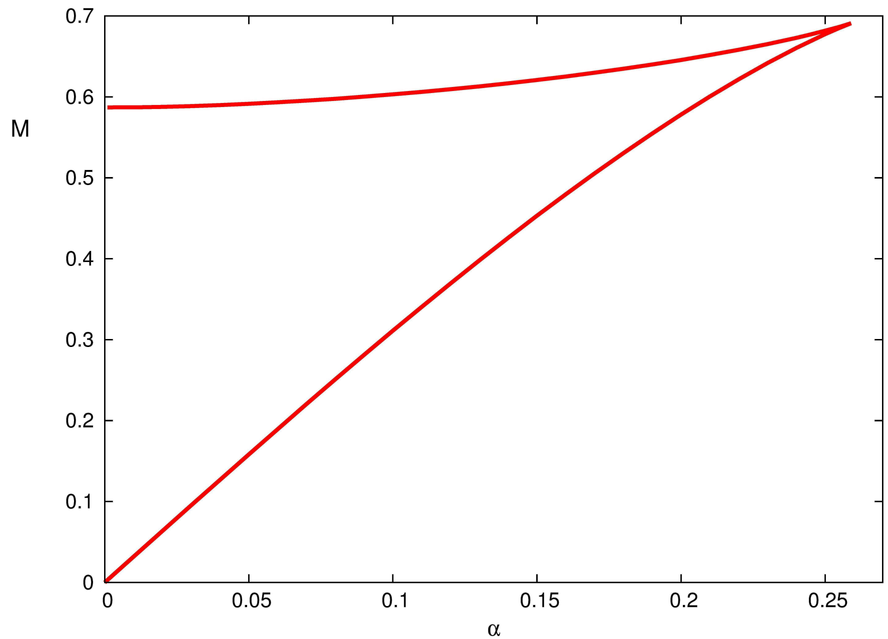
  $U: \ S^3 \to S^3$

Spherical symmetry:

$$ds^2 = -\sigma^2(r) N(r) dt^2 + \frac{1}{N(r)} dr^2 + r^2 (d\theta^2 + \sin^2\theta d\phi^2)$$

$$\mathcal{L}_{Sk} = \frac{1}{2} \text{Tr} \ \left(\partial_\mu U \partial^\mu U^\dagger\right) + \frac{1}{4} \text{Tr} \ \left(\left[U^\dagger \partial_\mu U, U^\dagger \partial_\nu U\right]^2\right) + m^2 \text{Tr} \ (U - \mathbb{I})$$



# Self-gravtating skyrmion+fermions

Yet another hedgehog $U(r) = \phi_0 + \phi^a \cdot \sigma^a = \cos F(r) + i\hat{n}^a \cdot \sigma^a \sin F(r)$

$$L_{Sk} = \partial_\mu \phi^a \partial^\mu \phi^a - \frac{1}{2}(\partial_\mu \phi^a \partial_\mu \phi^a)^2 + \frac{1}{2}(\partial_\mu \phi^a \partial_\nu \phi^a)(\partial^\mu \phi^b \partial^\nu \phi^b) - m^2(1 - \phi_0)$$

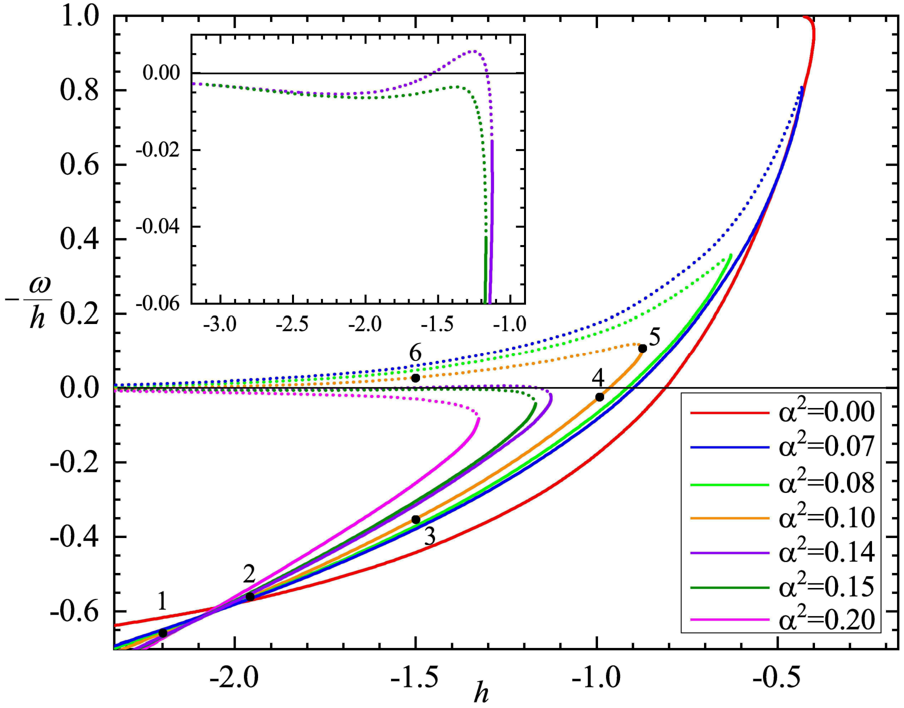
$$L_{\rm sp} = \frac{i}{2}\left((\hat{\not{D}}\bar{\psi})\psi - \bar{\psi}\hat{\not{D}}\psi\right) - h\bar{\psi}[\phi_0 + i\gamma_5(\phi^a \cdot \sigma^a)]\psi, \qquad \hat{D}_\mu \psi = (\partial_\mu - \Gamma_\mu)\psi$$

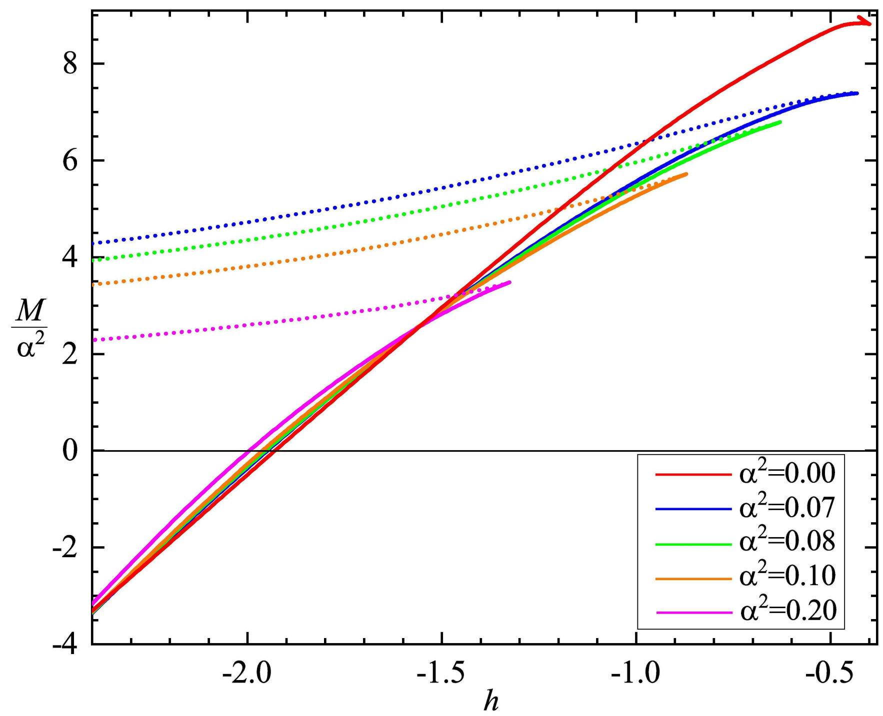
$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G \left[(T_{\mu\nu})_{Sk} + (T_{\mu\nu})_s\right]$$

$$T^{\mu\nu}_{\rm Sk} = 2\left[\partial^\mu \phi_a\, \partial^\nu \phi^a - \left(\partial^{[\mu} \phi^a\, \partial^{\alpha]} \phi^b\right)\left(\partial^{[\nu} \phi_a \partial_{\alpha]} \phi_b\right)\right] - g^{\mu\nu}\left[(\partial_\alpha \phi_a)^2 - \frac{1}{2}\left(\partial_{[\alpha} \phi_a\, \partial_{\beta]} \phi_b\right)^2\right]$$

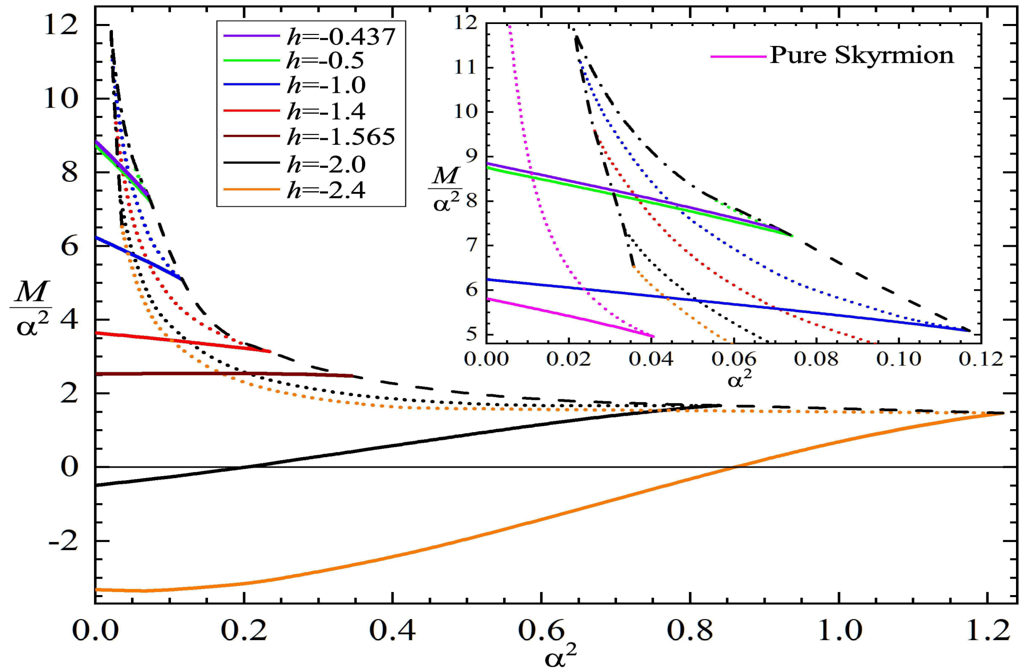
$$T^{\mu\nu}_{\rm sp} = \frac{i}{4}\left[\bar{\psi}\gamma^\mu(\hat{D}^\nu \psi) + \bar{\psi}\gamma^\nu(\hat{D}^\mu \psi) - (\hat{D}^\mu \bar{\psi})\gamma^\nu \psi - (\hat{D}^\nu \bar{\psi})\gamma^\mu \psi\right] - g^{\mu\nu}\mathcal{L}_{\rm s}$$

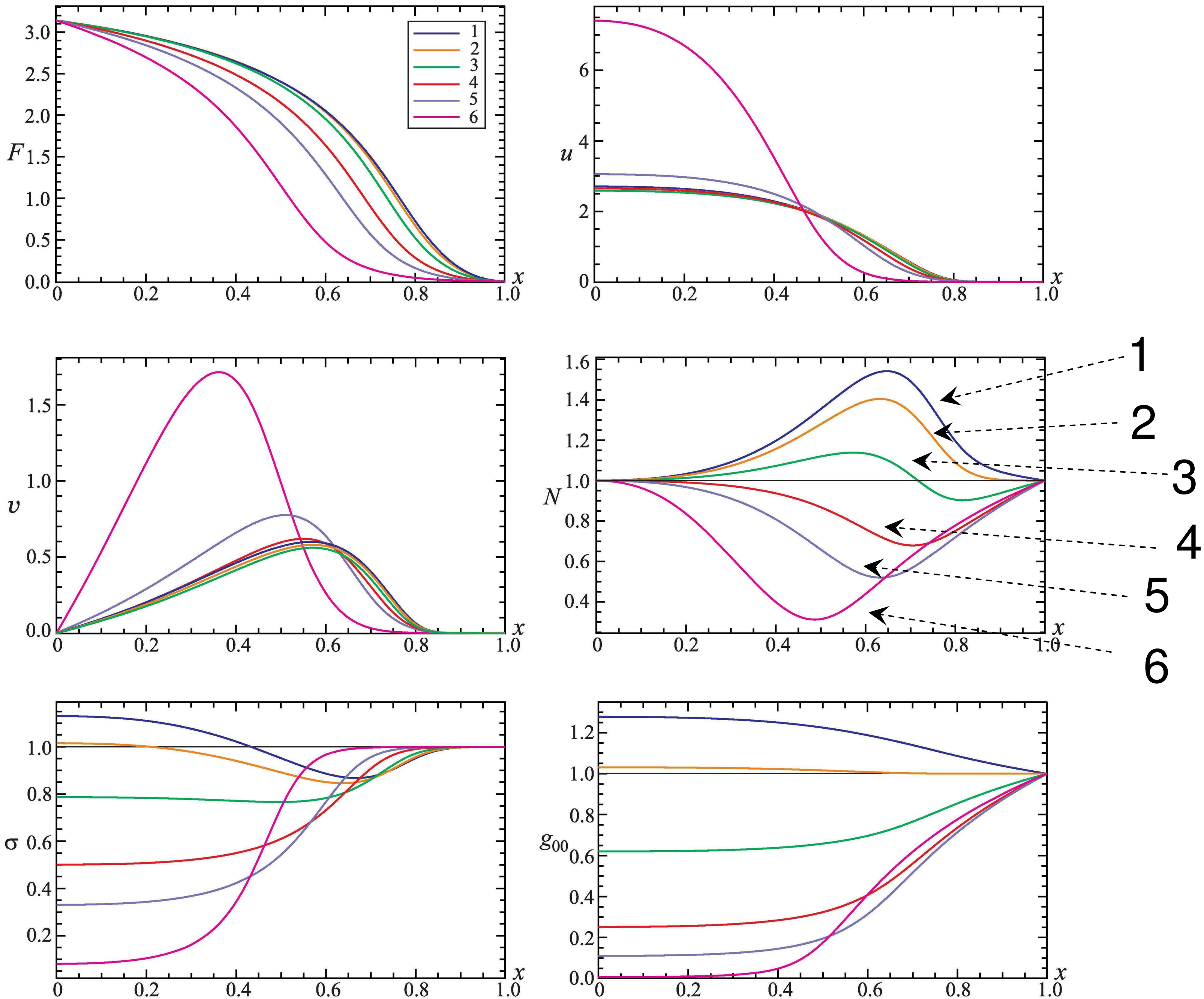
$$\psi = e^{-i\omega t}\begin{pmatrix}\chi \\ \eta\end{pmatrix} \quad \text{with} \quad \chi = \frac{u(r)}{\sqrt{2}}\begin{pmatrix}0 & -1 \\ 1 & 0\end{pmatrix}, \quad \eta = i\frac{v(r)}{\sqrt{2}}\begin{pmatrix}\sin\theta e^{-i\varphi} & -\cos\theta \\ -\cos\theta & -\sin\theta e^{i\varphi}\end{pmatrix}$$




$$\alpha^2 = 4\pi G f_\pi^2, \quad h \to h/(a_0 f_\pi)$$





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# Violation of the energy conditions

null and weak energy conditions:

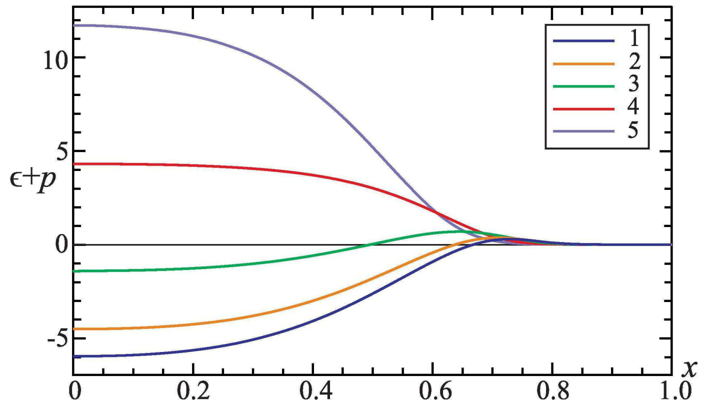
$$T_{\mu\nu}k^\mu k^\nu \geq 0 \quad \text{and} \quad T_{\mu\nu}V^\mu V^\nu \geq 0$$

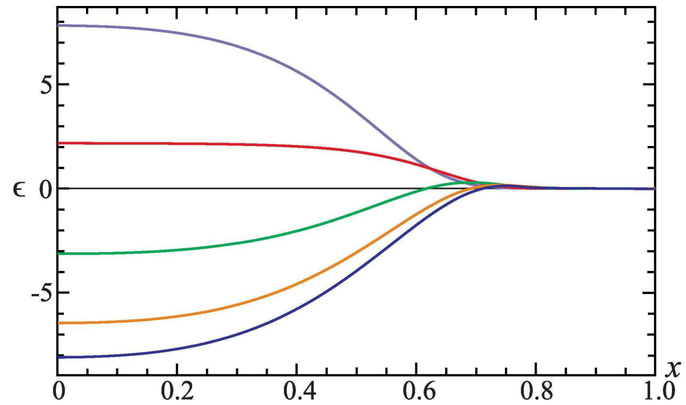
$$g_{\mu\nu}k^\mu k^\nu = 0, \qquad g_{\mu\nu}V^\mu V^\nu > 0$$

***Light-like vector*** ***timelike vector***

The null/weak energy conditions for gravitating Skyrmion-fermion system:

$$\epsilon + p \equiv T_0^0 - T_1^1 \geq 0, \quad \epsilon = T_0^0 > 0$$



# Summary

- Backreaction of the localized fermions may strongly affect the solitons inself, it breack the symmetry of the solutions.
- Localization of the fermions produces additional channels of interaction between the solitons, it may bound solitons with repulsive scalar interactions
- Dynamics of the solitons with localized fermionic modes?
- There are spinning Dirac stars, they possess non-zero angular momentum J=nQ with half-integer n
- The fermion zero mode localized on the gravitating monopole is fully absorbed into the interior of the forming RN black hole
- Localization of the backreacting fermionic mode on a self-gravitating Skyrmion violates energy conditions, configuration may possess a negative ADM mass
- No-go for BHs with fermionic hairs in asymptotically flat 3+1 dim?
- Other examples of violation of energy conditions related to self-gravitating fermions localized on a soliton?

# Thank you!