Closed 4-braids and the Jones unknot conjecture

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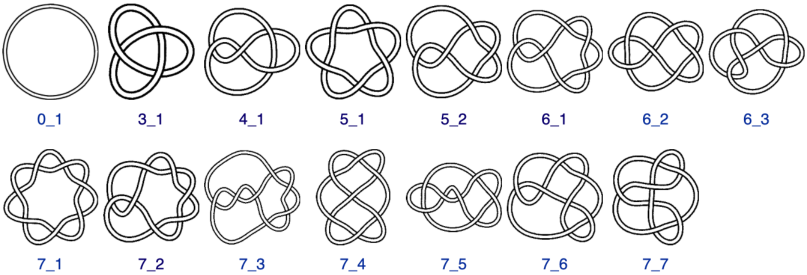
Content

Plan of my talk:

1. Introduction and brief review
2. HOMFLY polynomial $H^{\mathcal{K}}(q, A)$ and braid representation π
3. Braid group B_4, fundamental representation
4. Let $H^{\mathcal{K}}(q, A) \neq 1$. Does there exist $\mathcal{K} : J^{\mathcal{K}}(q) = 1$?

Knots classification

A knot is an embedding of a circle into three-dimensional Euclidean space making without self-intersections.



Knot invariants

Knot invariant I :

$$I : \mathcal{K} \to \mathbb{C}, \text{ or } \mathrm{Pol}(q, a), \ldots \quad (1)$$

if for equivalent knots, it holds:

$$I(\mathcal{K}) = I(\mathcal{K}') \quad (2)$$

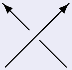
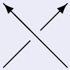
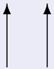
The main property of **invariants**:

$$\boxed{I(\mathcal{K}) \neq I(\mathcal{K}') \quad \Longrightarrow \quad \mathcal{K} \neq \mathcal{K}'} \quad (3)$$

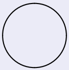
Jones polynomial

Definition

The Jones polynomial $J(q)$ is a one variable knot invariant, defined by the following skein relation:

$q^2 \cdot J\Big($  $\Big) - q^{-2} \cdot J\Big($  $\Big) = (q - q^{-1}) \cdot J\Big($  $\Big)$ (4)

with the requirement for the unknot:

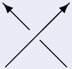
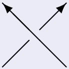
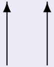
$J\Big($  $\Big) = 1.$ (5)

Jones unknotting problem. Does there exist a nontrivial knot with the trivial Jones polynomial?

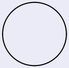
HOMFLY polynomial

Definition

The HOMFLY polynomial $H(q, A)$ is a two variable knot invariant that generalizes both the Alexander and the Jones polynomials. It can be defined with the use of the following skein relation:

$A \cdot H\Big($  $\Big) - A^{-1} \cdot H\Big($  $\Big) = (q - q^{-1}) \cdot H\Big($  $\Big)$, (6)

with the requirement for the unknot:

$H\Big($  $\Big) = 1.$ (7)

$J(q) = H(q, A = q^2)$. The same unknotting problem can be formulated for the HOMFLY polynomial.

Some known results

1. The HOMFLY polynomial is not a complete knot invariant. For example, up to 10 crossings there are 5 pairs: $H^{5_1} = H^{10_{132}}$, $H^{8_8} = H^{10_{129}}$, $H^{8_{16}} = H^{10_{156}}$, $H^{10_{25}} = H^{10_{56}}$, $H^{10_{40}} = H^{10_{103}}$. The HOMFLY polynomial is more powerful than the Jones polynomial: $J^{10_{41}} = J^{10_{94}}$, but $H^{10_{41}} \neq H^{10_{94}}$.

2. **Anstee-Przytycki-Rolfsen** (1989) and **Jones-Rolfsen** (1994): suggested generalized mutations, which preserve Jones polynomial.

3. **Tuzun-Sikora** (2021): up to 24 crossings Jones detects the unknot.

4. **Thislethwaite** (2001): $\exists\, \mathcal{L} \neq \bigcirc\bigcirc : \; J^{\mathcal{L}} = J^{\bigcirc\bigcirc}$; **Eliahou-Kauffman-Thislethwaite** (2003): extension to the k-component links.

5. **Kronheimer-Mrowka** (2011): Khovanov homology detects the unknot.

Quantum sl_N braid representation

Definition

The braid group on m strands is
$B_m = \{\sigma_1, \ldots, \sigma_{m-1} \Big| \sigma_i\sigma_j = \sigma_j\sigma_i,\ |i-j| \geq 2;\ \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}\}$.

N.Yu. Reshetikhin and V.G. Turaev (1987-1990) constructed its representation with the use of quantum algebras $U_q(sl_N)$:

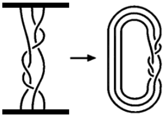
$$\begin{array}{rcl} \pi : B_m & \to & \mathrm{End}(V_1 \otimes \ldots \otimes V_m) \\ \pi(\sigma_i) & = & \mathcal{R}_i, \end{array} \tag{8}$$

where V_i are irr. f.-d. representations of $U_q(sl_N)$ and

$$\mathcal{R}_i = 1_{V_1} \otimes \ldots \otimes \check{\mathcal{R}}_{i,i+1} \otimes \ldots \otimes 1_{V_m} \quad \in \mathrm{End}(V_1 \otimes \ldots \otimes V_m), \tag{9}$$

where $\check{\mathcal{R}}_{i,i+1} : V_i \otimes V_{i+1} \to V_{i+1} \otimes V_i$ is the universal R-matrix.

Quantum sl_N knot invariants



Any knot is the closure of the corresponding braid $\mathcal{K} = \hat{\beta}$.
Reshetikhin–Turaev defined quantum knot invariants:

$$H_\lambda^{\mathcal{K}}(q, A = q^N) = \frac{\left(A^{|\lambda|} q^{4c_2(\lambda)}\right)^{-w(\mathcal{K})}}{H_\lambda^{\bigcirc}(q, A)} \cdot {}_q\mathrm{tr}_{V_1 \otimes \cdots \otimes V_m} \left(\pi(\beta^{\mathcal{K}}) \right). \tag{10}$$

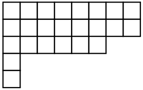
where $V_1 = \ldots = V_m = V_\lambda$, because a knot has only 1 component; $w(\mathcal{K})$ is the algebraic number of crossings (writhe number); $c_2(\lambda)$ is the eigenvalue of the quadratic Casimir element of sl_N; $|\lambda|$ is the number of boxes in the corresponding Young diagram.

This invariant is called the colored HOMFLY polynomial or quantum sl_N invariant or Reshetikhin-Turaev invariant. In the simplest case $\lambda = \Box$ it is the usual HOMFLY polynomial defined via the skein relation.

Quantum sl_N knot invariants

F.-d. irreps of highest weights are enumerated by Young diagrams:

$$\mu = [r_1 \geq r_2 \geq \ldots \geq r_{N-1}] =$$



The decomposition in irreducible representations for quantum knot invariants – the so called character expansion:

$$H_\lambda^{\mathcal{K}}(q, A = q^N) \;\sim\; {}_q\mathrm{tr}_{V_\lambda^{\otimes m}}\left(\pi(\beta^{\mathcal{K}})\right) = \sum_\mu \mathrm{tr}_\mu\left(\pi(\beta^{\mathcal{K}})\right) \cdot \mathrm{qdim}\left(V_\mu\right). \quad (11)$$

HOMFLY polynomial for 4-strand knots

Let $\lambda = [1]$, then $V_{[1]}^{\otimes 4} = V_{[4]} \oplus 3V_{[3,1]} \oplus 2V_{[2,2]} \oplus 3V_{[2,1,1]} \oplus V_{[1,1,1,1]}$. In this case, the HOMFLY polynomial has 5 corresponding summands:

$$\begin{aligned} H_{[1]}^{\mathcal{K}}(q,A) &= \frac{q-q^{-1}}{A-A^{-1}} A^{-w(\mathcal{K})} \times \\ &\times \Big(\mathrm{tr}_{[4]} \left(\pi(\beta^{\mathcal{K}})\right) \mathrm{qdim}\left(V_{[4]}\right) + \mathrm{tr}_{[3,1]} \left(\pi(\beta^{\mathcal{K}})\right) \mathrm{qdim}\left(V_{[3,1]}\right) + \\ &+ \mathrm{tr}_{[2,2]} \left(\pi(\beta^{\mathcal{K}})\right) \mathrm{qdim}\left(V_{[2,2]}\right) + \mathrm{tr}_{[2,1,1]} \left(\pi(\beta^{\mathcal{K}})\right) \mathrm{qdim}\left(V_{[2,1,1]}\right) + \\ &+ \mathrm{tr}_{[1,1,1,1]} \left(\pi(\beta^{\mathcal{K}})\right) \mathrm{qdim}\left(V_{[1,1,1,1]}\right) \Big). \end{aligned} \tag{12}$$

It is known that $V_\lambda \simeq V_{\lambda^t}$, $q \to -q^{-1}$. Therefore, in order to determine $H_{[1]}^{\mathcal{K}}(q,A)$ we have to determine only 3 terms

$$\mathrm{tr}_{[4]} \left(\pi(\beta^{\mathcal{K}})\right), \quad \mathrm{tr}_{[3,1]} \left(\pi(\beta^{\mathcal{K}})\right), \quad \mathrm{tr}_{[2,2]} \left(\pi(\beta^{\mathcal{K}})\right).$$

Explicit R-matrices

Recall the decomposition $V_{[1]}^{\otimes 4} = V_{[4]} \oplus 3V_{[3,1]} \oplus 2V_{[2,2]} \oplus 3V_{[2,1,1]} \oplus V_{[1,1,1,1]}$

$$[4]:\pi(\sigma_1)=\pi(\sigma_2)=\pi(\sigma_3)=q \quad\Rightarrow\quad \mathrm{tr}_{[4]}\left(\pi(\beta^{\mathcal{K}})\right)=q^{w(\mathcal{K})} \tag{13}$$

$$[3,1]:\pi(\sigma_1)=\begin{pmatrix} q & 0 & 0\\ 0 & q & 0\\ 0 & 0 & -\frac{1}{q}\end{pmatrix},\ \pi(\sigma_2)=\begin{pmatrix} q & 0 & 0\\ 0 & -\frac{1}{q^2[2]_q} & -\frac{\sqrt{[3]_q}}{[2]_q}\\ 0 & -\frac{\sqrt{[3]_q}}{[2]_q} & \frac{q^2}{[2]_q}\end{pmatrix}$$

$$\pi(\sigma_3)=\begin{pmatrix} -\frac{1}{q^3[3]_q} & -\frac{[2]_q\sqrt{q^2+q^{-2}}}{[3]_q} & 0\\ -\frac{[2]_q\sqrt{q^2+q^{-2}}}{[3]_q} & \frac{q^3}{[3]_q} & 0\\ 0 & 0 & q\end{pmatrix} \tag{14}$$

$$[2,2]:\pi(\sigma_1)=\pi(\sigma_3)=\begin{pmatrix} q & 0\\ 0 & -\frac{1}{q}\end{pmatrix},\ \pi(\sigma_2)=\begin{pmatrix} -\frac{1}{q^2[2]_q} & -\frac{\sqrt{[3]_q}}{[2]_q}\\ -\frac{\sqrt{[3]_q}}{[2]_q} & \frac{q^2}{[2]_q}\end{pmatrix} \tag{15}$$

R-matrices for $[2,2]$ are equal to R-matrices for $[2,1]$ on 3 strands. (16)

Nontrivial HOMFLY on B_4

Consider $\beta \in B_4$, $\mathcal{K} = \hat{\beta}$. Suppose $H^{\mathcal{K}}(q, A) \neq 1$. Is it possible $J^{\mathcal{K}}(q) = 1$?

Theorem (Korzun-Lanina-Sleptsov'2024)

There does not exist a 4-strand knot with the trivial Jones and Alexander polynomials and the non-trivial HOMFLY polynomial.

Nontrivial HOMFLY on B_4

Theorem (KLS'2024)

If there exist 4-strand knots with the trivial Jones polynomial and the non-trivial HOMFLY polynomial, then the HOMFLY polynomials of such knots have the following form:

$$H^{\mathcal{K}}(q,A) = 1 + (A^2 - q^2)(A^2 - q^{-2})(A^2 - q^4)(A^2 - q^{-4}) \cdot F^{\mathcal{K}}(q,A) \quad (17)$$

with the following functions:

$$F^{\mathcal{K}_m} = -\sum_{n=0}^{m} \frac{[2n+4]_q [n+3]_q [n+1]_q}{[3]_q [4]_q} A^{-2n-8} \quad (18)$$

with an arbitrary non-negative integer number m.

MFW inequality: $\frac{max(deg_A H) - min(deg_A H)}{2} \leq b^{\mathcal{K}} - 1.$

Eigenvalues

Theorem (Wenzl'1992)

3-strand R-matrices (for $[2,2]$*) are unitary at the points* $q = \exp\frac{2\pi i}{k}$ *for* $|k| = 1, 2$ *and* $|k| \geq 6$*.*
4-strand R-matrices (for $[3,1]$*) are unitary at the points* $q = \exp\frac{2\pi i}{k}$ *for* $|k| = 1, 2$ $|k| \geq 8$*.*

Corollary

Since all R-matrices for 4-braid are unitary at the particular values $q = \exp\frac{2\pi i}{k}$ *for* $|k| = 1, 2$ $|k| \geq 8$*, the corresponding representation* $\pi(\beta)$ *for any braid* β *is also unitary at these points. Hence, its eigenvalues lie on the unit circle:* $|\lambda_i| = 1$*.*

Character expansion

Let us present the obtained conjectural HOMFLY polynomial

$$H^{\mathcal{K}}(q,A) = 1 + (A^2 - q^2)(A^2 - q^4)(A^2 - q^{-2})(A^2 - q^{-4}) \cdot F^{\mathcal{K}}(q,A),$$

$$F^{\mathcal{K}_m} = -\sum_{n=0}^{m} \frac{[2n+4]_q [n+3]_q [n+1]_q}{[3]_q [4]_q} A^{-2n-8} \qquad (19)$$

in the form

$$\begin{aligned} H^{\mathcal{K}}(q,A) &= \frac{q-q^{-1}}{A-A^{-1}} A^{-w(\mathcal{K})} \times \\ &\times \Big(\mathrm{tr}_{[4]}\left(\pi(\beta^{\mathcal{K}})\right) \mathrm{qdim}\left(V_{[4]}\right) + \mathrm{tr}_{[3,1]}\left(\pi(\beta^{\mathcal{K}})\right) \mathrm{qdim}\left(V_{[3,1]}\right) + \\ &+ \mathrm{tr}_{[2,2]}\left(\pi(\beta^{\mathcal{K}})\right) \mathrm{qdim}\left(V_{[2,2]}\right) + \mathrm{tr}_{[2,1,1]}\left(\pi(\beta^{\mathcal{K}})\right) \mathrm{qdim}\left(V_{[2,1,1]}\right) + \\ &+ \mathrm{tr}_{[1,1,1,1]}\left(\pi(\beta^{\mathcal{K}})\right) \mathrm{qdim}\left(V_{[1,1,1,1]}\right) \Big). \end{aligned} \qquad (20)$$

Character expansion

Let us denote $\mathrm{tr}_\mu\left(\pi(\beta^{\mathcal{K}})\right)$ by a_μ for simplicity. As a result, we get

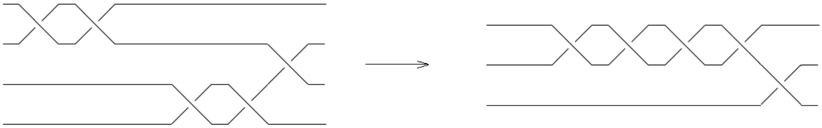
$$
\begin{aligned}
a_{[4]} &= q^{2m+5}, \\
a_{[1,1,1,1]} &= -q^{-2m-5}, \\
a_{[2,1,1]} &= -\frac{1+q^2 \;-\; q^{2+2m}\left(1+q^2+q^4+q^6+q^8\right) \;+\; a_{[2,2]}q^{11+4m}}{q^{9+4m}(1-q+q^2)(1+q+q^2)}, \\
a_{[3,1]} &= -\frac{a_{[2,2]}q^2 \;+\; q^{2m+3}\left(1+q^2+q^4+q^6+q^8\right) \;-\; q^{11+4m}\left(1+q^2\right)}{(1-q+q^2)(1+q+q^2)}.
\end{aligned}
\tag{21}
$$

A simple observation is the fact that our hypothetical 4-strand knots of the trivial Jones polynomial have the odd writhe number w with the possible minimal value $w_{\min} = 5$.

Comments on parametrization

We express $a_{[2,1,1]}$ and $a_{[3,1]}$ through $a_{[2,2]}$. The coefficient $a_{[2,2]}$ for a knot $\mathcal{K} = \hat{\beta}$ on 4 strands is equal to the coefficient $a_{[2,1]}$ for a knot $\mathcal{K}' = \hat{\beta}' = \hat{\beta}\big|_{\sigma_3 \to \sigma_1}$ on 3 strands (see example below). In other words, considering coefficient $a_{[2,2]}$ we effectively deal with 3-strand braids. Since w is odd, the closure of such 3-strand braids always gives a 2-component link.

Therefore, the coefficient $a_{[2,2]}$ is parameterized by 2-component links on 3-strands with the writhe number $w = 2m + 5$, $m = 0, 1, 2, \ldots$ The simplest example is provided by the torus link $T[2,4]$ that can be represented as the closure of the 3-strand braid $(4,1)$. This diagram of the torus link $T[2,4]$ has $w = 5$, $m = 0$.



System of equations on eigenvalues

So, we choose an arbitrary 2-link on 3-strand braid and fix $a_{[2,2]}$. Then we find

$$a_{[3,1]} = -\frac{a_{[2,2]}q^2 \; + \; q^{2m+3}\left(1+q^2+q^4+q^6+q^8\right) \; - \; q^{11+4m}\left(1+q^2\right)}{(1-q+q^2)(1+q+q^2)} \tag{22}$$

$$a_{[3,1]} \equiv \mathrm{tr}_{[3,1]}\left(\pi(\beta^{\mathcal{K}})\right) = \lambda_1(q)+\lambda_2(q)+\lambda_3(q), \tag{23}$$

$$[3,1] : \det\left(\pi(\sigma_i)\right) = (-q), \;\; i = 1,2,3. \tag{24}$$

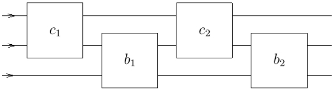
For representation $[3,1]$, the system takes the following form:

$$\begin{cases} \lambda_1(q)+\lambda_2(q)+\lambda_3(q) = a_{[3,1]}(q), \\ \lambda_1(q)\cdot\lambda_2(q)\cdot\lambda_3(q) = (-q)^{w(\mathcal{K})}, \\ \lambda_1^{-1}(q)+\lambda_2^{-1}(q)+\lambda_3^{-1}(q) = a_{[3,1]}(q^{-1}). \end{cases} \tag{25}$$

The third equation follows from the property $\pi(\sigma_i)^{-1}(q) = \pi(\sigma_i)(q^{-1})$.

Computer search

We considered 3-strand braids of the type $(c_1, b_1; c_2, b_2)$, $-7 \leq c_1, b_1, c_2, b_2 \leq 7$, with the writhe number $w = 2m + 5$, $m = 0, \dots, 5$. We computed absolute values of eigenvalues $\lambda_i(q)$, $q = \exp \frac{2\pi i}{k}$ for all integer values of $k \in [8, 150]$ numerically with the help of Wolfram Mathematica. It turns out that the condition $|\lambda_i| = 1$ for all values of k is fulfilled only for the following braids:



$(-5, 2; 6, 2)$, $(-2, -1; 1, 7)$, $(-2, -1; 3, 5)$, $(-2, 2; 3, 2)$, $(-1, 1; 2, 3)$, $(-2, 2; 1, 4)$. Hypothetically, each of them can correspond to a knot on 4 strands for which the HOMFLY polynomial coincides with our formula (17) for $m = 0$.

Thank you for your attention!