β-deformed $\widetilde{W}$ algebras

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May 21, 2024

Outline

Motivation

Matrix Models 101

Matrix Models capture the most interesting theoretical properties of generic field/string theory and allow to

- study the "non-perturbative" properties of integrals

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- study the "non-perturbative" properties of integrals
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- avoid additional complexities of functional integration

Matrix Models 101

Example

Partition function

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and potential

$$V(X) = \sum_{i \geq 0} t_k X^k \tag{3}$$

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$$L_n Z(\mathbf{t}) = 0, \qquad n \geq -1, \tag{3}$$

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where

$$L_n = \sum_{k\geq 0} k t_k \frac{\partial}{\partial t_{n+k}} + \sum_{k=0}^{n} \frac{\partial^2}{\partial t_k \partial t_{n-k}}. \tag{3}$$

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$$[L_n, L_m] = (n-m)L_{n+m}. \tag{3}$$

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where L_n-s form a simplest $\widetilde{W}$ algebra of constraints , which can be solved, giving W-representation of matrix model

$$Z(\mathbf{t}) = e^{W(\mathbf{t})} \cdot 1. \tag{3}$$

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- W-representation

$\widetilde{W}$ algebras

Two-matrix models possess W-representation

$$Z_n(\mathbf{p}) = \exp \operatorname{tr} \left(\Lambda \frac{\partial}{\partial \Lambda} \Lambda \right)^n \cdot 1, \qquad p_k = \operatorname{tr} \Lambda^k. \tag{4}$$

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$$\widetilde{W}_k^{(n)} Z_n = (k+n) \frac{\partial Z_n}{\partial p_{k+n}}, \qquad k+n \geq 1. \tag{5}$$

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with $\widetilde{W}$ operators defined as

$$\operatorname{tr} \left(\Lambda \frac{\partial}{\partial \Lambda} \Lambda \right)^n = \sum_k p_k \widetilde{W}_{k-n}^{(n)}. \tag{6}$$

$\widetilde{W}$ algebras

After all $\widetilde{W}$ algebra elements are explicitly given by recursion

$$\widetilde{W}_k^{(n+1)} = \sum_{m\geq 0} p_m \widetilde{W}_{k+m}^{(n)} + \sum_{m=1}^{k+n} m \frac{\partial}{\partial p_m} \widetilde{W}_{k-m}^{(n)} \quad \text{for} \quad k+n \geq 0, \tag{7}$$

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with the base

$$\widetilde{W}_k^{(0)} = \delta_{k,0}. \tag{9}$$

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which retrieved a lot of attention recently, is

$$Z_n^{(m)}(\mathbf{p}) = \exp \operatorname{tr} \left(\left(\Lambda \frac{\partial}{\partial \Lambda} \right)^m \Lambda \right)^n \cdot 1. \tag{11}$$

Generalized $\widetilde{W}$ algebras

The question arises: What is the algebra of constrains on

$$Z_n^{(m)}(\mathbf{p}) = \exp \operatorname{tr} \left(\left(\Lambda \frac{\partial}{\partial \Lambda} \right)^m \Lambda \right)^n \cdot 1? \qquad (12)$$

Generalized $\widetilde{W}$ algebras

Obtained algebras got the name generalized $\widetilde{W}$ algebras.
Non-recurive definition for this case turned out to be

$$\operatorname{tr}\left(\left(\Lambda\frac{\partial}{\partial\Lambda}\right)^m\Lambda\right)^n = \sum_k p_k \widetilde{W}^{(m,n)}_{k-n}. \tag{13}$$

Generalized $\widetilde{W}$ algebras

Recursive one is of new zig-zag type

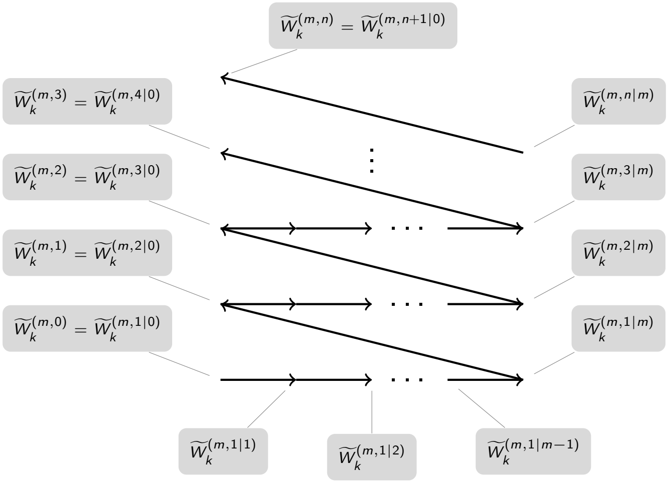

Figure: Structure of recursive procedure for $\widetilde{W}_k^{(m,n)}$ operators

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$$\widetilde{W}_k^{(m,n|l)} = 0 \qquad \text{otherwise.} \tag{18}$$

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Clearly, the transition was made from one series of operators $H_n^{(1)}$ to the set of similar ones $H_n^{(m)}$.

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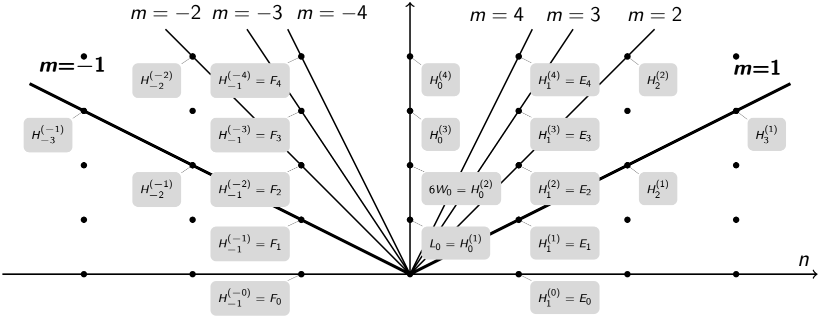

Figure: Commutative subalgebras (integer rays) of $W_{1+\infty}$ algebra depicted on a $2d$ lattice

β-deformation

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Averages in one-matrix models

$$\left\langle f(\operatorname{tr} X^k) \right\rangle = \int dX\, f(\operatorname{tr} X^k) e^{\operatorname{tr} V(X)} \tag{22}$$

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- more...

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Hamiltonians $H_n^{(m)}$ can be rewritten in terms of eigenvalues, too

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where

$$\mathfrak{d}_i^{\beta} = \frac{\partial}{\partial\lambda_i} + \sum_{j\neq i} \frac{1}{\lambda_i - \lambda_j}\left(1 - P_{ij}\right). \tag{26}$$

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Therefore, corresponding β-deformed $\widetilde{W}$-algebras should be non-recursively defined as

$$\sum_i \left(\left(\lambda_i \mathfrak{d}_i^{\beta} \right)^m \lambda_i \right)^n = \sum_{k,i} \lambda_i^k \widetilde{W}_{\beta,k}^{(m,n)}. \tag{27}$$

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How could they be recursively defined?

References

[1] Ya. Drachov, A. Mironov, and A. Popolitov. "$W_{1+\infty}$ and $\widetilde{W}$ Algebras, and Ward Identities". In: *Phys. Lett. B* 849 (2024), p. 138426. DOI: 10.1016/j.physletb.2023.138426. arXiv: 2311.17738 [hep-th].

Thank you for attention!