Simulation of gravitational wave emission after inflation

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Motivation & outline

Motivation:

- We expect that the next generation of gravitational wave detectors will be able to measure the relic gravitational background
- The type of spectrum depends on the initial conditions and how the structures are formed
- We can compare the spectrum results for different inflation models

Outline:

- We start with inflation and count the spectrum of scalar perturbations
- After inflation, perturbations evolve in a linear regime for some time
- After that, the linear approximation becomes inapplicable and the perturbations gather into structures that collapse

Inflation

- We have chosen the potential of Starobinsky

$$V(\phi) = \frac{3}{4} M_\phi^2 M_{pl}^2 \left(1 - e^{-\sqrt{\frac{3}{4}}\frac{\phi}{M_{pl}}}\right)^2 \tag{1}$$

- Background field dynamics is described by the equation:

$$\ddot{\phi} + 3H\dot{\phi} + \partial_\phi V = 0 \tag{2}$$

- The evolution of the scale factor is described by:

$$2\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} = -8\pi G \left(\frac{1}{2}\dot{\phi}^2 - V(\phi)\right) \tag{3}$$

Inflation

- We use slow roll approximation at the beginning of inflation to set initial conditions. In this case background equation is:

$$\dot{\phi} = -\frac{1}{3H}\partial_\phi V(\phi) \tag{4}$$

- And the Friedman equation:

$$H = \frac{1}{M_{pl}}\left(\frac{8\pi V}{3}\right)^{1/2} \tag{5}$$

- Initial condition for ϕ can be found from from the number of e-folds:

$$N_e(\phi) = \frac{8\pi}{M_{pl}^2}\int_{\phi_e}^{\phi_0}\frac{V'}{V}d\phi \tag{6}$$

- So, this three equations give us initial conditions for ϕ, $\dot{\phi}$ and $\dot{a}$

Perturbations

- We divide field into two parts: background and perturbations:

$$\phi(\vec{x}, t) = \phi_0(t) + \varphi(\vec{x}, t) \tag{7}$$

- We take the metric in the form

$$ds^2 = (1 + 2\Phi)dt^2 - a^2(1 - 2\Phi)d\vec{x}^2 \tag{8}$$

- Equation for perturbations evolution is:

$$\ddot{\varphi} + 3H\dot{\varphi} + \partial^2_\phi V\varphi + 2\Phi\partial_\phi V - 4\dot{\phi}\dot{\Phi} + \frac{k^2}{a^2}\varphi = 0 \tag{9}$$

- The scalar potential equation follows from the 0i linearized Einstein equation:

$$\dot{\Phi} + H\Phi = 4\pi G\dot{\phi}\varphi \tag{10}$$

- The initial conditions are such that deep beyond the horizon the perturbations coincide with a free scalar field:

$$\varphi = \frac{e^{-ik\tau}}{a\sqrt{2k}}, \;\; \Phi = 0 \tag{11}$$

Scalar perturbations spectrum

- Scalar perturbations spectrum is defined as:

$$\mathcal{P}_R = \frac{k^3 H^2}{2\pi^2 \dot{\phi}^2} |Q_k|^2 \tag{12}$$

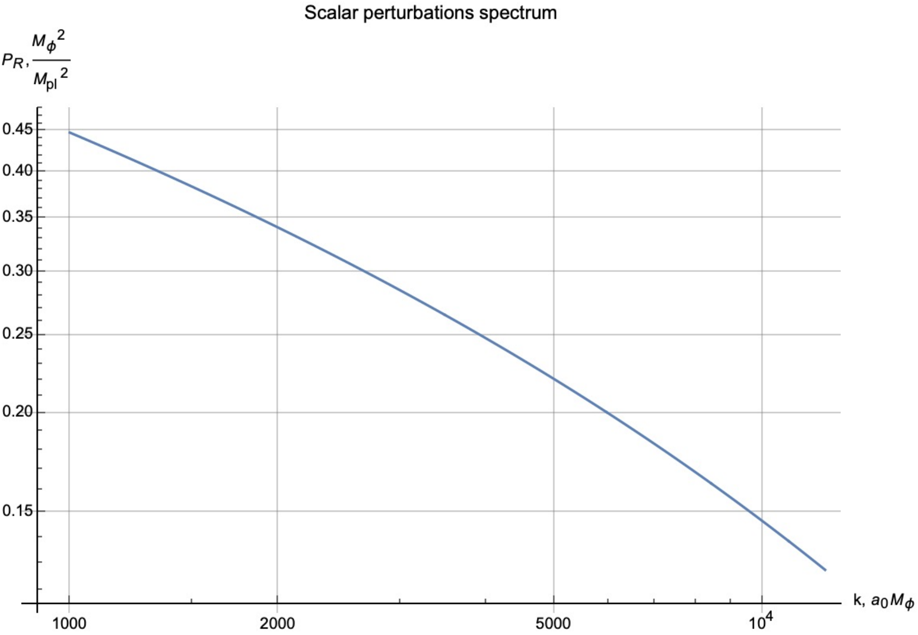
- Where Q_k is the Mukhanov-Sasaki variable, defined as

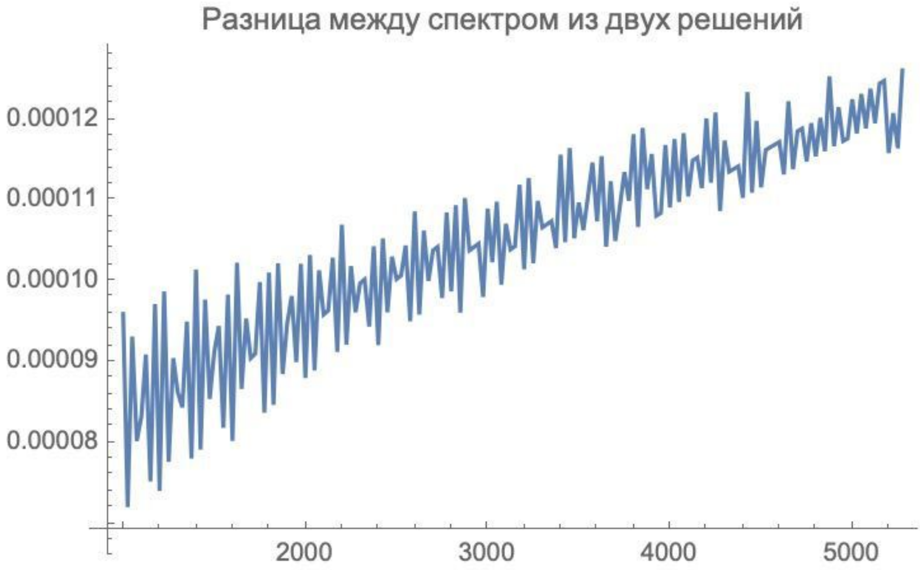
$$Q_k = \varphi_k + \frac{\dot{\phi}}{H}\Phi \tag{13}$$

- We can combine the equations for perturbations and scalar potential by writing an equation for the Mukhanov-Sasaki variable:

$$\ddot{Q}_k + 3H\dot{Q}_k + \left(\frac{k^2}{a^2} + \frac{3\dot{\phi}^2}{M_{pl}^2} - \frac{\dot{\phi}^4}{2H^2 M_{pl}^4} + 2\frac{\dot{\phi}\partial_\phi V}{HM_{pl}^2} + \partial_\phi^2 V \right) Q_k = 0 \tag{14}$$

- We solve both sets of equations to control the difference in solutions


Scalar perturbations spectrum
$P_R, \frac{M_\phi{}^2}{M_{pl}{}^2}$
0.45
0.40
0.35
0.30
0.25
0.20
0.15
1000
2000
5000
10^4
k, $a_0 M_\phi$


Разница между спектром из двух решений
0.00012
0.00011
0.00010
0.00009
0.00008
2000
3000
4000
5000

After inflation

- After inflation $\phi \ll M_{pl}$ and the potential can be considered to be quadratic.

$$S = \int \sqrt{-g} \left(\frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - \frac{1}{2} m^2 \phi^2 \right) d^4 x \quad (15)$$

- The field can be split into two parts: the background $\varphi_0(t)$, which is considered homogeneous and dependent only on time, and the perturbations $\varphi(t, \vec{x})$, i.e.:

$$\phi = \varphi_0(t) + \varphi(t, \vec{x}) \quad (16)$$

- The equation for φ is as follows:

$$\ddot{\varphi} + \ddot{\varphi}_0 - 4\dot{\Phi}\dot{\varphi}_0 + 3H(\dot{\varphi} + \dot{\varphi}_0) + 2\Phi m^2 \varphi_0 - \frac{\Delta\varphi}{a^2} + m^2(\varphi + \varphi_0) = 0 \quad (17)$$

Non-relativistic approximation

- The new variable ψ is defined as follows:

$$\varphi_{(0)} = \frac{e^{-imt}\psi_{(0)}}{\sqrt{2}a^{\frac{3}{2}}(t)} + h.c, \tag{18}$$

- We use a non-relativistic approximation in the form $m\psi \gg \dot{\psi}$:

$$\frac{e^{imt}}{\sqrt{2}a^{3/2}}\left(2m^2\Phi\psi_0 - 2im\dot{\psi} - \frac{\Delta\psi}{a^2}\right) + h.c + O(mt) = 0 \tag{19}$$

- The equation for scalar potential follows from Einstein's equations using the same approximations:

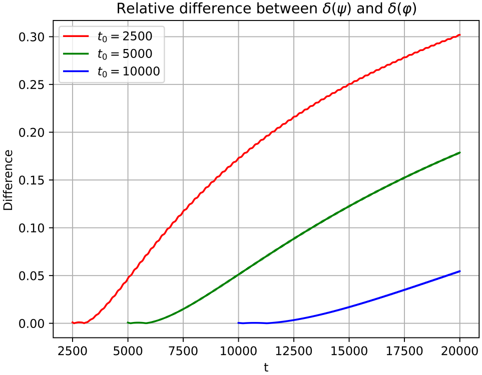
$$\Delta\Phi = 4\pi Gm^2 \frac{|\psi|^2 - |\psi_0|^2}{a(t)} \tag{20}$$

- To obtain linearized equations, we divide the ψ into the background part and perturbations and leave only linear contributions

- The difference in contrasts δ is defined as:

$$\delta \equiv \frac{\delta T_0^0}{T_0^0} \tag{21}$$

- We solved the linear equations for ψ and φ for a set of initial times and calculated the corresponding contrasts, which we denoted as $\delta(\psi)$ and $\delta(\varphi)$



$\left|\dfrac{\delta(\psi)-\delta(\varphi)}{\delta(\psi)}\right|$ for the set of initial times

Numerical methods

- We move on to dimensionless quantities and solve the equations using these quantities. There are two free parameters in the problem: the mass of the field (m) and the average value of the field ψ_0
- To solve the Schrodinger equation, we use the symplectic method of the 4th order:

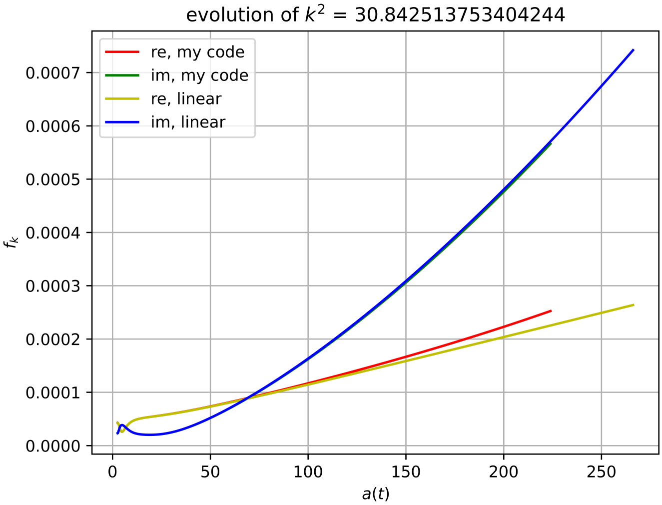
$$\mathbf{f}(t) = e^{-it\mathbf{H}} f_0 \quad (22)$$

$$e^{\tau\mathbf{H}} = e^{a_n\tau\mathbf{V}} \cdot e^{b_n\tau\mathbf{T}} + o(\tau^4), \quad (23)$$

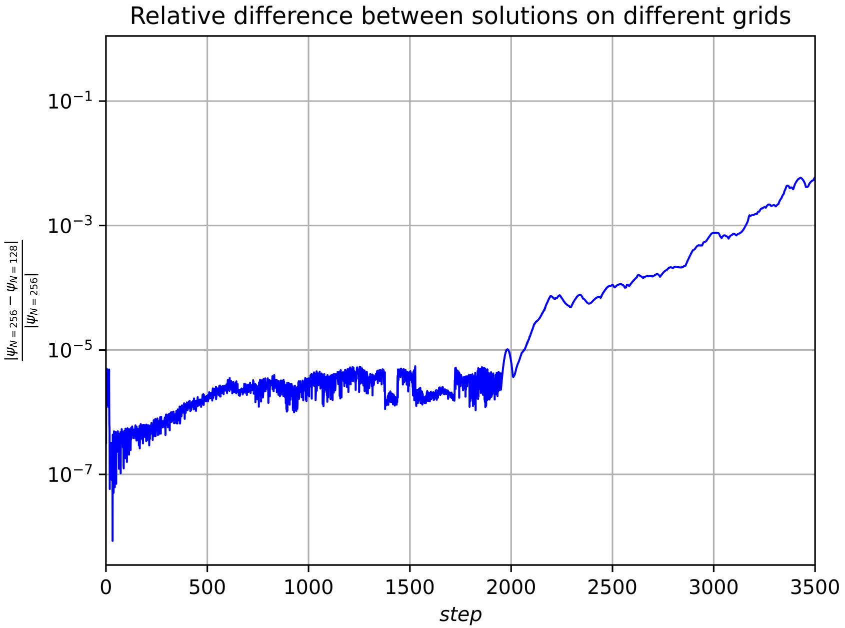
- To solve the Poisson equation, we will use the Fourier series expansion method:

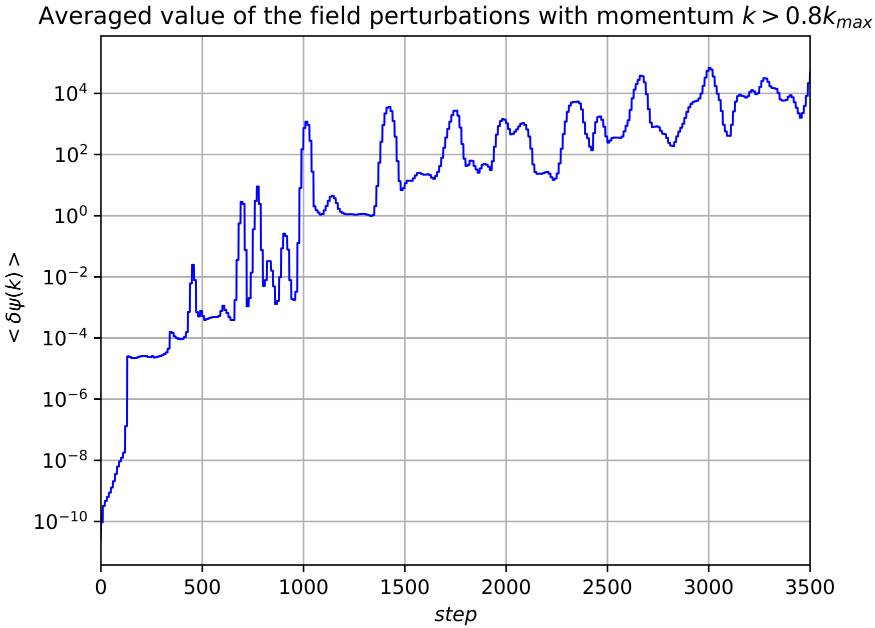
$$\tilde{\Phi}(\vec{k}) = -4\pi G \frac{\tilde{\varrho}(\vec{k})}{k^2} \quad (24)$$

- To check the correctness of the modelling, we compare solutions of linear and non-linear equations for the small perturbations



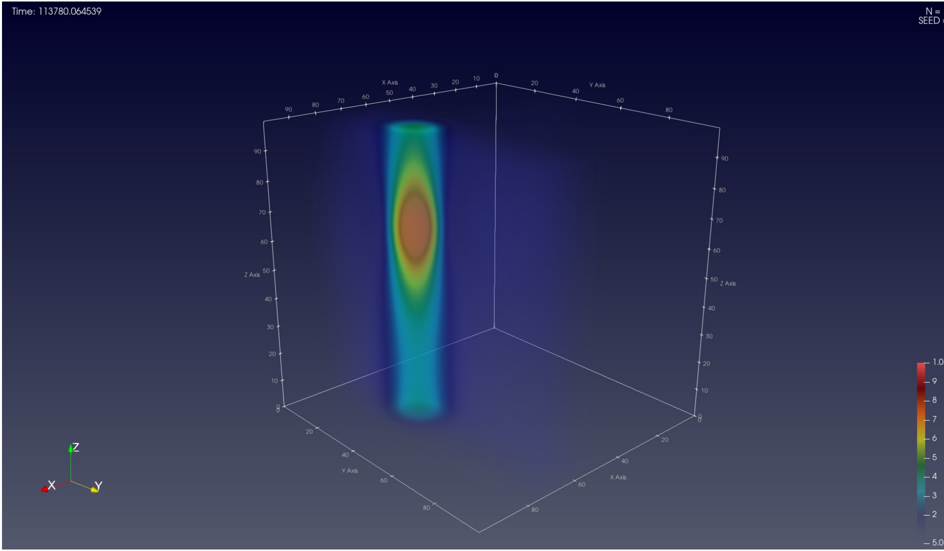
Solutions of linear and non-linear equation


Relative difference between solutions on different grids
$\frac{|\psi_{N=256} - \psi_{N=128}|}{|\psi_{N=256}|}$
10^{-1}
10^{-3}
10^{-5}
10^{-7}
0
500
1000
1500
2000
2500
3000
3500
step

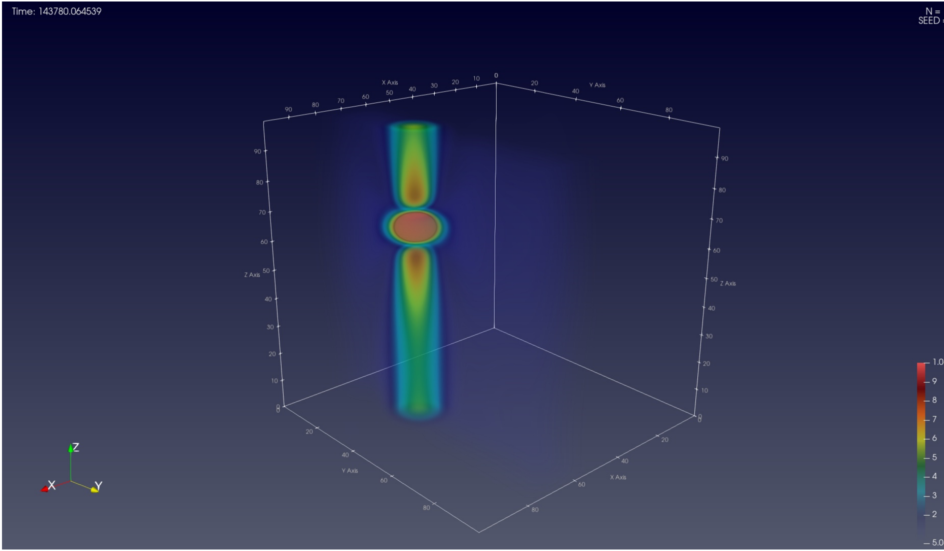


Averaged value of the field perturbations with momentum $k > 0.8k_{max}$
$< \delta\psi(k) >$
10^{4}
10^{2}
10^{0}
10^{-2}
10^{-4}
10^{-6}
10^{-8}
10^{-10}
0
500
1000
1500
2000
2500
3000
3500
step

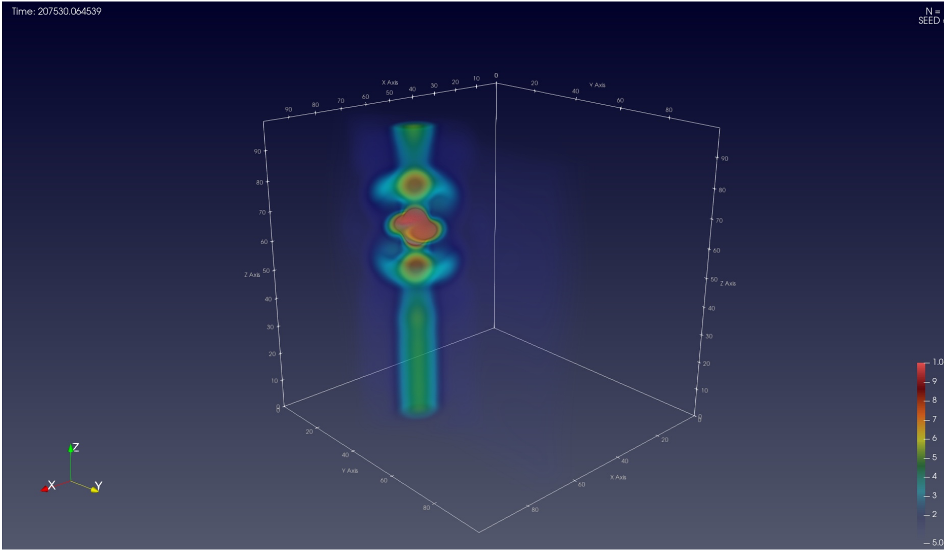
Evolution of the field. Color is $|\psi|^2$



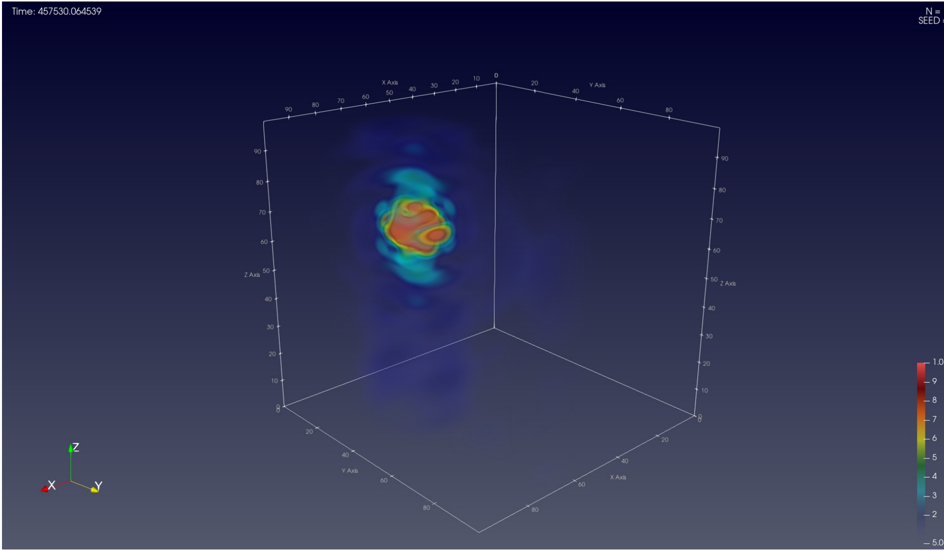
Evolution of the field. Color is $|\psi|^2$



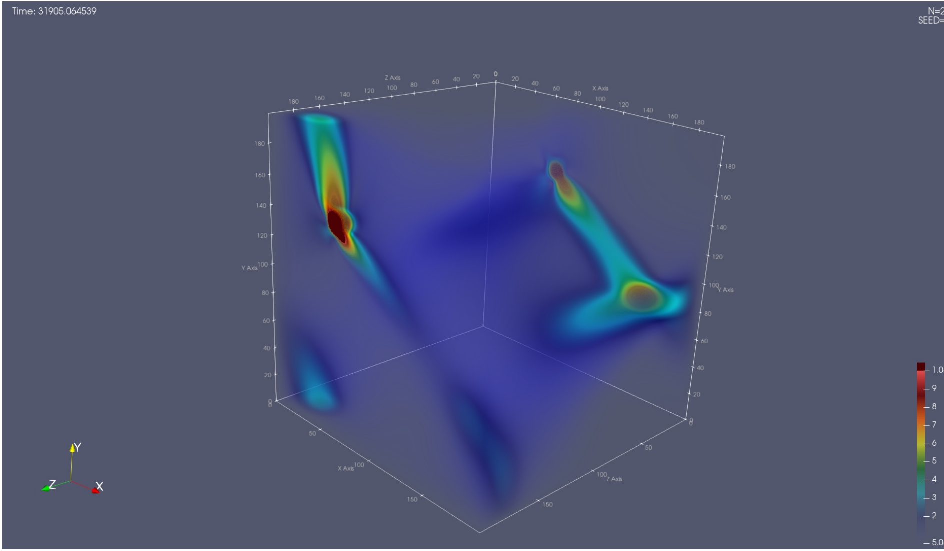
Evolution of the field. Color is $|\psi|^2$



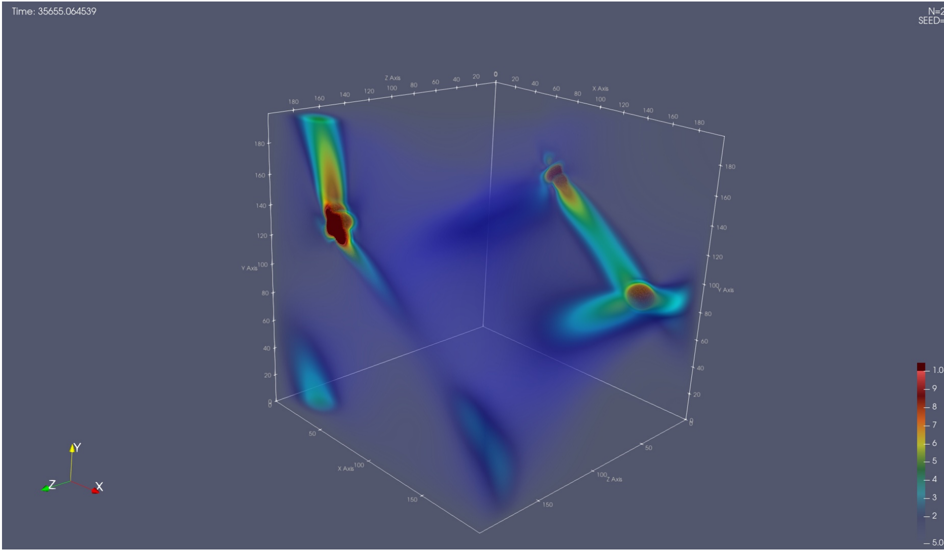
Evolution of the field. Color is $|\psi|^2$



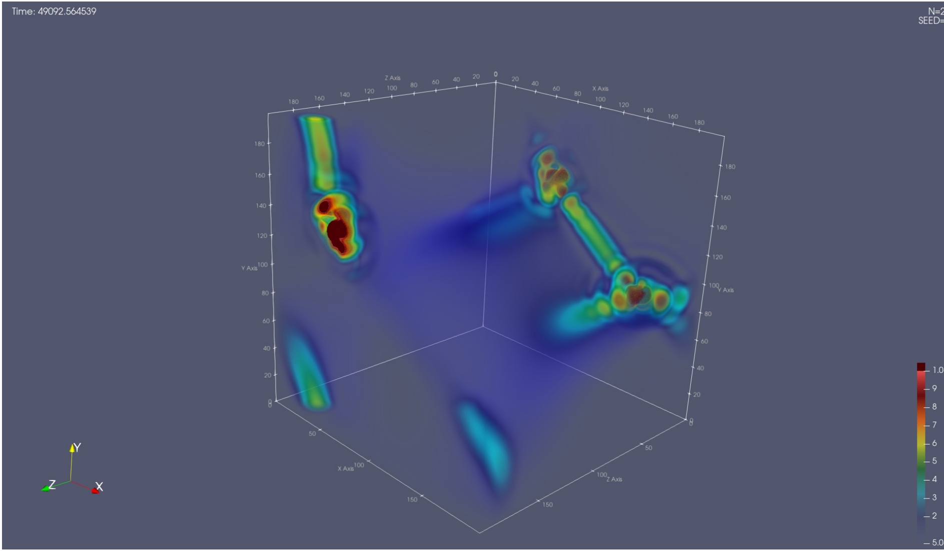
Evolution of the field. Color is $|\psi|^2$



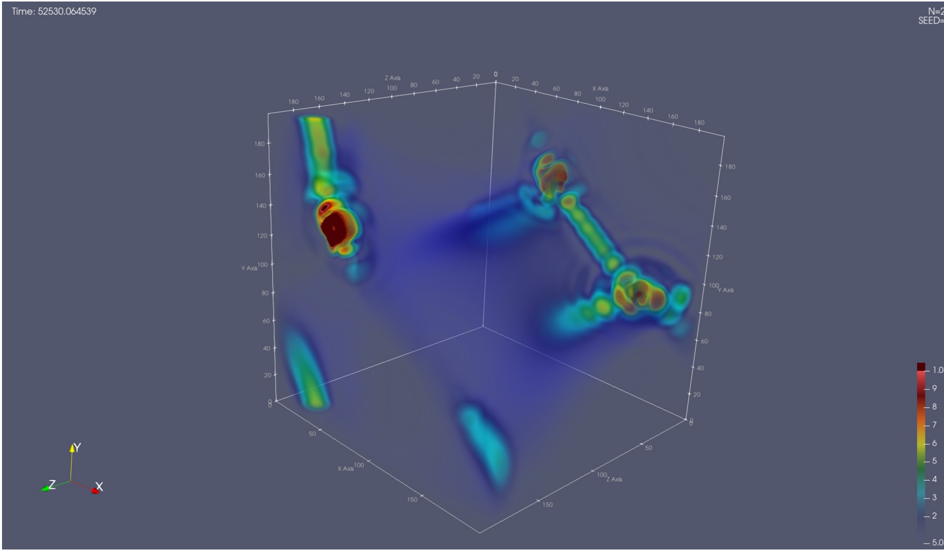
Evolution of the field. Color is $|\psi|^2$



Evolution of the field. Color is $|\psi|^2$



Evolution of the field. Color is $|\psi|^2$



- The equations for the tensor modes have the form:

$$\ddot{h}_{ij}(\vec{x},t) + 3H\dot{h}_{ij}(\vec{x},t) - \frac{\nabla^2}{a^2}h_{ij}(\vec{x},t) = 16\pi G \Pi_{ij}^{TT} \tag{25}$$

- The solution of such an equation for the quantity $\overline{h}$, defined as $\overline{h}_{ij}(\vec{k},\eta) = a h_{ij}(\vec{k},\eta)$ for the modes inside the horizon, assuming that no gravitational waves have been emitted up to the moment $\eta = \eta_i$ and after some moment $\eta = \eta_f$, is given by the Green's function:

$$h_{ij}(\eta,\vec{k}) = -\frac{16\pi G}{a(\eta)k}\int_{\eta_i}^{\eta_f} d\eta' e^{k(\eta-\eta')} a(\eta')\Pi_{ij}^{TT}(\eta',\vec{k}) \tag{26}$$

We define gravitational waves energy density over a volume V as:

$$\rho_{gw} = \frac{1}{32\pi G}\langle \dot{h}_{ij}(x,t)\dot{h}_{ij}(x,t)\rangle_V \tag{27}$$

- Gravitational waves spectrum is defined as:

$$\left(\frac{d\rho_{gw}}{d\ln k}\right)_{\eta>\eta_f} = \frac{S_k(\eta_f)}{a^4(\eta)} \tag{28}$$

- The expression for the spectrum of the gravitational waves:

$$S_k(\eta) = \frac{8\pi G k^3}{V}\int d\Omega \sum_{p=+,\times}\left|\int_{\eta_i}^{\eta_f} d\eta' e^{ik(\eta_f-\eta')}a(\eta')\Pi_p(\eta',\vec{k})\right|^2 \tag{29}$$

- where Π_p is projection of transverse and traceless part of the energy-momentum tensor onto gw's polarization vectors.

Conclusions

- We can simulate the nonlinear evolution of the field and the formation of structures
- We can set initial conditions which follows from exact inflationary model
- It is planned to obtain a spectrum of gravitational waves that are emitted due to the nonlinear evolution of the field