Entanglement Entropy In De Sitter Spacetime: No Pure States For Conformal Matter

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Based on arXiv:2304.12351
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Conference "Quarks-2024"
May 21, 2024

Big Picture Motivation

- Our universe approaches **de Sitter spacetime**.
- For a static observer, de Sitter spacetime contains **causally disconnected regions** separated by **cosmological horizon**.
- Cosmological and black hole horizons share similar properties: **temperature**, **entropy**, **Hawking radiation**.
- However, cosmological horizon is **observer-dependent**, in contrast to black hole geometry. Microscopic interpretation of the cosmological horizon is not clear.

Big Picture Motivation

The microscopics of cosmological horizon thermodynamics could potentially rely on **holography**. For example, the gravitational fine-grained entropy (for AdS black hole) was first derived via $\text{AdS}_{d+1}/\text{CFT}_d$ correspondence (Ryu-Takayanagi formula). However, the de Sitter holography is still a subject to debate (known examples are dS/CFT correspondence, different versions of static patch holography — worldline holography, stretched horizon holography).

Big Picture Motivation

- Much of a progress has been achieved in deriving gravitational fine-grained entropy in black hole backgrounds by performing calculations using **semi-classical gravitational path integral**.

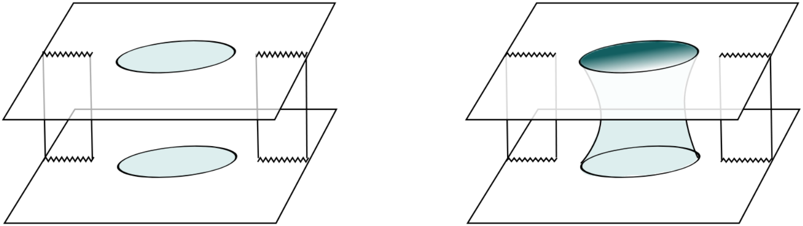


Рис.: Two different saddles in $n = 2$ Euclidean gravitational path integral in the presence of gravity in the shaded region. On the left is the Hawking saddle, on the right — the replica wormhole where gravity dynamically glues together the shaded regions. Replica wormholes reproduce the **island rule**.

Big Picture Motivation

- Non-trivial saddle points in the path integral (the so-called **replica wormholes**) define the gravitational von Neumann entropy (**QES/island rule**).
- The island rule is a **non-perturbative** effect.
- The original derivation of the island rule for AdS_2 black hole relies on the fact that the full manifold $\mathcal{M} \simeq \mathrm{AdS}_2 \cup M_2$. The AdS_2 is dynamical gravity, while Minkowski bath M_2 realizes the **asymptotically flat** b.c. The "radiation" (CFT_2) is collected in M_2. This allows to consider a replica manifold and use the replica trick.

Big Picture Motivation

- In dS, event horizon lies between the static observer and the null infinity.
- Therefore, there is no unambiguous way to couple the thermal bath in dS.
- Even if the bath is coupled to asymptotic infinity, the Euclidean gravitational path integral will not obtain the corresponding contribution, since only the static patch survives in Euclidean signature.
- Therefore, for dS the problem does not have a solution neither from holography nor from replica calculation.

Entropy In The Presence Of Gravity

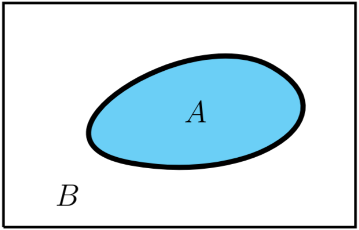
The **“island rule”** for entanglement entropy of QFT in gravitational systems reads

$$S(R) \simeq \min_{\mathcal{I}} \left\{ \text{ext}_{\mathcal{I}} \left[\frac{\text{Area}(\partial \mathcal{I})}{4G_N} + S_m(R \cup \mathcal{I}) \right] \right\}, \qquad (1)$$

where

- $\Sigma \simeq R \cup \overline{R}$ is the Cauchy surface, on which the state of QFT is defined;
- R is the region of interest, the entropy of which we aim to calculate;
- $\mathcal{I}$ — the “island” defined by extremization, $\partial \mathcal{I}$ — its boundary;
- S_m — entanglement entropy of QFT on the fixed classical background.

Purity And Complementarity Of Entanglement Entropy



- Let the total system ρ_{tot} consist of two subsystems: $A \cup B$. The Hilbert space of the total system $\mathcal{H}_{tot}$ is given by

$$\mathcal{H}_{tot} \simeq \mathcal{H}_A \otimes \mathcal{H}_B. \tag{2}$$

- The reduced density matrices of the subsystems are defined as

$$\rho_A = \text{Tr}_{\mathcal{H}_B} \rho_{tot}, \quad \rho_B = \text{Tr}_{\mathcal{H}_A} \rho_{tot}. \tag{3}$$

Purity And Complementarity Of Entanglement Entropy

- Entanglement entropy of a subsystem X is defined as

$$S_X = -\operatorname{Tr} \rho_X \log \rho_X, \quad X = A, B. \tag{4}$$

- Basic properties of entanglement entropy of interest:
  - **Purity**: if the system X is in a pure state, then $\rho_X^2 = \rho_X$, therefore, $S_X = 0$.
  - **Complementarity**: if the total state $X \cup \overline{X}$ is pure, then entanglement entropies of the subsystem X and its complement $\overline{X}$ are the same: $S_X = S_{\overline{X}}$.
  - **Araki-Lieb triangle inequality**: $|S_A - S_B| \leq S_{A \cup B}$.
  - $S_X \leq S_{\mathrm{GH}}(X) \propto \mathrm{Area}(X)$.

Setup

The metric of dS_4 is given by

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega_2^2, \quad f(r) = 1 - \frac{r^2}{\ell^2} \qquad (5)$$

The radial distance $d^2(\mathbf{x}, \mathbf{y})$ between two points can be calculated as

$$d^2(\mathbf{x}, \mathbf{y}) = \frac{2\sqrt{f(x)f(y)}}{\kappa_c^2}\Big[\cosh\kappa_c(r_*(x) - r_*(y)) - \cosh\kappa_c(t_x - t_y)\Big] \qquad (6)$$

where $\kappa_c = 1/\ell$ is surface gravity, and the tortoise coordinate $r_*(r)$ is

$$r_*(r) = \frac{\ell}{2}\log\frac{\ell + r}{|\ell - r|} = \begin{cases} \ell \,\mathrm{arctanh}\, r/\ell, & r < \ell, \\ \ell \,\mathrm{arctanh}\, \ell/r, & r > \ell. \end{cases} \qquad (7)$$

Setup

- We use **partial reduction** from $d = 4$ to $d = 2$.
- We add matter represented by c **free massless Dirac fermions**:

  $\mathcal{S}_{\text{pure dS}_4} \to \mathcal{S}_{\text{partially reduced dS}_2} + \mathcal{S}_{\text{CFT}_2\text{ fermions}}$.
- Entropy of conformal matter for one interval

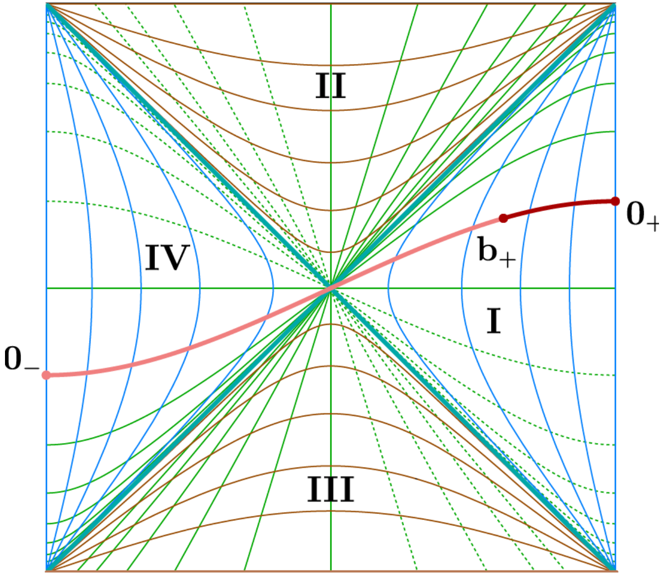
$$S_{\text{m}} = \frac{c}{6}\log\frac{d^2(\mathbf{x},\mathbf{y})}{\varepsilon^2}. \tag{8}$$

- Entropy of N intervals is given by

$$S_{\text{m}} = \frac{c}{6}\sum_{i,j}\log\frac{d^2(\mathbf{x}_i,\mathbf{y}_j)}{\varepsilon^2} - \frac{c}{6}\sum_{i<j}\log\frac{d^2(\mathbf{x}_i,\mathbf{x}_j)}{\varepsilon^2} - \tag{9}$$

$$-\frac{c}{6}\sum_{i<j}\log\frac{d^2(\mathbf{y}_i,\mathbf{y}_j)}{\varepsilon^2}. \tag{10}$$

Main Results



Main Results

- Cauchy surfaces Σ are **finite-sized**, hence, do not need IR regularization.
- $S_{\mathrm{m}}(\Sigma) \geq \frac{c}{3} \log \frac{2}{\kappa_c \varepsilon}$. **This result is in contradiction with pure state condition**: $S_{\mathrm{m}}(\Sigma) = 0$.
- Entropies for a finite interval and its complement are

$$S_{\mathrm{m}}(R) = \frac{c}{6} \log \left(\frac{2\sqrt{f(b)}}{\kappa_c^2 \varepsilon^2} \left[\cosh \kappa_c r_*(b) - \cosh \kappa_c (t_b - t_{0_+})\right] \right)$$

$$S_{\mathrm{m}}\left(\overline{R}\right) = \frac{c}{6} \log \left(\frac{2\sqrt{f(b)}}{\kappa_c^2 \varepsilon^2} \left[\cosh \kappa_c r_*(b) + \cosh \kappa_c (t_b - t_{0_-})\right] \right)$$

Therefore, **complementarity is explicitly violated**: $S_{\mathrm{m}}(R) \neq S_{\mathrm{m}}\left(\overline{R}\right)$.

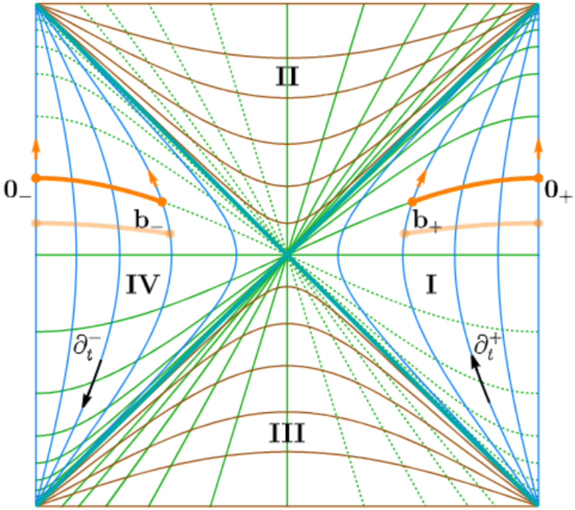
Main Results

Consider the finite region $R = [\mathbf{b}_+, \mathbf{0}_+]$.

- If $\kappa_c(t_b - t_{0_+}) = \text{const}$, then $S(R) = \text{const}$, and $S(R) \le S_{\text{GH}}$.
- Let $\kappa_c(t_b - t_{0_+}) \equiv g(t_b) \neq \text{const}$. Then
  - If $g'(t_b) > 0$, then Cauchy surface breaks, $S(R) \to \infty$, and the problem becomes ill-defined.
  - If $g'(t_b) < 0$, the endpoints $\mathbf{b}_+$ and $\mathbf{0}_+$ asymptotically approach the hypersurface of constant time at late times, and $S(R) \to \text{const}$.

Main Results

Consider $R = [0_-, \mathbf{b}_-] \cup [\mathbf{b}_+ 0_+]$ with $t_{b_+} = -t_{b_-} \equiv t_b$, $t_{0_+} = t_{b_+}$, $t_{0_-} = t_{b_-}$.



Main Results

- At early times $t_b \ll r_*(b)/2$, the entropy monotonically increases
$$S_{\rm m}\left(R\right) \simeq \frac{2c}{3}\kappa_c t_b + \text{const}$$
- At late times $t_b \gg r_*(b)/2$, the entropy saturates at the value
$$S_{\rm m}\left(R\right) \simeq \frac{c}{3}\log\left(\frac{2\sqrt{f\left(b\right)}}{\kappa_c^2\varepsilon^2}\right) + \frac{c}{3}\log\left(\cosh\kappa_c r_*(b) - 1\right)$$
- $\lim_{b\to\ell} r_*(b) \to \infty \implies$ as $b \to \ell$, the linear growth regime gets longer. **This might formally lead to the information paradox in dS.**
- Numerical analysis reveals **no island** for this configuration.

Conclusions

- As a consistency test of the setup, we studied basic properties of entanglement entropy for a pure state in dS. Both **pure state condition** and **complementarity** are **violated**. This disproves CFT_2 formulas for entanglement entropy to partially reduced dS geometry.
- We have given an example in this setup, which might **potentially lead to the information paradox**. Numerical calculations reveal **no non-trivial island** for this configuration. The **possible information paradox cannot be resolved** by the island formula.

Thank you for your attention!