# On the choice of variables for quantization conformal general relativity

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# Motivation

• All fundamental interactions, with the exception of gravity, are described using the probability amplitudes of various processes. Researchers are trying to resolve the emerging dual description of physical phenomena by constructing a quantum theory of gravity.

• The non-renormalizability of the standard general theory of relativity leads to the need to study alternative approaches to the quantization of gravity.

• Whether a particular theory can be quantized depends on the choice of the underlying variables to be quantized.

• It is necessary to investigate the possibility of quantizing the general theory of relativity and its modifications in other variables (not the metric tensor).

In our work, we investigated the model specified by the action [Deser '1970, Dirac '1973]

$$S_{\rm CGR} = \int d^4\chi\sqrt{-\widetilde{g}}\left[\frac{\widetilde{M}_P^2}{16\pi}\left(\widetilde{R}-2\widetilde{\Lambda}\right)\right. \\ \left. +\frac{3\widetilde{M}_P^2}{8\pi}(\widetilde{g}^{\mu\nu}\nabla_\mu D\nabla_\nu D) + L_{\rm matter}(\widetilde{g}_{\mu\nu})\right] \quad (1)$$

Here $\widetilde{M}_P = M_P e^{-D}$ is the conformal Planck mass, $\widetilde{\Lambda} = e^{-2D}\Lambda$ is the conformal cosmological constant, $\Lambda$ is the standard cosmological constant, $M_P$ is the standard Planck mass.

$$ds^2 = g_{\mu\nu}dx^\mu \otimes dx^\nu = e^{-2D}\ \widetilde{g}_{\mu\nu}\ d\chi^\mu \otimes d\chi^\nu \quad (2)$$

# Variables $\omega^R_{(c)(b),(a)}$ and $\omega^L_{(b)(a),(c)}$

In earlier works [A.Arbuzov, B. Latosh '2018], it was shown that metric differential can be represented as

$$d\widetilde{g}_{\mu\nu} = \left(e_\mu{}^{(b)} e_\nu{}^{(a)} + e_\mu{}^{(a)} e_\nu{}^{(b)}\right) \left(\omega^R{}_{(b)(a),}(dx^\alpha)\right.$$

$$\left. + \, \omega^L{}_{(b)(a),}(dx^\alpha)\right) = \left(e_\mu{}^{(b)} e_\nu{}^{(a)} + e_\mu{}^{(a)} e_\nu{}^{(b)}\right) \omega^R{}_{(b)(a),\alpha} dx^\alpha \tag{3}$$

where the variables $\omega^R_{(b)(a),(c)}$ and $\omega^L_{(b)(a),(c)}$ were introduced along with a nonlinear realization of conformal symmetry,

$$\omega^R_{(a)(c),(b)} = \frac{1}{2} e^\alpha{}_{(b)} \left(e^\beta{}_{(a)} \partial_\alpha e_{(c)\beta} + e^\beta{}_{(c)} \partial_\alpha e_{(a)\beta}\right), \tag{4}$$

$$\omega^L_{(a)(c),(b)} = \frac{1}{2} e^\alpha{}_{(b)} \left(e^\beta{}_{(a)} \partial_\alpha e_{(c)\beta} - e^\beta{}_{(c)} \partial_\alpha e_{(a)\beta}\right), \tag{5}$$

Note that $\omega^L{}_{(b)(a),}(dx^\alpha)$ does not contribute to the metric differential due to permutation symmetry in $(b)(a)$. Consequently, the dynamic part can only be contained in $\omega^R{}_{(b)(a),(c)}$.

$$\frac{\partial \widetilde{g}_{\mu\nu}}{\partial x^{(c)}} = \left(e_\mu{}^{(b)} e_\nu{}^{(a)} + e_\mu{}^{(a)} e_\nu{}^{(b)}\right) \omega^R_{(b)(a),(c)}, \tag{6}$$

# Non-holonomy coefficients

The decomposition of tetrad vectors and co-tetrads into basis 1-forms with respect to basis in tangent space and cotangent space has the form

$$e_{(a)} = e^{\alpha}{}_{(a)}\partial_{\alpha}, \;\; e^{(a)} = e_{\alpha}{}^{(a)}dx^{\alpha} \tag{7}$$

The decomposition of a cotetrad into basic 1-forms (covectors) from the cotangent space has the form

$$e^{\alpha}{}_{(a)}e^{(b)}{}_{\alpha} = \delta^{(b)}_{(a)}, \tag{8}$$

$$e^{\alpha}{}_{(a)}e_{\beta}{}^{(a)} = \delta^{\alpha}_{\beta}, \tag{9}$$

$$\left[e_{(a)}, e_{(b)}\right] = c_{(a)(b)}{}^{(c)}e_{(c)}. \tag{10}$$

The coefficients $c_{(a)(b)}{}^{(c)}$ are called non-holonomy coefficients. Explicit

$$c_{(a)(b)}{}^{(c)} = \left(e^{\alpha}{}_{(a)}\partial_{\alpha}e^{\beta}{}_{(b)} - e^{\alpha}{}_{(b)}\partial_{\alpha}e^{\beta}{}_{(a)}\right) e_{\beta}{}^{(c)}. \tag{11}$$

# Spin connections

We are interested in the case when the connection is metric,

$$\omega_{(a)(b)(c)} = \frac{1}{2}\left(c_{(a)(b)(c)} - c_{(b)(c)(a)} + c_{(c)(a)(b)}\right), \tag{12}$$

where the notation $c_{(a)(b)(c)} := g_{(c)(d)} c_{(a)(b)}{}^{(d)}$ is used. Expression for spin connection components

$$\begin{aligned} \omega_{(a),(b)(c)} = {} & \frac{1}{2} e^{\alpha}{}_{(a)} \left(e^{\beta}{}_{(c)} \partial_\alpha e_{(b)\beta} - e^{\beta}{}_{(b)} \partial_\alpha e_{(c)\beta}\right) \\ & + \frac{1}{2} e^{\alpha}{}_{(b)} \left(e^{\beta}{}_{(a)} \partial_\alpha e_{(c)\beta} + e^{\beta}{}_{(c)} \partial_\alpha e_{(a)\beta}\right) \\ & - \frac{1}{2} e^{\alpha}{}_{(c)} \left(e^{\beta}{}_{(b)} \partial_\alpha e_{(a)\beta} + e^{\beta}{}_{(a)} \partial_\alpha e_{(b)\beta}\right). \end{aligned} \tag{13}$$

Using $e_{\beta(c)} \partial_\alpha e^{\beta}{}_{(b)} = -e^{\beta}{}_{(b)} \partial_\alpha e_{\beta(c)}$ we can express spin connections via $\omega^{R,L}_{(c)(b),(a)}$ and obtain formula

$$\omega_{(a),(b)(c)} = \omega^{L}_{(c)(b),(a)} + \omega^{R}_{(a)(c),(b)} - \omega^{R}_{(b)(a),(c)}. \tag{14}$$

The conformal metric in the ADM formalism has the form

$$\widetilde{ds}^2 = \widetilde{g}_{ij}(d\chi^i + N^i dt)(d\chi^j + N^j dt) - (Ndt)^2. \tag{15}$$

Transition to tetrad representation of metrics

$$\begin{aligned} g_{\mu\nu} dx^\mu \otimes dx^\nu &= e^{-2D}\widetilde{g}_{\mu\nu} d\chi^\mu \otimes d\chi^\nu = e^{-2D}\eta_{(a)(b)} e^{(a)} \otimes e^{(b)} \\ &= e^{-2D}\eta_{(a)(b)}(e_\mu{}^{(a)} d\chi^\mu) \otimes (e_\nu{}^{(b)} d\chi^\nu). \end{aligned} \tag{16}$$

$$\begin{cases} e^{(0)} = Nd\chi^0, \\ e^{(j)} = e_i{}^{(j)} \left[ d\chi^i + N^i d\chi^0 \right]. \end{cases} \tag{17}$$

The quantities $N$ and $N^i$ are called of the lapse function and the shift vector, respectively. Here $e^0$, $e^{(j)}$ are a set of basic co-tetrads $e_i{}^{(j)}$.

# Plane, nonlinear gravitational wave

The metric of a nonlinear, plane gravitational wave has the form

$$\widetilde{g} = -d\chi^0 \otimes d\chi^0 + d\chi^3 \otimes d\chi^3 + e^{\Sigma}\left[e^{\sigma} d\chi^1 \otimes d\chi^1 + e^{-\sigma} d\chi^2 \otimes d\chi^2\right]. \quad (18)$$

We choose the Lichnerovich gauge $N = 1$, $N^i = 0$ and $\gamma = 1$. Therefore $e^{\Sigma} = 1$, $\Sigma = 0$, $\sqrt{-\widetilde{g}} = 1$.

$$S_{\text{Gravitons}} = \int d\chi^4 \, \frac{1}{2}\left[\left(\frac{\partial\sigma}{\partial\chi^0}\right)^2 - \left(\frac{\partial\sigma}{\partial\chi^3}\right)^2\right]. \quad (19)$$

Representation of $\omega^R_{(a)(b),(c)}$ in the form of expansion in plane waves

$$\omega^R_{(a)(b),(c)} = \int \frac{d^3k}{(2\pi)^3}\frac{1}{\sqrt{2\omega_k}} \, ik_{(c)}\left[\epsilon^R_{(a)(b)}(k) g^+_k e^{ik\cdot x} + \epsilon^R_{(a)(b)}(-k) g^-_k e^{-ik\cdot x}\right], \quad (20)$$

where $g^{\pm}_k$ pretend to be creation and annihilation operators

The action for gravity in the chosen calibration has the form

$$S_{\text{Gravitons}} = \int d\chi^4 \ \frac{\widetilde{M}_P^2}{16\pi}\widetilde{R}. \tag{21}$$

The components of the curvature tensor in a nonholonomic basis will have the form

$$\begin{aligned} R_{(a)(b)(c)}{}^{(d)} &= \partial_{(a)}\omega_{(b)(c)}{}^{(d)} - \partial_{(b)}\omega_{(a)(c)}{}^{(d)} - \omega_{(a)(c)}{}^{(e)}\omega_{(b)(e)}{}^{(d)} \\ &+ \omega_{(b)(c)}{}^{(e)}\omega_{(a)(e)}{}^{(d)} - c_{(a)(b)}{}^{(e)}\omega_{(e)(c)}{}^{(d)}, \end{aligned} \tag{22}$$

The dependence of the spin connection components $\omega_{(b),(c)}{}^{(d)}$ on $\omega^R_{(b)(a),(c)}$ is linear:

$$\omega_{(b),(c)}{}^{(d)} = \eta^{(d)(a)} \left(\omega^L_{(a)(c),(b)} + \omega^R_{(b)(a),(c)} - \omega^R_{(c)(b),(a)}\right). \tag{23}$$

Substitution of (23) into (22) leads to a bilinear representation of $\widetilde{R}$ in $\omega^R$.

We considered the assumption that the components of $\omega^{R}_{(a)(c),(b)}$ can play the role of basic variables in the quantization of conformal general relativity and have an advantage over the metric in the matter of renormalization.

• It is shown that to introduce a special set of dynamic variables $\omega^{R}_{(a)(c),(b)}$ and $\omega^{L}_{(a)(c),(b)}$ does not require the use of either conformal symmetry or a nonlinear realization of a bigger symmetry group. Thus, this method was generalized to a much wider class of theories, namely to all in which the spin connection is metric.

• The Lagrangian of the theory contains derivatives of $\omega^{R}_{(a)(c),(b)}$ no higher than the first order and therefore the theory does not have a conformal graviton propagator in these variables.

• The resulting theory turns out to be trivial if we take $\omega^{R}_{(a)(c),(b)}$ as the basic variables of quantum gravity. The implications of this need to be explored.

• One might consider considering conformal gravitons similarly to the case of a spinor field, in the Lagrangian, which also lacks a kinetic term.