Stability of cosmological solutions in Horndeski theory with respect to background anisotropy

ArXiv: 2212.03285, 2305.19171

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on QUARKS'24
24.05.2024

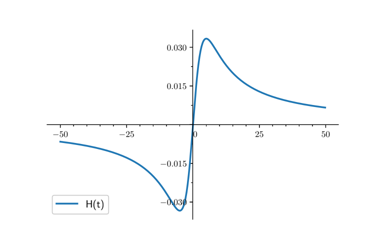
» Some problems of Inflation

- η-problem. In large field models of inflation, the inflaton has to traverse a distance in field space larger than the Planck mass M_{pl} in natural units. This has been argued to be problematic, since non-renormalizable quantum corrections to the field's action arise. In the absence of functional fine-tuning or additional symmetries, inflation would be spoiled;
- The presence of eternal inflation in almost all proposals has been argued to lead to a possible loss of predictability due to our inability to prescribe a unique measure: this is the so-called measure problem.
- Inflation does not provide a theory of initial conditions that would explain why the inflaton field starts out high in its potential.

A. Linde (2014), 1402.0526

Alternative scenarios

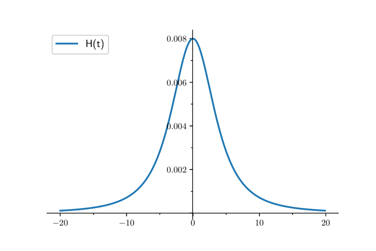
Bounce



Starts from contracting stage ⇒ bounce ⇒ expansion

M. Novello, S. E. Perez Bergliaffa (2008), arXiv: 0802.1634

Genesis



Starts from Minkowski, empty space, then energy density builds up, Universe starts to expand, expansion accelerates.

Creminelli, Nicolis, Trincherini (2010), arXiv: 1007.0027

Both can be viewed as alternatives to, or completion of inflation.

» Null energy condition (NEC)

$$T_{\mu\nu} n^\mu n^\nu \geq 0$$

for any null vector n^μ, such that $n_\mu n^\mu = 0$

* Quite robust
* Implies a number of properties. For example: Penrose theorem.

Penrose' 1965

In cosmology: if the NEC holds, and spatial curvature is negligible, there is initial singularity
$\Rightarrow$ No Bounce or Genesis.

» Horndeski theory

$$S = \int \mathrm{d}^4x \sqrt{-g}\, (\mathcal{L}_2 + \mathcal{L}_3 + \mathcal{L}_4 + \mathcal{L}_5)\,,$$

$$\mathcal{L}_2 = F(\pi, X),$$

$$\mathcal{L}_3 = K(\pi, X)\Box\pi,$$

$$\mathcal{L}_4 = -G_4(\pi, X)R + 2G_{4X}(\pi, X)\left[(\Box\pi)^2 - \pi_{;\mu\nu}\pi^{;\mu\nu}\right],$$

$$\mathcal{L}_5 = G_5(\pi, X)G^{\mu\nu}\pi_{;\mu\nu} + \frac{1}{3}G_{5X}\left[(\Box\pi)^3 - 3\Box\pi\pi_{;\mu\nu}\pi^{;\mu\nu} + 2\pi_{;\mu\nu}\pi^{;\mu\rho}\pi_{;\rho}{}^{\nu}\right],$$

where π is the scalar field, $X = g^{\mu\nu}\pi_{,\mu}\pi_{,\nu}$, $\pi_{,\mu} = \partial_\mu\pi$, $\pi_{;\mu\nu} = \nabla_\nu\nabla_\mu\pi$, $\Box\pi = g^{\mu\nu}\nabla_\nu\nabla_\mu\pi$, $G_{4X} = \partial G_4/\partial X$, etc.

» Perturbations above FLRW background

$$ds^2 = dt^2 - a^2(t)\left(dx^2 + dy^2 + dz^2\right).$$

The decomposition of the perturbation of the metric $h_{\mu\nu}$ by helicities has the following form

$$h_{00} = 2\Phi$$
$$h_{0i} = -\partial_i \beta + Z_i^T,$$
$$h_{ij} = -2\Psi\delta_{ij} - 2\partial_i\partial_j E - \left(\partial_i W_j^T + \partial_j W_i^T\right) + h_{ij}^T,$$

where Φ, β, Ψ, E - scalar fields, Z_i^T, W_i^T - transverse vector fields, $(\partial_i Z_i^T = \partial_i W_i^T = 0)$, h_{ij}^T - transverse traceless tensor. Also we denote $\delta\pi = \chi$.

» The second-order action

Standart way:

- Use the unitary gauge $E = \chi = 0$.
- Solve constraints on Φ and β.

$$S^{(2)} = \int \mathrm{d}t \; \mathrm{d}^3 x a^3 \left[\frac{\mathcal{G}_T}{2} \left(\dot{h}_{ij} \right)^2 - \mathcal{F}_T \frac{\left(\overrightarrow{\nabla} h_{ij} \right)^2}{a^2} + \mathcal{G}_S \left(\dot{\zeta} \right)^2 - \mathcal{F}_S \frac{\left(\overrightarrow{\nabla} \zeta \right)^2}{a^2} \right].$$

$\mathcal{G}_T, \mathcal{F}_T, \mathcal{G}_S, \mathcal{F}_S$ - functions depending on F, K, G_4, G_5 and their derivatives.

» No-go theorem

To avoid ghost and gradient instabilities one requires $\mathcal{G}_T > 0$, $\mathcal{G}_S > 0$ and $\mathcal{F}_T > 0$, $\mathcal{F}_S > 0$.

$\mathcal{F}_S$ has a structure:

$$\mathcal{F}_S = \frac{1}{a}\frac{\mathrm{d}}{\mathrm{d}t}\xi - \mathcal{F}_T$$

$$\Rightarrow \frac{d}{dt}\xi = a \cdot (\mathcal{F}_S + \mathcal{F}_T) > 0,$$

The point is that

$$\xi = \frac{a\mathcal{G}_T^2}{2\theta},$$

is, therefore, a monotonously growing function, which means it must cross zero at some point, but we have $\mathcal{G}_T^2$ in the numerator of ξ. These two statements contradict each other.

Furthermore, at the point $\xi = 0$ $\theta \to \infty$ which means that the background fields H and π diverge, and that there is a singularity in the theory.

» Ways to avoid No-go theorem

1. Beyond Horndeski theory:

$$\mathcal{L} = \mathcal{L}_H + F_4(\pi, X)\epsilon^{\mu\nu\rho}{}_{\sigma}\epsilon^{\mu'\nu'\rho'\sigma}\pi_{,\mu}\pi_{,\mu'}\pi_{;\nu\nu'}\pi_{;\rho\rho'}$$
$$+F_5(\pi, X)\epsilon^{\mu\nu\rho\sigma}\epsilon^{\mu'\nu'\rho'\sigma'}\pi_{,\mu}\pi_{,\mu'}\pi_{;\nu\nu'}\pi_{;\rho\rho'}\pi_{;\sigma\sigma'},$$

then $\xi = \frac{a\mathcal{G}_T(\mathcal{G}_T+D)}{2\theta}$ and one can construct stable non-singular solution.

S. Mironov, V. Rubakov, and V. Volkova (2018 - 2020), arXiv: 1807.08361, 1905.06249, 1910.07019.

2. Naive strong coupling:

$$\int_{-\infty}^{t} \frac{d\xi}{dt} dt = \int_{-\infty}^{t} a(t)\left[\mathcal{F}_T(t) + \mathcal{F}_S(t)\right] dt < \infty$$

This implies that $\mathcal{F}_T \to 0, \mathcal{F}_S \to 0$ as $t \to -\infty$; One also has $\mathcal{G}_T \to 0, \mathcal{G}_S \to 0$ as $t \to -\infty$. In this case, the coefficients in the quadratic action for perturbations about the classical solution tend to zero as $t \to -\infty$. It has been shown that, if we consider the next order of action for perturbations, the strong coupling can be avoided for a specific choice of the parameters of theory.

Y. Ageeva, P. Petrov and V. Rubakov (2020-2022), arXiv: 2009.05071, 2003.01202, 2104.13412

» $\theta = 0$

The quadratic action for the scalar perturbations has the form

$$S^{(2)} = \int \mathrm{d}t\,\mathrm{d}^3x\,a^3 \Bigg(A_1\,\dot{\Psi}^2 + A_2\,\frac{(\vec{\nabla}\Psi)^2}{a^2} + A_3\,\Phi^2 + A_4\,\Phi\frac{\vec{\nabla}^2\beta}{a^2} + A_5\,\dot{\Psi}\frac{\vec{\nabla}^2\beta}{a^2}$$
$$+A_6\,\Phi\dot{\Psi} + A_7\,\Phi\,\frac{\vec{\nabla}^2\Psi}{a^2} + A_8\,\Phi\frac{\vec{\nabla}^2\chi}{a^2} + A_9\,\dot{\chi}\frac{\vec{\nabla}^2\beta}{a^2} + A_{10}\,\chi\ddot{\Psi} + A_{11}\,\Phi\dot{\chi}$$
$$+A_{12}\,\chi\frac{\vec{\nabla}^2\beta}{a^2} + A_{13}\,\chi\frac{\vec{\nabla}^2\Psi}{a^2} + A_{14}\,\dot{\chi}^2 + A_{15}\,\frac{(\vec{\nabla}\chi)^2}{a^2} + A_{17}\,\Phi\chi$$
$$+A_{18}\,\dot{\Psi}\chi + A_{19}\,\Psi\chi + A_{20}\,\chi^2\Bigg)$$

where $A_4 = \theta, A_6 = 3\theta$, and $E = 0$ - partial gauge fix.

» Gauge invariant variables

This action is invariant with respect to small coordinate transformations:

$$x^\mu \to x^\mu - \xi^\mu,$$

where $\xi^\mu = \left(\xi_0, \xi_T^i + \delta^{ij}\partial_j \xi_S\right)^{\mathrm{T}}$. In which the fields change as:

$$\Phi \to \Phi + \dot{\xi}_0, \quad \beta \to \beta - \xi_0 + a^2 \dot{\xi}_S, \quad \chi \to \chi + \xi_0 \dot{\pi}, \quad \Psi \to \Psi + \xi_0 H, \quad E \to E - \xi_S.$$

The action can be rewritten in explicitly gauge-invariant form by introducing new variables (Bardeen variables):

$$\mathcal{X} = \chi + \dot{\pi}\left(\frac{\beta}{a^2} + \dot{E}\right),$$

$$\mathcal{Y} = \Psi + H\left(\frac{\beta}{a^2} + \dot{E}\right),$$

$$\mathcal{Z} = \Phi + \frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{\beta}{a^2} + \dot{E}\right].$$

» Three variables action

In terms of these variables, the action will take the form

$$S^{(2)} = \int \mathrm{d}t\, \mathrm{d}^3x\, a^3 \left(A_1 \left(\dot{\mathcal{Y}}\right)^2 + A_2 \frac{(\vec{\nabla}\mathcal{Y})^2}{a^2} + A_3\, \mathcal{Z}^2 + A_6\, \mathcal{Z}\dot{\mathcal{Y}} + A_7\, \mathcal{Z} \frac{\vec{\nabla}^2\mathcal{Y}}{a^2} \right.$$
$$+ A_8\, \mathcal{Z} \frac{\vec{\nabla}^2\mathcal{X}}{a^2} + A_{10}\, \mathcal{X}\ddot{\mathcal{Y}} + A_{11}\, \mathcal{Z}\dot{\mathcal{X}} + A_{13}\, \mathcal{X} \frac{\vec{\nabla}^2\mathcal{Y}}{a^2} + A_{14} \left(\dot{\mathcal{X}}\right)^2$$
$$\left. + A_{15} \frac{(\vec{\nabla}\mathcal{X})^2}{a^2} + A_{17}\mathcal{Z}\mathcal{X} + A_{18}\, \mathcal{X}\dot{\mathcal{Y}} + A_{20}\mathcal{X}^2 \right)$$

At this point it is clearly seen that the field Z is non-dynamic and we can derive a Z–constraint which has the following form:

$$\mathcal{Z} = \frac{1}{2A_3} \left(-A_7 \frac{\vec{\nabla}^2\mathcal{Y}}{a^2} - A_8 \frac{\vec{\nabla}^2\mathcal{X}}{a^2} + 3A_4\dot{\mathcal{Y}} - A_{11}\dot{\mathcal{X}} - A_{17}\mathcal{X} \right)$$

We used that $A_6 = -3A_4$.

» Brief summary

Due to the following relations for the coefficients:

$$A_3 = \frac{3}{2} A_4 H - \frac{1}{2} A_{11} \dot{\pi},$$

Our options:

<table>
<tr><td>$A_4 \neq 0$</td><td colspan="2">$c_\infty^2 = \mathcal{F}_S / \mathcal{G}_S$</td></tr>
<tr><td rowspan="2">$A_4 \equiv 0$</td><td>$\dot{\pi} \neq 0$</td><td>no dynamics in scalar sector</td></tr>
<tr><td>$\dot{\pi} = 0, H = 0$</td><td>$c_\infty^2 = 1$</td></tr>
</table>

Thus, we obtained that $A_4 = 0$ everywhere, always leads to a stable solution in the scalar perturbation sector. In the case of non-trivial field π there are no dynamical scalar perturbations, and thus the stability condition does not arise at all, and in the case of a static background field π, we obtain a scalar perturbation with the sound speed squared $c_\infty^2 = 1$.

» Reconstruction of Lagrangian functions

Without loss of generality we choose the following form of the scalar field

$$\pi(t) = t,$$

so that $X = 1$. To reconstruct the theory which corresponds some solution we use the following ansatz for the Lagrangian functions

$$F(\pi, X) = f_0(\pi) + f_1(\pi) \cdot X,$$
$$K(\pi, X) = k_1(\pi) \cdot X,$$
$$G_4(\pi, X) = \frac{1}{2}.$$

We are interested to consider the case $G_4 = \text{const}$, which corresponds to GR.

Only the equations of motion and the condition $A_4 = 0$ remain as possible constraints:

$$f_0 = -\dot{H},$$
$$f_1 = -3H^2,$$
$$k_1 = H.$$

» Bouncing solution

Hubble parameter can be choosen in the following form for the case of the bounce:

$$H(t) = \frac{t}{3\left(\tau^2 + t^2\right)},$$

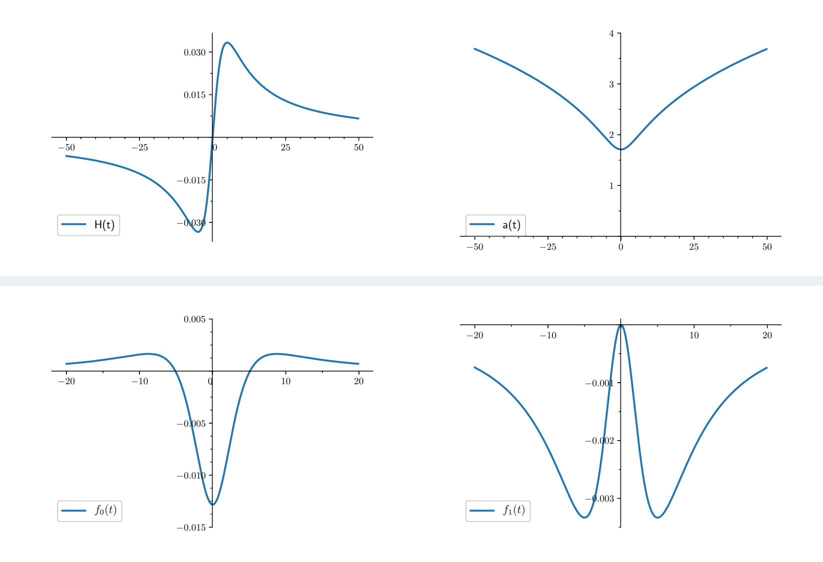
so that

$$a(t) = \left(\tau^2 + t^2\right)^{\frac{1}{6}},$$

and the bounce occurs at $t = 0$. In what follows we take $\tau \gg 1$ to make this scale safely greater than Planck time. The parameter τ determines the duration of the bouncing stage.

Corresponding Lagrangian reads

$$\mathcal{L} = \frac{\pi^2 - \tau^2}{3\left(\tau^2 + \pi^2\right)^2} - \frac{\pi^2 X}{\left(\tau^2 + \pi^2\right)^2} + \frac{\pi X}{3\left(\tau^2 + \pi^2\right)} \Box\pi + \frac{1}{2} R.$$



Hubble parameter $H(t)$, scale factor $a(t)$ and the Lagrangian functions $f_0(t)$, $f_1(t)$ of the bouncing scenario with parameter $\tau = 25$ (recall that $k_1(t) = H(t)$).

» Anisotropic Bianchi I background

Background metric:

$$ds^2 = dt^2 - \left(a^2(t)dx^2 + b^2(t)dy^2 + c^2(t)dz^2\right).$$

The decomposition of metric perturbations $h_{\mu\nu}$ into helicity components in this case has the form

$$\begin{aligned}
h_{00} &= 2\Phi \\
h_{0i} &= -\partial_i \beta + Z_i^T, \\
h_{ij} &= -2\frac{H_i}{H}\Psi g_{ij} - 2\partial_i\partial_j E - \left(\partial_i W_j^T + \partial_j W_i^T\right) + h_{ij}^T,
\end{aligned}$$

where Φ, β, Ψ, E - scalar fields, H_i - Hubble parametres$(i = a, b, c)$ and $H = \frac{1}{3}(H_a + H_b + H_c)$, Z_i^T, W_i^T - are transverse vector fields, $(\partial_i Z_i^T = \partial_i W_i^T = 0)$, h_{ij}^T - is transverse traceless tensor.

» The second-order action

Then the second-order action for scalar sector has the form:

$$
\begin{aligned}
S^{(2)} = \int \mathrm{d}x\, abc \Bigg(& \frac{1}{6} A_1 \sum_{i \neq j} \dot{\Psi}_i \dot{\Psi}_j + \frac{A_2}{2} \sum_{\substack{i=a,b,c \\ i \neq j \neq k}} \Delta_i \Psi_j \Delta_i \Psi_k + A_3 \Phi^2 \\
& + \Phi \left(A_4^i \Delta_i^2 \beta \right) + A_5 \sum_{\substack{i=a,b,c \\ i \neq j \neq k}} \dot{\Psi}_i \left(\Delta_j^2 \beta + \Delta_k^2 \beta \right) + \Phi \left(A_6^i \dot{\Psi}_i \right) + \frac{A_7}{2} \Phi \sum_{\substack{i=a,b,c \\ i \neq j \neq k}} \Delta_i^2 \left(\Psi_j + \Psi_k \right) \\
& + \Phi \left(A_8^i \Delta_i^2 \chi \right) + \dot{\chi} \left(A_9^i \Delta_i^2 \beta \right) + \chi \left(A_{10}^i \ddot{\Psi}_i \right) + A_{11} \Phi \dot{\chi} + \chi \left(A_{12}^i \Delta_i^2 \beta \right) \\
& + \chi \sum_{i,j} \frac{1}{2} A_{13}^{ij} \left(\Delta_i^2 \Psi_j + \Delta_j^2 \Psi_i \right) + A_{14} (\dot{\chi})^2 + A_{15}^i \left(\Delta_i \chi \right)^2 + A_{17} \Phi \chi + \chi \left(A_{18}^i \dot{\Psi}_i \right) \\
& + A_{20} \chi^2 + \frac{1}{2} \sum_{\substack{i,j=a,b,c \\ i \neq j}} B^{ij} \Psi_i \dot{\Psi}_j - \Psi_a \left(B^{ab} \Delta_y^2 \beta + B^{ac} \Delta_z^2 \beta \right) + \Psi_b \left(B^{ab} \Delta_x^2 \beta + B^{bc} \Delta_z^2 \beta \right) \\
& + \Psi_c \left(B^{ac} \Delta_x^2 \beta - B^{bc} \Delta_y^2 \beta \right) \Bigg),
\end{aligned}
$$

where $\Psi_i = \bar{H}_i \Psi$ и $\bar{H}_i = H_i / H$.

» The stability of the Bounce solution with respect to small anisotropy

We consider the action in the unitary gauge $\chi = 0$ and direct the momentum $\bar{k}$ along the x-axis, so $\bar{k} = (k_x, 0, 0)$. Then

$$S^{(2)} = \int \mathrm{d}x \; abc \left(\frac{1}{6} A_1 \sum_{i \neq j} \dot{\Psi}_i \dot{\Psi}_j - A_2 k_x^2 \Psi_b \Psi_c + A_3 \Phi^2 + A_4^a {k_x}^2 \Phi \beta \right.$$

$$+ A_5 \beta {k_x}^2 \left(\dot{\Psi}_b + \dot{\Psi}_c \right) + \Phi \left(A_6^i \dot{\Psi}_i \right) + A_7 \Phi {k_x}^2 \left(\Psi_b + \Psi_c \right)$$

$$\left. + \frac{1}{2} \sum_{\substack{i,j=a,b,c \\ i \neq j}} B^{ij} \Psi_i \dot{\Psi}_j - k_x^2 \beta \left(B^{ab} \Psi_b + B^{ac} \Psi_c \right) \right)$$

After removing the constraints, we get the following action on the Ψ variable

$$S^{(2)} = \int \mathrm{d}t\, \mathrm{d}^3x\, abc \left(\mathcal{G}_S \left(\dot{\Psi} \right)^2 + M\Psi^2 + \mathcal{F}_S \frac{k_x^2}{a^2} \Psi^2 \right),$$

where

$$\mathcal{G}_S = \frac{2}{9} \frac{A_3 {A_1}^2}{(A_4^x)^2} \left(\bar{H}_b + \bar{H}_c \right)^2 - \frac{2}{3} \frac{A_1}{A_4^x} \left(A_4^y \bar{H}_b + A_4^z \bar{H}_c \right) \left(\bar{H}_b + \bar{H}_c \right) + \frac{2}{3} A_1 \bar{H}_b \bar{H}_c,$$

$$\mathcal{F}_S = -2A_2 \bar{H}_b \bar{H}_c - \frac{1}{9a^3} \left(\bar{H}_b + \bar{H}_c \right)^2 \frac{d}{dt} \left[\frac{A_1^2 a^3}{A_4^x} \right] + \frac{{A_1}^2}{9A_4^x} \left(\bar{H}_b^2 - \bar{H}_c^2 \right) \left(H_b - H_c \right),$$

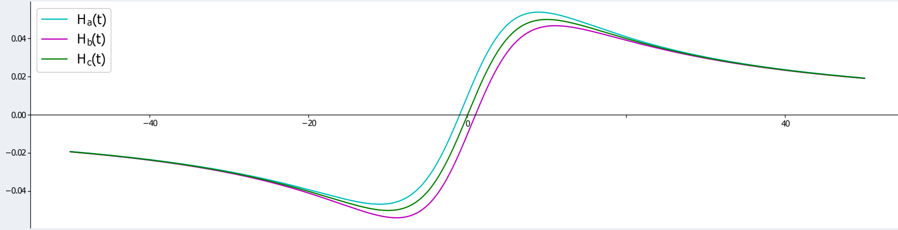
» Anisotropic bounce

$$\mathcal{L} = \frac{\pi^2 - \tau^2}{3\left(\tau^2+\pi^2\right)^2} - \frac{\pi^2 X}{\left(\tau^2+\pi^2\right)^2} + \frac{\pi X}{3\left(\tau^2+\pi^2\right)}\Box\pi + \frac{1}{2}R.$$

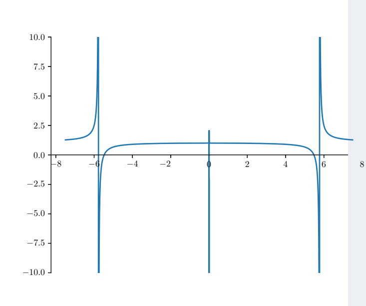
Anisotropic bounce:

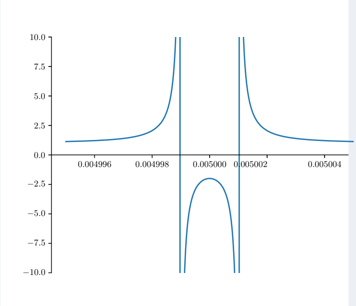
$$H_a = \frac{t}{(\tau^2+t^2)} + \frac{\alpha}{(\tau^2+t^2)^{3/2}}, \quad H_b = \frac{t}{(\tau^2+t^2)} - \frac{\alpha}{(\tau^2+t^2)^{3/2}}, \quad H_c = \frac{t}{(\tau^2+t^2)}.$$

Here the parameter τ defines the bounce amplitude and α the degree of deviation from the isotropic case



To analyze the stability of the scalar field, we numerically plot the square of the speed of sound c_S^2:





The square of the speed of sound c_S^2, when we choose $\alpha = 0.1, \tau = 10$. In this case, the square of the speed of sound will have at least 2 symmetric singular points and tends to 0 as univerce becomes isotropic.

Thank you for your attention!

* A combination of Einstein equations (spatially flat):

$$\frac{dH}{dt} = -4\pi G(\rho + p)$$

$\rho = T_{00} =$ energy density; $T_{ij} = \delta_{ij} p =$ effective pressure.

* The Null Energy Condition:

$$T_{\mu v} n^{\mu} n^{\nu} \geq 0, n^{\mu} = (1,1,0,0) \Longrightarrow \rho + p \geq 0 \Longrightarrow dH/dt \leq 0,$$

Hubble parameter was greater early on. **No bounce**

* Another side of the NEC: Covariant energy-momentum conservation:

$$\frac{d\rho}{dt} = -3H(\rho + p)$$

NEC: energy density decreases during expansion, except for $p = -\rho$, cosmological constant. **No Genesis**

» An example of an attempt to violate the NEC

Let's consider Minkowski background with one scalar field π, a spatially homogeneous classical solution $\pi_c(t)$ may or may not be pathological. The pathology, if any, shows up in the behavior of small perturbations about this background, $\pi = \pi_c + \chi$. Assuming that the linearized field equation for χ is of the second order in derivatives, the quadratic Lagrangian for χ is always given by

$$L_\chi^{(2)} = \frac{1}{2} U \dot{\chi}^2 - \frac{1}{2} V \left(\partial_i \chi\right)^2 - \frac{1}{2} W \chi^2$$

Rubakov V A "The Null Energy Condition and its violation"Phys. Usp. 57 128–142 (2014)

The dispersion relation is

$$U\omega^2 = V\mathbf{p}^2 + W,$$

1. Stable background: $U > 0, V > 0, W \geqslant 0$ energy density for perturbations

$$T_{00}^{(2)} = \frac{1}{2}U\dot{\chi}^2 + \frac{1}{2}V\left(\partial_i\chi\right)^2 + \frac{1}{2}W\chi^2 > 0$$

   1.1 $V < U$ - subluminal speed - OK
   1.2 $V > U$ - super-luminal speed - theory cannot be UV-completed in a Lorentz- invariant way.
   1.3 $U = V$ - potentially problematic.
   1.4 $U > 0, V > 0, W < 0$ - Tachyonic instability.
2. $U > 0, V < 0$ or $U < 0, V > 0$ - Gradient instability.
3. $U < 0, V < 0$ - Ghost instability.

» Only $A_4 = 0$ case

After integrating out Z, introducing

$$\zeta = \mathcal{Y} + \eta \mathcal{X}, \quad \eta = \frac{3A_{11}A_4 - 2A_{10}A_3}{4A_1A_3 - 9{A_4}^2},$$

and integrating out $\mathcal{X}$ variable, we get the following action:

$$S^{(2)} = \int \mathrm{d}t\, \mathrm{d}^3x\, a^3 \left(A_2 \frac{\left(\overrightarrow{\nabla}\zeta\right)^2}{a^2} - \frac{1}{9}\frac{{A_1}^2}{A_3}\frac{\left(\overrightarrow{\nabla}^2\zeta\right)^2}{a^4} \right)$$

which means the absence of dynamics of the field ζ.

» Additional options

From the view of the Z–constraint,

$$\mathcal{Z} = \frac{1}{2A_3}\left(-A_7\frac{\overrightarrow{\nabla}^2\mathcal{Y}}{a^2} - A_8\frac{\overrightarrow{\nabla}^2\mathcal{X}}{a^2} + 3A_4\dot{\mathcal{Y}} - A_{11}\dot{\mathcal{X}} - A_{17}\mathcal{X}\right)$$

we can also distinguish the case $A_3 = 0$ as a singular point. By reason of the following ratios on the coefficients

$$A_3 = \frac{3}{2}A_4H - \frac{1}{2}A_{11}\dot{\pi},$$

we have two options: $A_4 = 0, A_{11} = 0$ and $A_4 = 0, \dot{\pi} = 0$.

» $A_4 = 0, A_{11} = 0$

In this case, the Z–constraint gives us the condition:

$$\mathcal{X} = -\frac{A_7}{A_8}\mathcal{Y}$$

Which brings the action into the following form:

$$S^{(2)} = \int \mathrm{d}t\, \mathrm{d}^3x\, a^3\, m\mathcal{Y}^2$$

where

$$m = \text{(Some VERY big expression)}$$

» $A_4 = 0, \dot{\pi} = 0$

In this case, the condition $A_4 = 0$ takes the form of:

$$G_4 H = 0$$

For $A_4 = 0$ it is also necessary to impose the condition $H = 0$. And the action takes the form:

$$S^{(2)} = \int \mathrm{d}t\,\mathrm{d}^3x\,a^3 \left(\mathcal{G}_S \left(\dot{\mathcal{Y}}\right)^2 + m\mathcal{Y}^2 - \mathcal{F}_S \frac{\left(\overrightarrow{\nabla}\mathcal{Y}\right)^2}{a^2} \right)$$

Where $\mathcal{F}_S = \mathcal{G}_S$ The case of the Minkowski space in GR ($G_4 = \frac{1}{2}$) is a special case of this solution.