Nonlocal Potts model on random lattice and chromatic number of the plane

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Classic Potts model

$$H = \sum_{i,k \in M} J_{ik} \delta(\sigma_i, \sigma_k)$$

J_{ik} – interaction, $\sigma \in S, S = \{1, 2, ..., q\}$

Defined on the **regular** lattice

Common problems

1. Finding the vacuum state of the system
2. Calculating partition function $Z = \sum_{\{\sigma_i\}} e^{-\beta H}$

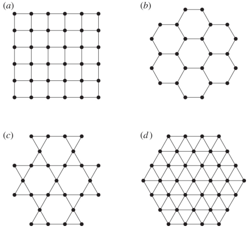


Figure 1: Typical regular lattices

Defined on a random lattice

$$H = \sum_{i,k \in M} J_{ik} \delta(\sigma_i, \sigma_k),$$

$$J_{ik} = \begin{cases} J, & R - \frac{\delta}{2} \leq |i - k| \leq R + \frac{\delta}{2} \\ 0, & \text{otherwise}, \end{cases}$$

$\delta \ll R$.

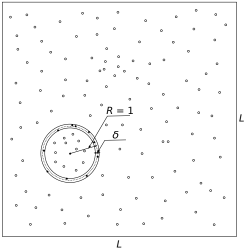


Figure 2: Schematic representation of the model on 2d random lattice

The number of neighbors (in the context of chosen metric)

- **Regular** lattice – the same for any spin, pretty small, depends on the geometry of the lattice
- **Random** lattice – obeys Gaussian distribution, not bounded

- **Interaction with closest neighbors (local)**
  - ➔ Ising model
  - ➔ Potts model
  - ➔ Z_N model
- **Interaction with all particles (massless field)**
  - ➔ Gravitational interaction
  - ➔ Electromagnetic interaction
- **Interaction at finite distance**
  - ➔ network models
  - ➔ chromatic clusters in images of real objects
  - ➔ problems of combinatorial topology

Erdos-Hadwiger-Nelson (EHN) problem

What is the minimal number of colors q one must use to color the R^d space in such a way that no two points at unit distance are colored identically?

Solutions

- R^1: $q = 2$
- R^2: $5 \leqslant q \leqslant 7$
- R^3: $6 \leqslant q \leqslant 15$

Figure 3: R^1

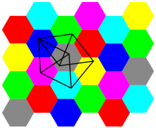


Figure 4: R^2: Moser's spindle and 7-color hexagon tiling *wikipedia.org*

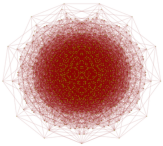


Figure 5: R^2: The unit-distance graph, $q = 5$ *arXiv:1804.02385*

Hamiltonian

$H = A \sum_{x,y} J_{xy} \delta_{i(x)i(y)}$,

A – normalization factor, $A = 1$

The goal

Using supercomputer simulations find minima of H and corresponding vacuum configurations for q in range [2,7] for two dimentional random lattice, starting from random configuration.

Vacuum states of such models have not been investigated before.

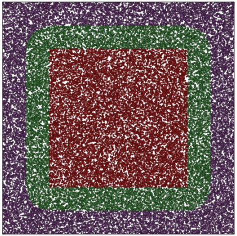


Figure 6: The schematic picture of energy zones.

$\langle n \rangle = 2\pi R\delta \frac{N}{L^2}$ – average number of neighbors.

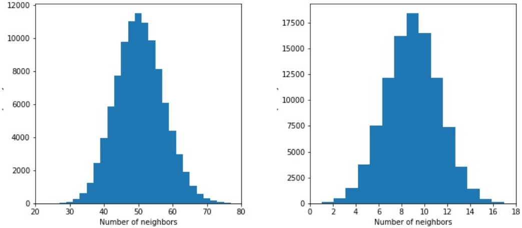


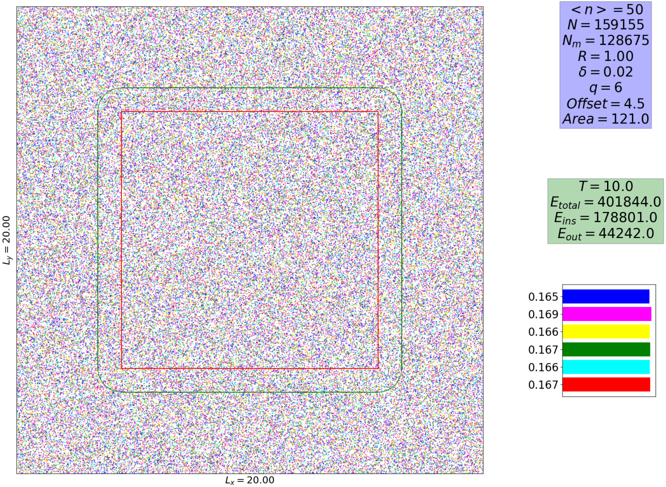
Figure 7: $L = 20, q = 4, N = 159155, \delta = 0.02$. **Left**: distribution of number of neighbors, $\langle n \rangle = 50$; **right**: distribution of number of neighbors which have the least represented color.

Minimization algorithm (simulated annealing)

Steps

1. Generate random configuration
2. Calculate its energy
3. Choose temperature reduction function, define T_{ini} and T_{fin}
4. Choose random order of picking up particles using Fisher-Yates shuffle algorithm
5. For each particle randomly pick new color. If the energy decreased or stayed unchanged then accept new configuration. In the case the energy increased configuration is accepted with probability $P = e^{-\frac{E' - E}{T}}$
6. Reduce temperature
7. Repeat steps starting from **4**, while $T > T_{fin}$.

An illustration of evolution of the system for $q = 6$



Examples of configurations with minimal energies

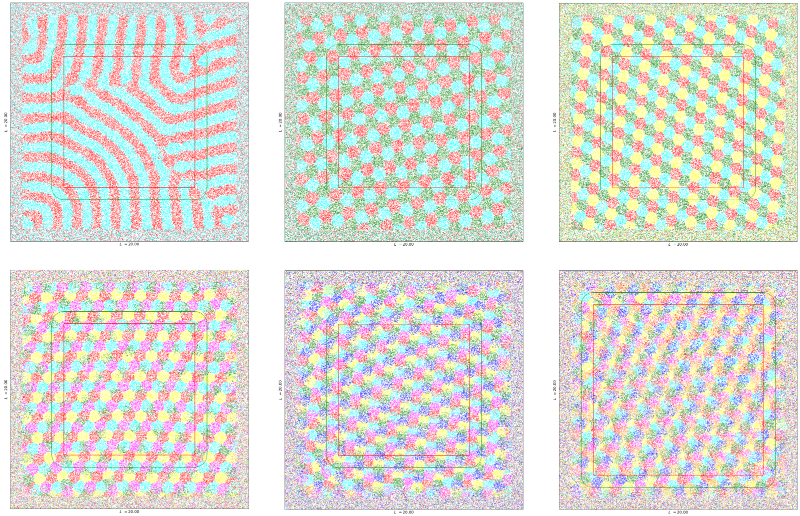


Figure 8: Top row: q = 2,3,4; bottom row: q = 5,6,7

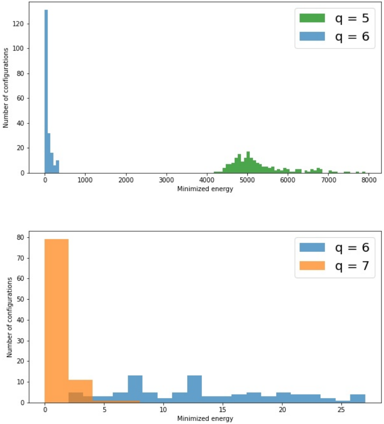


Figure 9: **Top**: $q = 5$ and $q = 6$; **bottom**: $q = 6$ and $q = 7$. On the base of 200 configurations for each color. The number of configurations with zero energy is 0 for $q = 5$, 1 for $q = 6$, 170 for $q = 7$.

Color symmetry breaking (static)

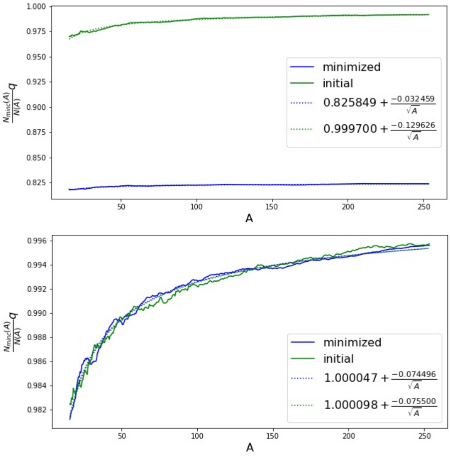


Figure 10: The ratio of number of particles with least number of particles to the whole number of particles inside particular area. **Top**: $q = 5$, **bottom**: $q = 3$.

Color symmetry breaking (static): summary

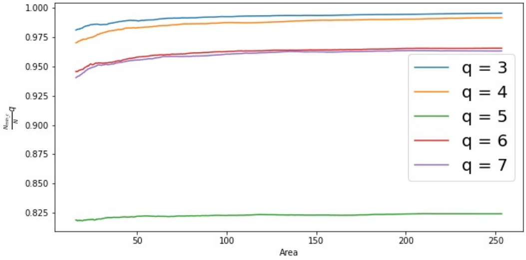


Figure 11: The ratio of least represented color to whole number of particles scaled by q for $L = 20$, $N = 159155$

Color symmetry breaking (dynamic)

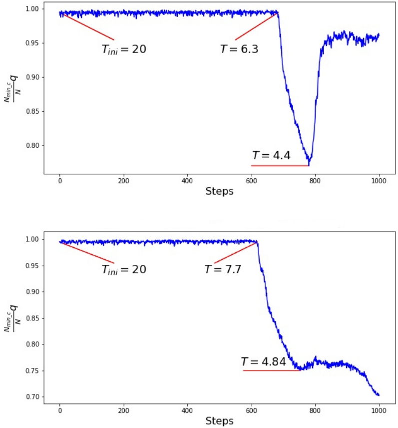


Figure 12: The dynamic of changing in ratio of least represented color to whole number of particles scaled by q in the process of minimization. **Top**: $q = 6$, **bottom**: $q = 5$.

Color symmetry breaking (dynamic): summary

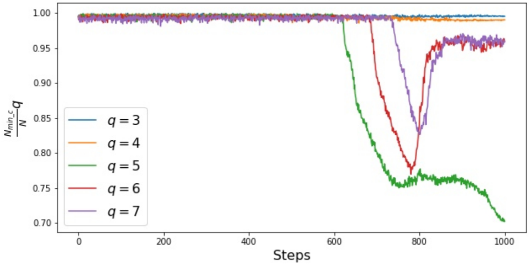


Figure 13: The dynamic of changing in ratio of least represented color to whole number of particles scaled by q in the process of minimization

- The simulated annealing algorithm for finding ground states of nonlocal Potts model is devised and implemented
- The result was improved sufficiently comparing to greedy algorithm.
- For $q = 5$ no state with zero energy was found
- The study of vacuum energy for $q = 6$ requires additional analysis.
- The breaking of color symmetry emerges for ground states and it is the most prominent for $q = 5$

Thank you for attention!