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# Renormalization group inspired autonomous equations for secular effects in de Sitter space

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# Content

- The de Sitter universe is a spacetime with positive constant 4-curvature that is homogeneous and isotropic in both space and time.
- It is completely characterized by only one constant and has as many symmetries as the flat (Minkowski) spacetime.
- De Sitter universe plays a central role in understanding the properties of cosmological inflation.
- Inflation is a stage of accelerated expansion of the early Universe. The expansion is quasi-exponential, and at lowest order it can be approximated by de Sitter space.
- The inflationary stage allows the growth of quantum fluctuations, which are necessary to explain observed large-scale structure of the Universe. So it is important to study quantum field theory in de Sitter background.

# Introduction and Motivations

- It has been known for some time that perturbatively calculated correlation functions of certain quantum field theories set in an expanding background grow secularly (infinitely) with time.
- This growth can lead to a breakdown of the perturbation theory past a certain point in time.
- The renormalization group method is very effective in quantum field theory.
- Dynamical renormalization group permits to improve perturbative solutions of some complicated differential equations.
- The attempts to apply the dynamical renormalization group to the treatment of secular effects in de Sitter spacetime did not reproduce known results.

## Introduction and Motivations

- In this work we considered a massless scalar field with quartic self-interaction in de Sitter spacetime and developed a semi-heuristic method for taking the late-time limit of a series of secularly growing terms obtained from quantum perturbative calculations.
- We compared our results with the well-known stochastic approach (Starobinsky, 1986, Starobinsky and Yokoyama, 1994).

## De Sitter Space in Flat Coordinates

We consider the de Sitter spacetime represented as an expanding spatially flat homogeneous and isotropic universe with the following metric

$$ds^2 = dt^2 - a^2(t)(dx^2 + dy^2 + dz^2) \ ,$$

where the scale factor $a(t)$ is

$$a(t) = e^{Ht} \ , -\infty < t < \infty,$$

and $H$ is the Hubble constant that characterizes the rate of expansion.

If we introduce a conformal time coordinate, given by $\eta(t) \equiv \int dt\, a^{-1}(t)$

$$ds^2 = a^2(\eta)(d\eta^2 - dx^2 - dy^2 - dz^2),$$
$$a(\eta) = -\frac{1}{H\eta}, \; -\infty < \eta < 0.$$

Physical distances: $\ell_{phys} = a(\eta)\ell = -\ell/(H\eta)$.

Physical energy or momentum: $k_{phys} = k/a(\eta) = -kH\eta$.

## Massless Scalar Field in de Sitter: Equation of Motion, Basis Functions and Quantization

$$S = \int d^4x \sqrt{-g} \left( \frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - V(\phi) \right)$$

We consider

$$V(\phi) = \frac{\lambda}{4}\phi^4, \;\; \lambda \ll 1.$$

The equation of motion for the free theory ($\lambda = 0$) is

$$\ddot{\phi}(\vec{x}, t) + 3H\dot{\phi}(\vec{x}, t) - \frac{\nabla^2}{a^2}\phi(\vec{x}, t) = 0 \; .$$

Transitioning to conformal time and making Fourier transformation

$$\phi(\vec{x}, t) = \int \frac{d^3\vec{k}}{(2\pi)^3} \phi_k(\eta) e^{i\vec{k}\cdot\vec{x}},$$

$$\phi_k''(\eta) - \frac{2}{\eta}\phi_k'(\eta) + k^2\phi_k(\eta) = 0, \quad k = |\vec{k}|.$$

## Equation of Motion, Basis Functions and Quantization

General solution of the Klein-Gordon equation

$$\phi_k(\eta) = C_1(1 + ik\eta)e^{-ik\eta} + C_2(1 - ik\eta)e^{ik\eta} \ .$$

Now, $\phi$ can be decomposed as

$$\phi(\vec{x}, t) = \int \frac{d^3\vec{k}}{(2\pi)^3} \left\{ u_k(\eta)e^{i\vec{k}\cdot\vec{x}} a_{\vec{k}} + u_k^*(\eta)e^{-i\vec{k}\cdot\vec{x}} a_{\vec{k}}^\dagger \right\} \ ,$$

where

$$a_{\vec{k}}|0\rangle = 0,$$

and

$$u_k(\eta) \sim \phi_k(\eta).$$

How to choose $C_1$ and $C_2$ ?

For modes with large momentum,

$$k_{phys} = (-kH\eta) \gg H \;\longleftrightarrow\; \text{short physical wavelength,}$$

the theory should behave like in flat spacetime, hence

$$u_k(\eta) = \frac{iH}{\sqrt{2k^3}}(1 + ik\eta)e^{-ik\eta} \; .$$

Such a choice is called the Bunch-Davies vacuum.

## Expectation Values of the Free Theory ($\lambda = 0$) and Secular Growth

$$\begin{aligned} \langle 0|\phi(\vec{x},t)\phi(\vec{y},t)|0\rangle &= \int \frac{d^3\vec{k}}{(2\pi)^3} u_k(\eta) u_k^*(\eta) e^{i\vec{k}\cdot(\vec{x}-\vec{y})} \\ &= \frac{H^2}{2} \int \frac{d^3\vec{k}}{(2\pi)^3} \frac{(1+k^2\eta^2)}{k^3} e^{i\vec{k}\cdot(\vec{x}-\vec{y})} \ . \end{aligned} \quad (1)$$

We would like to find the late-time behavior of the long-wavelength part of (1), i.e., the part coming from the modes with physical momenta much less than the Hubble scale, $-kH\eta \ll H$.

In the case of coinciding spatial points ($\vec{x} = \vec{y}$)

$$\langle 0|\phi^2(\vec{x},t)|0\rangle_L = \frac{H^2}{4\pi^2}\int_{\kappa}^{-1/\eta}\frac{dk}{k}(1+k^2\eta^2)$$
$$= -\frac{H^2}{4\pi^2}\left(\ln(-\kappa\eta) - \frac{1}{2} + \frac{\kappa^2\eta^2}{2}\right), \quad (2)$$

where we introduced an infrared cutoff $\kappa$ for the comoving momentum $k$, since the integral is divergent at $k = 0$.

## Expectation Values of the Free Theory ($\lambda = 0$) and Secular Growth

For $t \to \infty$ (i.e., $-\kappa\eta \ll 1$), the first term in (2) dominates, so in the late-time limit we have

$$\langle 0|\phi^2(\vec{x}, t)|0\rangle_L = \frac{H^3}{4\pi^2}(t - t_0) \ ,$$

where $t_0 \equiv (1/H) \ln(\kappa/H)$; thus, it grows linearly with time.

- This time-dependence is a sign of breakdown of de Sitter invariance.
- It was shown by B. Allen (1985)[1], and subsequently by B. Allen and A. Folacci (1987)[2] that no de Sitter invariant state exists for a massless scalar field.

[1]B. Allen, Phys. Rev. D **32** (1985), 3136

[2]B. Allen and A. Folacci, Phys. Rev. D **35**, 3771 (1987)

# Expectation Values and Secular Growth

In the presence of quartic interaction, $V(\phi) = \frac{\lambda}{4}\phi^4$, the leading late-time behavior of $\langle \phi^2(\vec{x}, t) \rangle_L$ can be calculated using perturbation theory and the "in-in" (Schwinger - Keldysh) formalism

$$\langle \phi^2(\vec{x}, t) \rangle_L = \frac{H^3}{4\pi^2} t - \lambda \frac{H^5}{24\pi^4} t^3 + \lambda^2 \frac{H^7}{80\pi^6} t^5 + \mathcal{O}(\lambda^3). \quad (3)$$

- The perturbative calculation of the long-wavelength part of $\langle \phi^2(\vec{x}, t) \rangle$ gives a series with terms that behave like $\lambda^n (Ht)^{2n+1}$.
- When $Ht > 1/\sqrt{\lambda}$, the perturbation theory breaks down, so it can't make reliable predictions at late times.

- It is disturbing that expectation values like (3) do not approach a finite limit, since they can enter in the expressions for energy density and observable correlation functions.
- There are also subdominant secular terms present: they are suppressed by additional powers of $\lambda$ with respect to the leading terms, so we ignore them here.

## Autonomous Equations

- We would like to calculate the expectation value of a product of field operators.
- We do not have the dynamical equation governing this function, but we have some information obtained by perturbative methods. For $\lambda \ll 1$,

$$f(t) = At - \lambda B t^3 + \mathcal{O}(\lambda^2) \ , \tag{4}$$

  where $A$ and $B$ are some known positive constants.
- As $t$ grows, the perturbation theory breaks down and the expansion (4) can no longer be trusted.
- From (4), $f(t) \to \infty$ as $t \to \infty$.
- But we can guess from physical considerations that $f(t)$ should be finite.
- How can we model the correct behavior and follow what happens at late times?

# Autonomous Equations

Our proposal: construct a simple autonomous first-order differential equation

$$\frac{df}{dt} = F[f(t)]$$

that produces the first two terms of the expression (4) by iterations. Denote $y(t) \equiv A(t - t_0)$, then
$f(t) = At - \lambda Bt^3 + \mathcal{O}(\lambda^2) \to$

$$f(t) = y(t) - \lambda \frac{B}{A^3} [y(t)]^3 + \mathcal{O}(\lambda^2) \ . \qquad (5)$$

Differentiating (5),

$$\frac{df}{dt} = A - \lambda \frac{3B}{A^2} y^2 + \mathcal{O}(\lambda^2) \ .$$

Within the given accuracy, $y^2$ on the right side of this equation can be replaced by $f^2$,

$$\frac{df}{dt} = A - \lambda \frac{3B}{A^2} f^2. \tag{6}$$

The iterative solution of (6) gives us the original expansion.

## Autonomous Equations: $\lambda$-order

But it's also possible to solve the equation

$$\frac{df}{dt} = A - \lambda \frac{3B}{A^2} f^2$$

explicitly. The solution, with initial condition $f(0) = 0$, is

$$f(t) = \sqrt{\frac{A^3}{3\lambda B}} \tanh\left[\sqrt{\frac{3\lambda B}{A}}(t)\right] .$$

The remarkable feature of this expression is that it is regular for all values of $t$, and when $t \to \infty$, one has

$$f(t) \to \sqrt{\frac{A^3}{3\lambda B}} .$$

## Autonomous Equations: $\lambda^2$-order

In principle, this procedure can be generalized for the situation when we have more than two terms coming from perturbation theory

$$f(t) = At - \lambda B t^3 + \lambda^2 C t^5 + \mathcal{O}(\lambda^3) \, .$$

The corresponding autonomous equation is

$$\frac{df}{dt} = A - \lambda \frac{3B}{A^2} f^2 + \lambda^2 \frac{6B^2}{A^5} \left( \frac{5AC}{6B^2} - 1 \right) f^4 \, . \qquad (7)$$

This equation is also integrable and we can obtain its implicit solution $t = t(f)$. In general it is not possible to find the explicit form of $f(t)$. But in some cases we can obtain the solution of Eq. (7) in the form of perturbative expansion in a small parameter.

## Autonomous Equations: $\lambda^2$-order

The corresponding autonomous equation is

$$\frac{df}{dt} = A - \lambda \frac{3B}{A^2} f^2 + \lambda^2 \frac{6B^2}{A^5} \left( \frac{5AC}{6B^2} - 1 \right) f^4 \, .$$

If $\frac{5AC}{6B^2} - 1 = 0$, we are back to the previous equation and its solution. This is not surprising, since

$$f(t) = \sqrt{\frac{A^3}{3\lambda B}} \tanh \left[ \sqrt{\frac{3\lambda B}{A}} (t) \right]$$

$$\to f(t) = At - \lambda B t^3 + \lambda^2 \frac{6B^2}{5A} t^5 \, .$$

## Autonomous Equations: $\lambda^2$-order

Let us denote $\epsilon \equiv \frac{5AC}{6B^2} - 1$, and rescale $F(t) \equiv \sqrt{\frac{3\lambda B}{A^3}} f(t)$. Then

$$\frac{df}{dt} = A - \lambda \frac{3B}{A^2} f^2 + \lambda^2 \frac{6B^2}{A^5} \left( \frac{5AC}{6B^2} - 1 \right) f^4 \; .$$

$$\rightarrow \frac{dF}{dt} = \sqrt{\frac{3\lambda B}{A}} \left( 1 - F^2 + \frac{2}{3} \epsilon F^4 \right) \; . \qquad (8)$$

If $\epsilon$ is small, we can look for the solution of this equation in the form of the perturbative expansion

$$F(t) = F_0(t) + \epsilon F_1(t) + \mathcal{O}(\epsilon^2) \; .$$

# Autonomous Equations: $\lambda^2$-order

To first-order in $\epsilon$, the solution is

$$f(t) = \left(1 + \frac{\epsilon}{3}\right)\sqrt{\frac{A^3}{3\lambda B}}\tanh\left[\sqrt{\frac{3\lambda B}{A}}t\right] + \frac{\epsilon\left\{\frac{2}{3}\sqrt{\frac{A^3}{3\lambda B}}\tanh\left[\sqrt{\frac{3\lambda B}{A}}t\right] - At\right\}}{\cosh^2\left[\sqrt{\frac{3\lambda B}{A}}t\right]},$$

- Expansion in small $\lambda$ reproduces the original series

$$f(t) = At - \lambda Bt^3 + \lambda^2 Ct^5 + \mathcal{O}(\lambda^3)\,.$$

- As $t \to \infty$, it approaches a finite limit

$$f(t) \to \sqrt{\frac{A^3}{3\lambda B}}\left(1 + \frac{\epsilon}{3}\right)\sqrt{\frac{A^3}{3\lambda B}}\left(\frac{2}{3} + \frac{5}{18}\frac{AC}{B^2}\right).$$

# Back to de Sitter

Recall that

$$\langle \phi^2(\vec{x},t)\rangle_L = \frac{H^3}{4\pi^2}t - \lambda\frac{H^5}{24\pi^4}t^3 + \lambda^2\frac{H^7}{80\pi^6}t^5 + \mathcal{O}(\lambda^3) \qquad (9)$$

We can identify the expression (9) with the general expression for the function $f(t)$. The coefficients $A, B$ and $C$ for $f(t) = \langle \phi^2(\vec{x},t)\rangle_L$ are

$$A = \frac{H^3}{4\pi^2}\ , \quad B = \frac{H^5}{24\pi^4}\ , \quad C = \frac{H^7}{80\pi^6}\ .$$

Solutions of the autonomous equations:

- $\lambda$: $\langle \phi^2(\vec{x},t)\rangle_L = \frac{H^2}{\sqrt{8\lambda}\pi}\tanh\left[\sqrt{\frac{H^2\lambda}{2\pi^2}}t\right] \to \frac{H^2}{\sqrt{8\lambda}\pi}$.
- $\lambda^2$: $\langle \phi^2(\vec{x},t)\rangle_L =$

$$\frac{7}{6}\frac{H^2}{\sqrt{8\lambda}\pi}\tanh\left[\sqrt{\frac{H^2\lambda}{2\pi^2}}t\right] + \frac{\frac{1}{3}\frac{H^2}{\sqrt{8\lambda}\pi}\tanh\left[\sqrt{\frac{H^2\lambda}{2\pi^2}}t\right] - \frac{H^3}{8\pi^2}t}{\cosh^2\left[\sqrt{\frac{H^2\lambda}{2\pi^2}}t\right]} \to \frac{7}{6}\frac{H^2}{\sqrt{8\lambda}\pi}.$$

## Comparison with the Stochastic Approach

The stochastic approach argues[3] that the behavior of the long-wave part of the quantum field $\phi(\vec{x}, t)$ can be modelled by an auxiliary classical stochastic variable $\varphi$ with a probability distribution $\rho(\varphi, t)$ that satisfies the Fokker-Planck equation

$$\frac{\partial \rho}{\partial t} = \frac{H^3}{8\pi^2} \frac{\partial^2 \rho}{\partial \varphi^2} + \frac{1}{3H} \frac{\partial}{\partial \varphi} \left( \frac{\partial V}{\partial \varphi} \rho(t, \varphi) \right) . \qquad (10)$$

$$\to \langle \varphi^n \rangle = \langle \phi^n(\vec{x}, t) \rangle_L \, , \langle \varphi^n \rangle = \int_{-\infty}^{\infty} d\varphi \, \varphi^n \rho(\varphi, t)$$

A well-known case that is described by a Fokker-Planck equation is the Brownian motion of particles.

[3] A. A. Starobinsky and J. Yokoyama, Phys. Rev. D **50**, 6357 (1994)

# Comparison with the Stochastic Approach

At late times any solution of the equation (10) approaches the static solution

$$\rho(\varphi) = \left(\frac{32\pi^2\lambda}{3}\right)^{\frac{1}{4}} \frac{1}{\Gamma\left(\frac{1}{4}\right) H} \exp\left(-\frac{2\pi^2\lambda\varphi^4}{3H^4}\right)$$

$$\to \langle\varphi^2\rangle = \sqrt{\frac{3}{2\pi^2}} \frac{\Gamma\left(\frac{3}{4}\right)}{\Gamma\left(\frac{1}{4}\right)} \frac{H^2}{\sqrt{\lambda}} .$$

Recall the result obtained from autonomous equations $\langle\phi^2\rangle_L = \frac{7}{6}\frac{H^2}{\sqrt{8\lambda\pi}}$,

$$\to \frac{\langle\varphi^2\rangle - \langle\phi^2\rangle_L}{\langle\varphi^2\rangle} \approx 0.0036 = 0.36\% .$$

We also compared the results for the product of four fields

$$\frac{\langle\varphi^4\rangle - \langle\phi^4\rangle_L}{\langle\varphi^4\rangle} \approx 0.0905 = 9.05\% .$$

How can one calculate the perturbative series for the correlators ?

There are two methods:

Schwinger–Keldysh technique.

Yang–Feldman equation.

# Schwinger–Keldysh technique

- Schwinger-Keldysh or "in-in" or "closed time path" formalism serves for the calculations of expectation values of operators when only the initial state of the system is given.
- In contrast to the "in-out" formalism there are four types of the propagators and two types of vertices, characterizing the quantum fields on the way forward in time and "back in time".
- After some calculations one remains with the integrals including Wightman functions and theta-functions.
- Diagrams that correspond to these integrals look similar to Feynman diagrams.

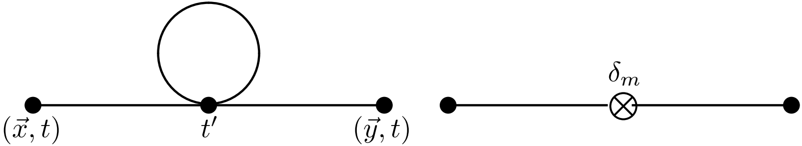


Figure: One-loop and counterterm diagrams.

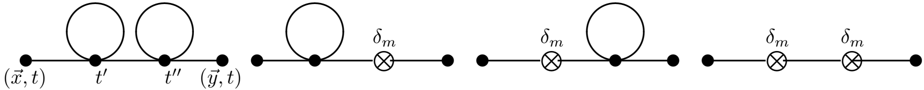


Figure: Diagram with two independent loops and and the counterterm diagrams.

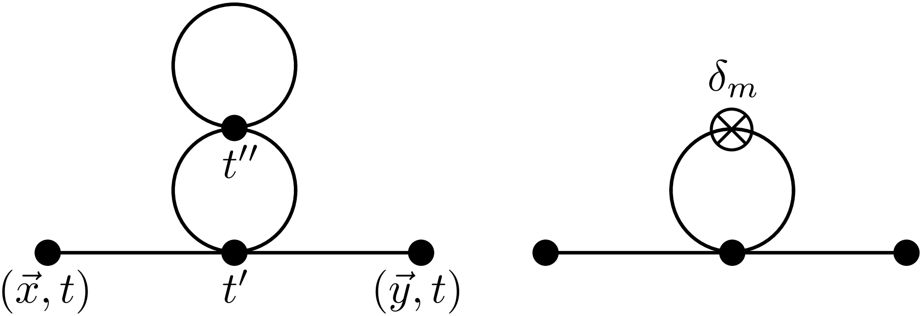


Figure: Snowman diagram and the corresponding counterterm diagram with the mass counterterm insertion in its loop.

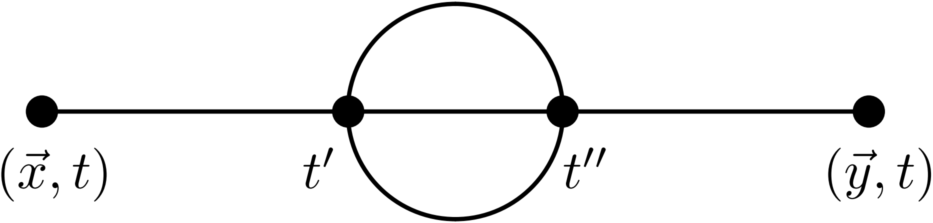


Figure: Sunset diagram.

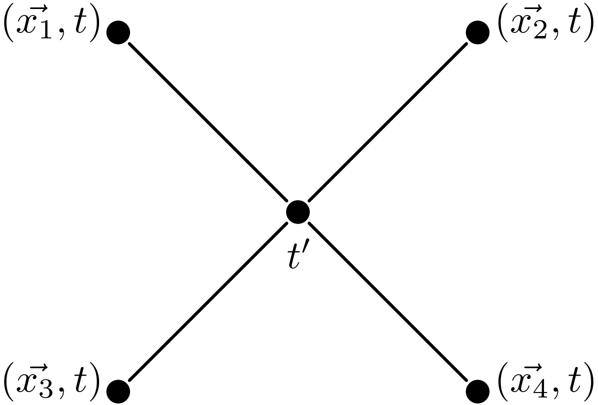


Figure: Connected diagram for the four-point function - cross.

## Yang-Feldman equation for the long-wavelength part of the scalar field on de Sitter background

Yang-Feldman equation, connecting the interacting and non-interacting quantum fields was suggested in
C. N. Yang and D. Feldman, The S Matrix in the Heisenberg Representation, Phys. Rev. **79**, 972 (1950).
Its application to a long-wave (infrared) part of a scalar field was suggested in
R. P. Woodard, Generalizing Starobinskii's Formalism to Yukawa Theory and to Scalar QED, J. Phys. Conf. Ser. **68**, 012032 (2007).
This application is described in detail in
V. K. Onemli, Vacuum Fluctuations of a Scalar Field during Inflation: Quantum versus Stochastic Analysis, Phys. Rev. D **91**, 103537 (2015).
It is a very efficient method for calculation of correlators!

There are two kinds of the infrared reduced scalar fields: the free field $\phi_0(\vec{x}, t)$, which satisfies the Klein-Gordon equation in the absence of the self-interaction, and the full infrared reduced scalar field $\phi(\vec{x}, t)$. These two fields are connected by the equation

$$\begin{aligned}\phi(t, \vec{x}) &= \phi_0(t, \vec{x}) \\ &\quad - \int_0^t dt' a^3(t') \int d^3x \, G(t, \vec{x}; t'\vec{x}') \frac{V'(\phi)}{1 + \delta Z}.\end{aligned}$$

Green's function $G$ satisfies the retarded boundary conditions, $Z$ is the renormalization constant of the scalar field and the potential $V$ includes the mass and coupling constant counterterms. Simple considerations show that the counterterms do not give contributions to the leading infrared part of the correlators.

The leading infrared part of the retarded Green's function has the form

$$G(t,\vec{x};t'\vec{x'}) = \frac{1}{3H}\theta(t-t')\delta^3(\vec{x}-\vec{x}')\left[\frac{1}{a^3(t')} - \frac{1}{a^3(t)}\right].$$

Then

$$\begin{aligned}\phi(t,\vec{x}) &= \phi_0(t,\vec{x}) - \frac{1}{3H}\int_0^t dt' V'(\phi(t',\vec{x}) \\ &= \phi_0(t,\vec{x}) - \frac{\lambda}{3H}\int_0^t dt' \phi^3(t',\vec{x}).\end{aligned}$$

Solving this equation by iterations, we obtain (up to the order $\lambda^2$):

$$\phi(t) = \phi_0(t) - \frac{\lambda}{3H}\int_0^t dt'\phi_0^3(t') + \frac{\lambda^2}{3H^2}\int_0^t dt'\phi_0^2(t')\int_0^{t'} dt''\phi_0^3(t'').$$

Substituting this expression into the expressions for the correlators of the scalar field, we reduce their calculations to the application of some analogue of the Wick theorem and to the calculations of the integrals, where integrands include only free two-point correlators.

$$\langle\phi_0(t,\vec{x})\phi_0(t',\vec{x})\rangle = \frac{H^2}{4\pi^2}\ln(a(t')) = \frac{H^3 t'}{4\pi^2},$$

where

$$t' \leq t,$$

For example:

$$\langle\phi^2(t)\rangle_\lambda = -\frac{\lambda}{3H}\left[\langle\phi_0(t)\int_0^t dt'\phi_0^3(t')\rangle + \langle\int_0^t dt'\phi_0^3(t')\phi_0(t)\rangle\right]$$
$$= -\frac{\lambda}{3H}\left[3\int_0^t dt'\langle\phi_0(t)\phi_0(t')\rangle\langle\phi_0^2(t')\rangle\right.$$
$$\left.+3\int_0^t dt'\langle\phi_0^2(t')\rangle\langle\phi_0(t')\phi_0(t)\rangle\right].$$

Then

$$\langle\phi^2(t)\rangle_\lambda = -\frac{2\lambda}{H}\left(\frac{H^2}{4\pi^2}\right)^2\int_0^t dt' H^2 t'^2 = -\frac{\lambda H^5}{24\pi^4}t^3,$$

which coincides with the known result.

## Conclusions

- The quantum field theory on the de Sitter background is very interesting.
- One can generalize the developed methods to the case of a massive scalar field (work in progress).
- It would be interesting but more difficult to consider similar problems on more general backgrounds.