some progress towards the

Path Integral for Gravity

Neil Turok

work with J. Feldbrugge

a l l   k n o w n   p h y s i c s

Schrödinger Feynman Einstein Maxwell-Yang-Mills Kobayashi-Maskawa Lagrange

$$\Psi = \int e^{\frac{i}{\hbar}\int\left(\frac{R}{16\pi G} - \frac{1}{4}F^2 + \overline{\psi} i \not{D}\psi - \lambda H\overline{\psi}\psi + |DH|^2 - V(H)\right)}$$

Euler Planck Newton Dirac Yukawa Higgs

dark energy

$$\psi = (u_L, d_L, u_R, d_R, u_L, d_L, u_R, d_R, u_L, d_L, u_R, d_R, e_L, \nu_L, e_R, \nu_R)\times 3$$

dark matter?
Boyle, Finn, NT
2018

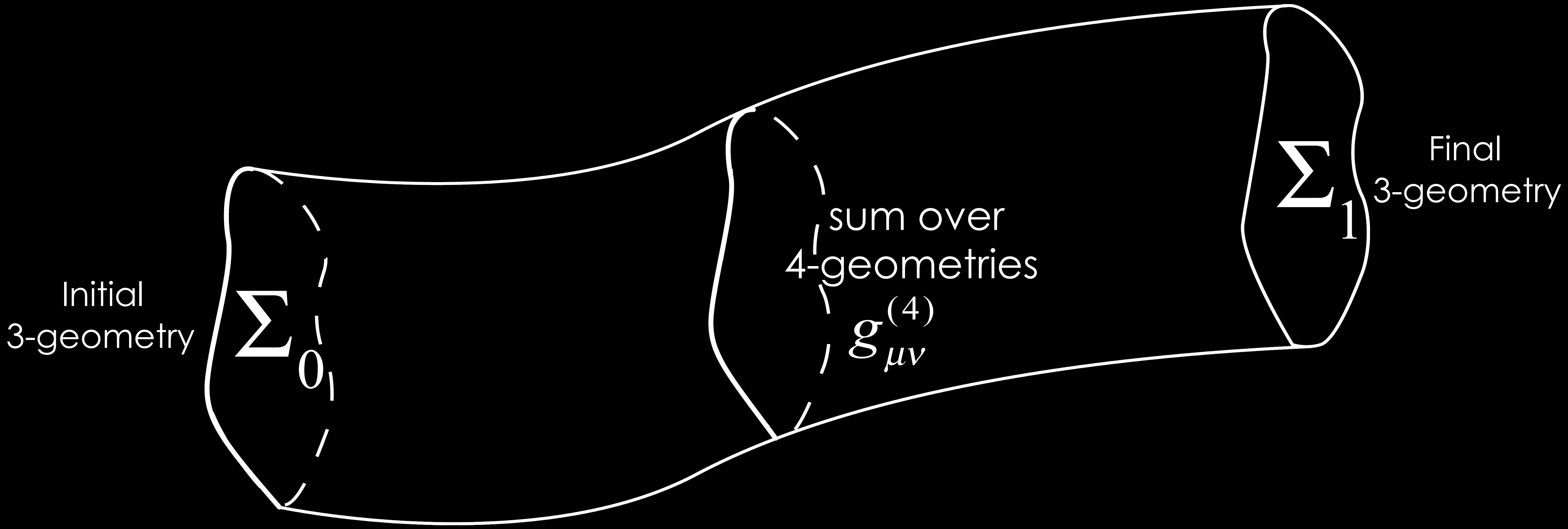
interference is

basic to quantum physics

universal

Wheeler, Feynman,
De Witt, Teitelboim …

quantum geometrodynamics



traditional approach:

$$\langle x_1 | e^{-iHt/\hbar} | x_0 \rangle \;\; = \langle x_1 | e^{-\beta H} | x_0 \rangle$$

$i.e.$, analytically continue to $t = -i\hbar$

"Euclidean time" =inverse temperature

This trick has dominated rigorous QFT

However, interference and thermal equilibrium are quite different physically

and gravity is in general incompatible with thermal equilibrium (Jeans instability)

ON DISTRIBUTIONS OF CERTAIN WIENER FUNCTIONALS(1)

BY

M. KAC

1. **Introduction.** The purpose of this paper is to present a unified approach toward the problem of calculating the distribution function of the Wiener functional

$$\int_0^t V(x(\tau))d\tau \tag{1.1}$$

where $x(t)$ is an element of Wiener's space $(0 \leqq t < \infty)$ and $V(x)$ is subject to certain restrictions. The most severe of these restrictions is that $V(x)$ be non-negative, or somewhat more generally, bounded from below.

Our principal result is the following: if $\sigma(\alpha; t)$ is the distribution function of (1.1), then

$$\int_0^\infty \int_0^\infty \exp(-u\alpha - st) d_\alpha \sigma(\alpha; t) dt = \int_{-\infty}^\infty \psi(x) dx, \tag{1.2}$$

where $\psi(x)$ is the fundamental solution (Green's function) of the differential equation

$$\frac{1}{2}\frac{d^2\psi}{dx^2} - (s + uV(x))\psi = 0, \qquad x \neq 0, \tag{1.3}$$

subject to the conditions

$$\begin{aligned} \psi(x) &\to 0, & x &\to \pm\infty, \\ |\psi'(x)| &< M, & x &\neq 0, \\ \psi'(+0) - \psi'(-0) &= -2 \end{aligned}$$

The existence and uniqueness of such a fundamental solution are parts of the assertion.

The differential equation (1.3) is quite similar to the equation of Schrödinger in quantum mechanics. In fact, the results of this paper were strongly influenced by the derivation of Schrödinger's equation which we found in a hitherto unpublished Princeton Thesis of R. P. Feynman. The principal motivation behind the investigation of the distribution functions of functionals (1.1) is the following: Let $X_1, X_2, \cdots$ be identically distributed random variables each having mean 0 and variance 1. Let furthermore

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(1) This investigation was begun while the author was a John Simon Guggenheim Memorial Fellow. It was completed under an ONR contract.

Instead of rotating time we deform integration contour in the path integral, exploiting a method for performing highly oscillatory integrals due to

Picard-Lefschetz

who gave a general criterion for whether a given saddle is n relevant to a real integral (in arbitrary finite dimension)

our work: flow the contour to find the relevant “Lefschetz thimbles” (or steepest descent contours)

new approach to defining Lorentzian (real time) path integrals

(J. Feldbrugge, NT in prep)

Our definition implies the following exact formula

Real, positive
probability measure

$$\int Dx\, e^{i\frac{S[x]}{\hbar}} \text{“="} \sum_{n_C} e^{i\frac{S[x_C]}{\hbar}} \int_{J_{n_C}} e^{i\theta_{n_C}(\delta x)}\, d\mu_{n_C}(\hbar, \delta x)$$

sum over relevant
classical solutions

phase, reduces to Maslow
in semiclassical limit

contour in space of complexified paths,
over which PI is absolutely convergent

classical theory organizes the quantum theory
classical solutions can still interfere
the formula should apply to gravity

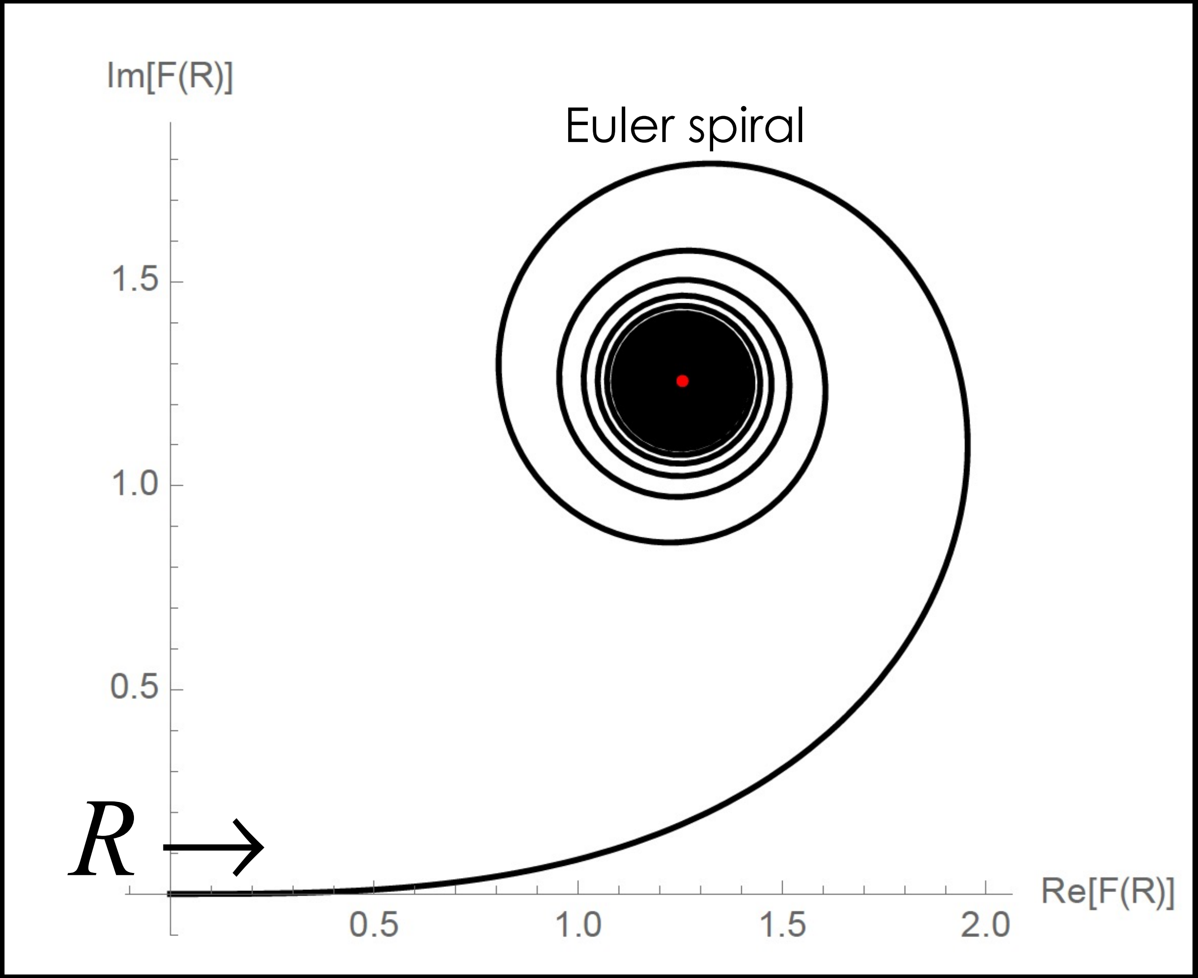
Highly oscillatory integrals

e.g., Gaussian (Fresnel integral)

$$F(R) = \int_{-R}^{+R} e^{ix^2}\, dx$$

$$I = \lim_{R\to\infty} F(R) = e^{i\frac{\pi}{4}}\sqrt{\pi}$$

Conditionally, not absolutely convergent



$\lfloor x$

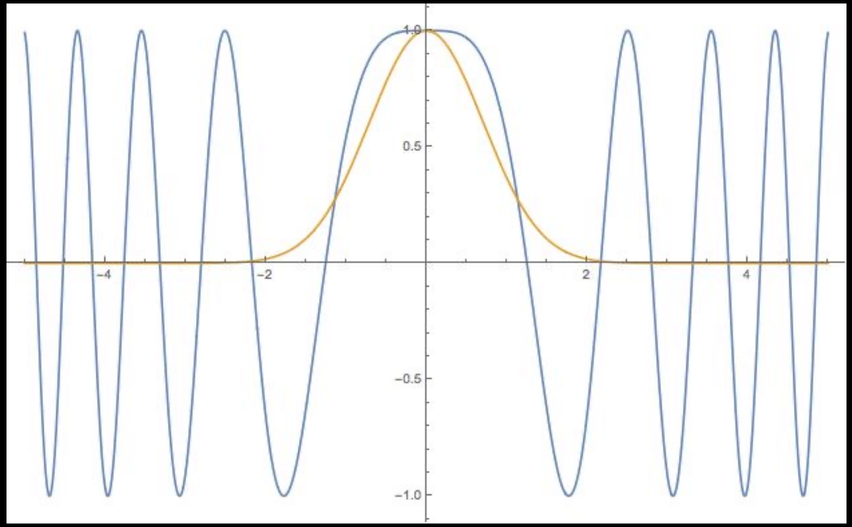
$$e^{i\frac{\pi}{4}}\int_{-R}^{R} ds\, e^{-s^2}$$

$$\left|\int R\, d\phi\, e^{iR^2 e^{2i\phi}}\right| < \frac{1}{R}$$

$-R$ $\qquad$ $+R$

Lefschetz thimble

$$x = e^{i\frac{\pi}{4}} s$$



What about higher dimensions? Infinite dimensions?

D=2 : square cutoff $$\lim_{R\to\infty}\int_{-R}^{R} dx \int_{-R}^{R} dy e^{i(x^2+y^2)} = \lim_{R\to\infty} F(R)^2 = i\,\pi$$

D=2 : round cutoff $$\overset{?}{\lim_{R\to\infty}} 2\pi \int_0^R r dr\, e^{ir^2} = \frac{\pi}{i}(e^{iR^2} - 1) \quad \text{NO LIMIT}$$

D>2 : sharp cutoff $$\int_0^R r^{D-1} dr\, e^{ir^2} \sim -\frac{i}{2} e^{iR^2} R^{D-2} + .. + \frac{e^{\frac{i\pi D}{4}} \Gamma(\frac{D}{2})}{2} + ..$$

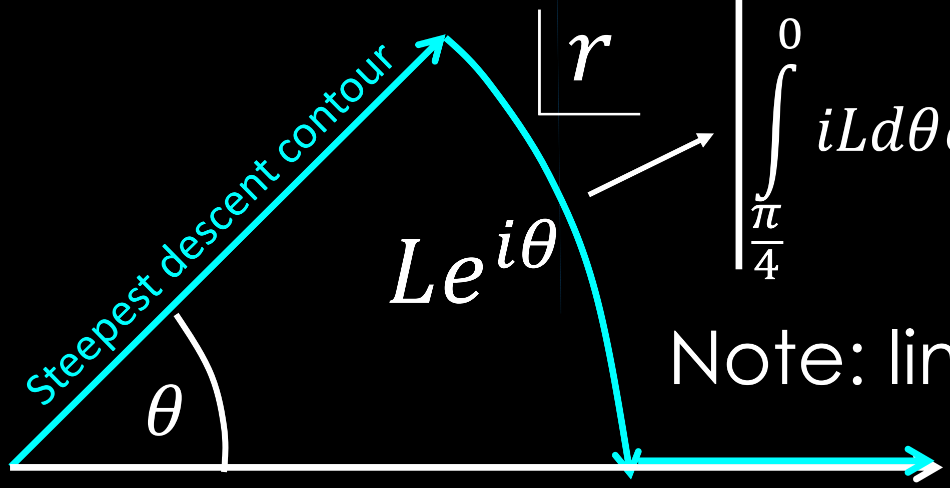
D>2 : smooth cutoff
(allows cancellations, which are physical) $$\int_0^\infty r^{D-1} dr\, e^{ir^2} e^{-\left(\frac{r}{R}\right)^2} \sim \frac{e^{\frac{i\pi D}{4}} \Gamma\left(\frac{D}{2}\right)}{2} (1 - \frac{i\,D}{2\,R^2} + ..)$$

The result for a smooth cutoff (taken to infinity at the end) is obtained without using a cutoff at all, by using Cauchy's theorem.

Assuming cutoff function is singular only at infinity, can deform contour to steepest descent contour and then take cutoff to infinity

For example, define "$\int_0^\infty dr r^{D-1} e^{ir^2}$" as $\int_0^\infty dr r^{D-1} e^{ir^2} e^{-\left(\frac{r}{R}\right)^2} \equiv \int_0^\infty dr f(r)$

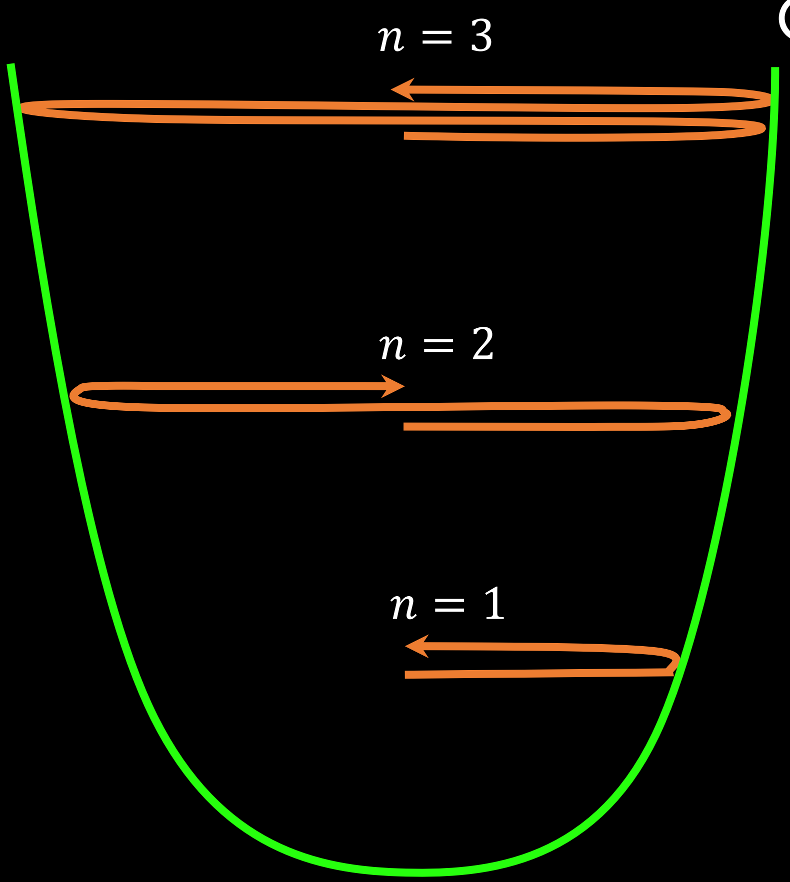
and deform contour to steepest descent plus "arc at infinity":



$$\left|\int_{\frac{\pi}{4}}^{0} iLd\theta e^{i\theta} f(L e^{i\theta})\right| \leq \frac{\pi L^d}{4} e^{-L^2/R^2}$$

Note: limits $R \to \infty, L \to \infty$ don't commute e.g., quartic oscillator $S = \int_0^1 dt\, \frac{1}{2}\left(\frac{\dot{x}^2}{T} - x^4 T\right)$; $H = -\frac{\partial S}{\partial T} = \frac{1}{2}\left(\frac{\dot{x}^2}{T^2} + x^4\right)$

Classical equations of motion: $\frac{\ddot{x}}{T^2} = -2x^3$

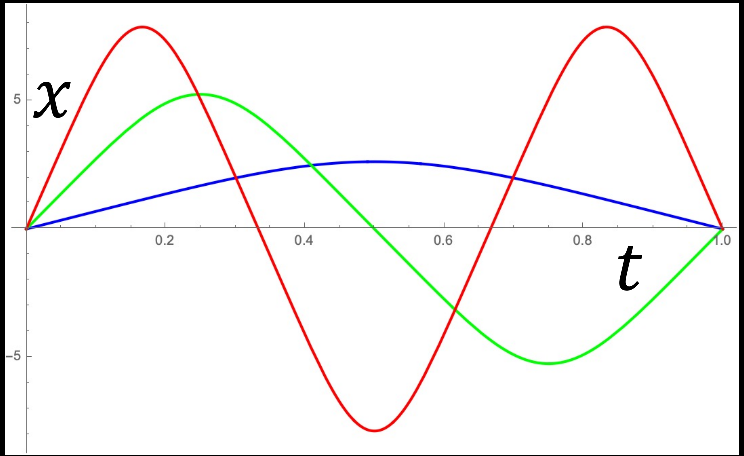


countable infinity of classical solutions

e.g., for *BCs* $x(0) = x(1) = 0,$

$x_{C,n} = n\kappa\, \mathrm{sn}(n\kappa\, t, -1),\ n = 1,2..$

Jacobi elliptic function $(\kappa \equiv 2K(-1) \approx 2.622)$

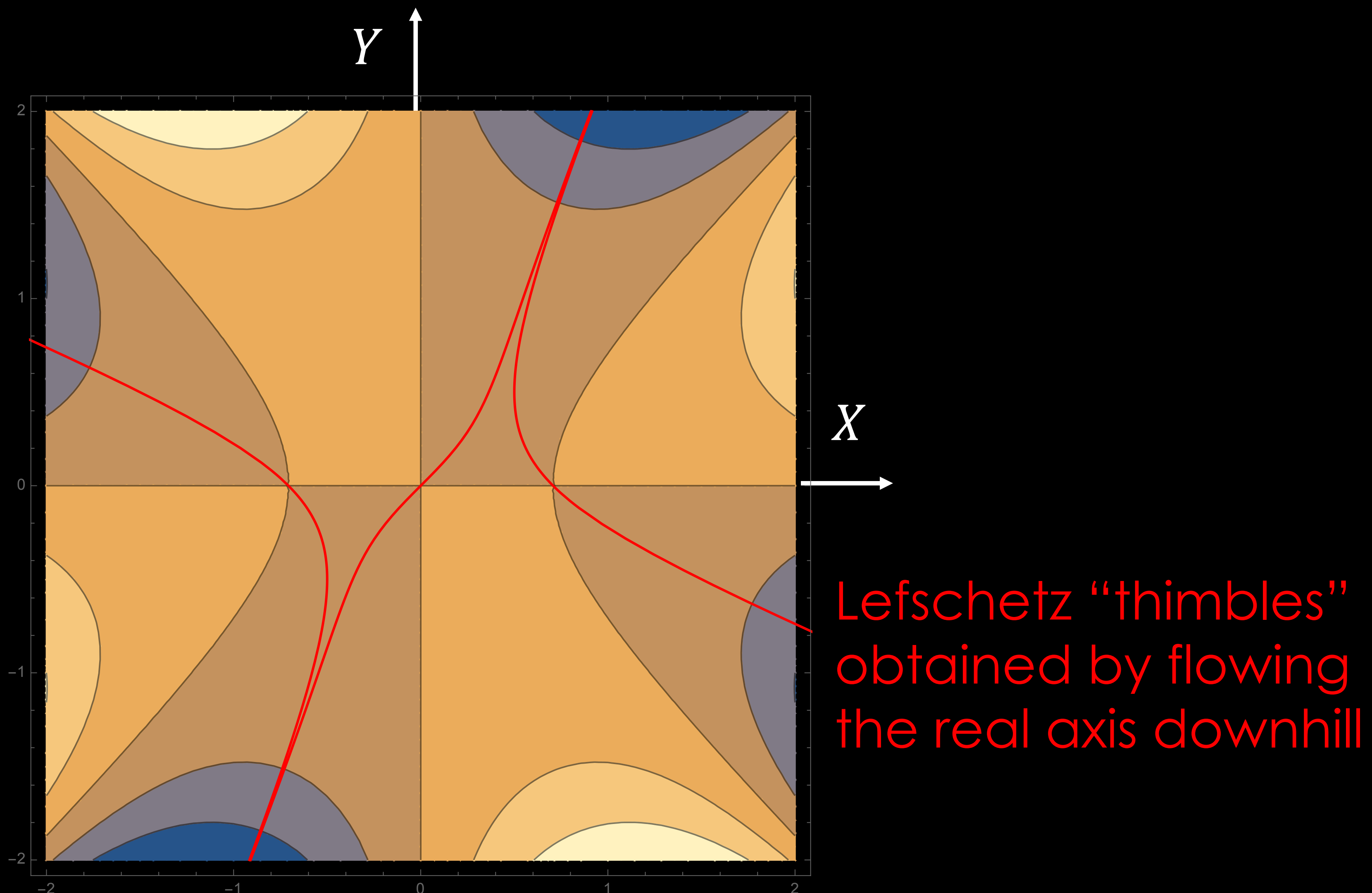


Energy $E = \frac{n^4 \kappa^4}{2T^4}$

Action $S = \frac{n^4 \kappa^4}{6T^3}$

picture of the complex plane for each mode coefficient, showing "height function" h with saddles and steepest descent contours

$$h = Re[i(x^2\text{-}x^4)]$$
$$x = X + iY$$

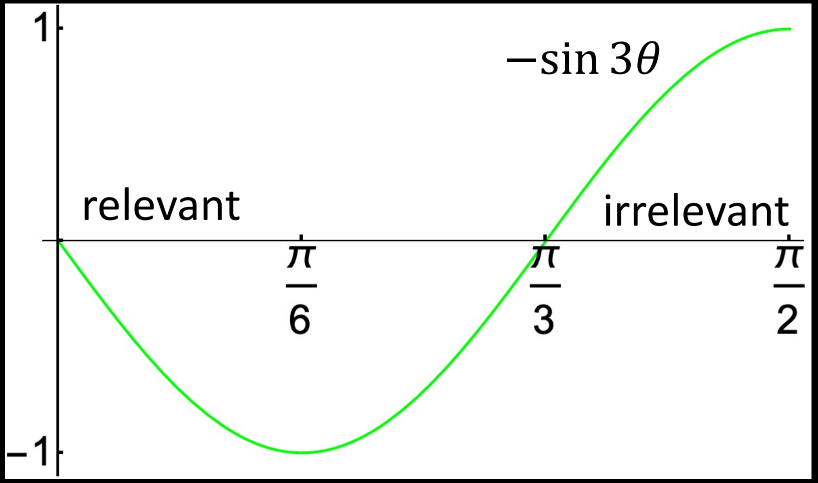


Wick Rotation

Take the Lorentzian theory and rotate T clockwise, $T \to e^{-i\theta}T,\ \ 0 \le \theta \le \frac{\pi}{2}$

Classical solutions still satisfy the boundary conditions

Classical action $S_C \propto \frac{1}{T^3}\ \Longrightarrow Re[i\, S_C] \propto -\sin 3\theta$
but relevant saddles must have $Re[i\, S_C] \le 0$



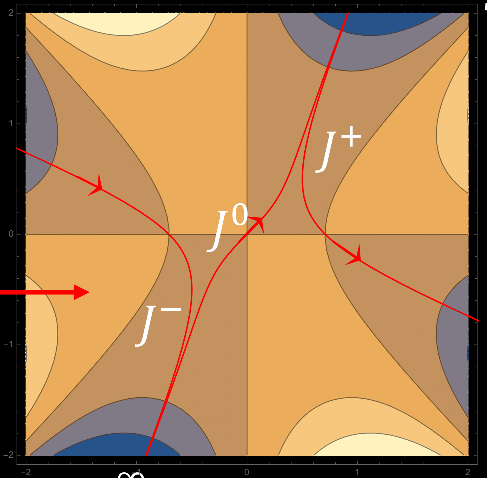
So the nontrivial classical saddles all disappear
in the rotation to imaginary time

Conversely, recovering their effect from
an imaginary time calculation would be exponentially hard

Expand fluctuations in Fourier modes

eigenfunctions of $\hat{O}$, where $S^{(2)} = \frac{1}{2}\int \delta x \hat{O} \delta x$

$$x(t) = x_C(t) + \delta x(t); \quad \delta x(t) = \sum_{m=1} \delta x_m \, \psi_m(t)$$



deform contour

$$K(x_1, x_0, T) = \frac{e^{i\frac{(x_1 - x_0)^2}{2\hbar T}}}{\sqrt{2i\pi\hbar T}} \frac{\int \prod d\delta x_m e^{\frac{i}{\hbar}S[x]}}{\int \prod d\delta x_m e^{\frac{i}{\hbar}S_0[x]}}$$

Free particle

$$\int_{J^0} \prod_{m=1} dx_m + \sum_{n_C=1}^{\infty} \left(\int_{J^+} dx_{n_C} + \int_{J^-} dx_{n_C}\right) \int_{J^0} \prod_{m \neq n_C, 1} dx_m$$

trivial thimble

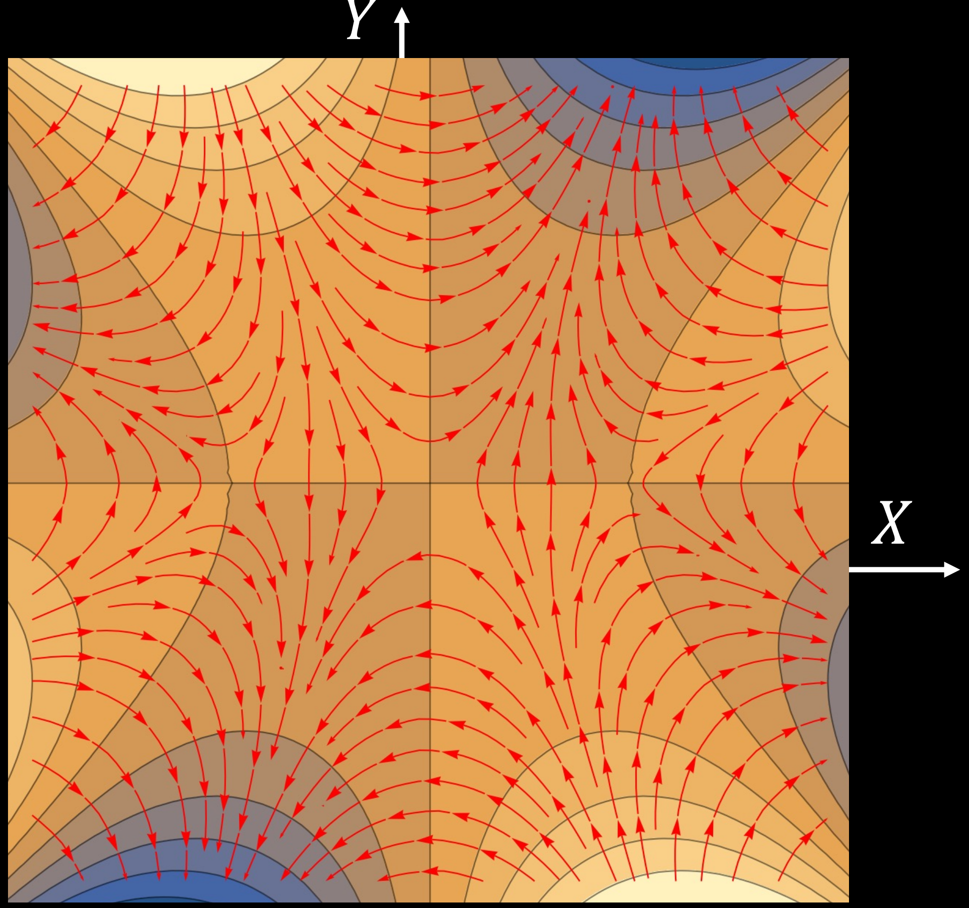
nontrivial thimbles

$$iS = \frac{i}{2}\int_0^1 dt(x(t)^2 \ - V(x)) \quad h = Re[i\, S[x]] = \frac{i}{2}(S[x] - S[\bar{x}])$$

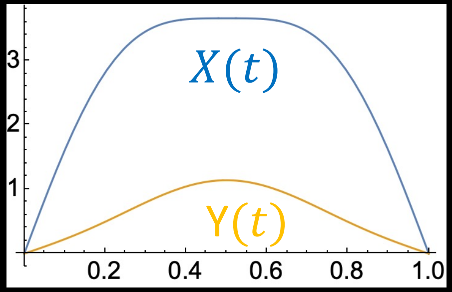
gradient flow
$$\partial_\tau x = -2\partial_{\bar{x}} h = -i(\ddot{\bar{x}} + \partial_{\bar{x}} V[\bar{x}])$$
$$\partial_\tau \bar{x} = -2\partial_x h = -i(\ddot{x} + \partial_x V[x])$$

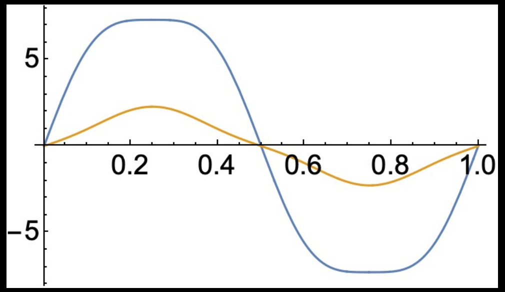
$$x(t) = X(t) + iY(t)$$

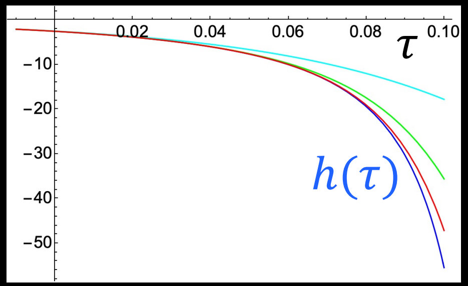
gradient flow for each mode: can solve analytically at small and large $|x|$



thimble:





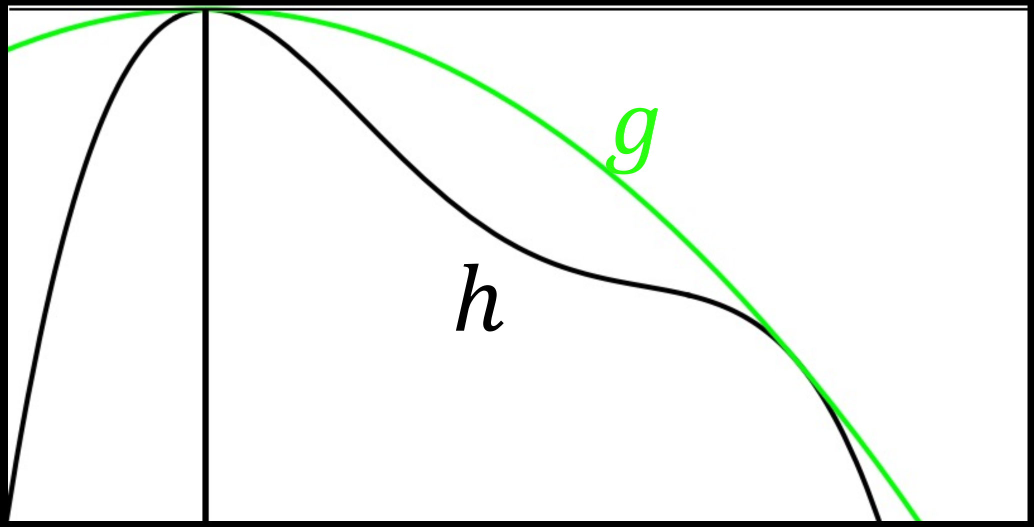


Since, at large $|x|$, h decreases faster than quadratically on the thimble , there exists a bounding Gaussian theory g,

satisfying

$h \leq g \ \leq 0:$

for all $x(t) \in J$



This suffices to prove that the path integral exists

1)Only include modes $m \leq N$ in nonlinear terms and take limit $N \to \infty$

Lebesgue's dominated convergence theorem shows the limit exists
Bochner-Minlos theorem shows the measure $d\mu_{n_C}(\hbar, \delta x)$ exists,
phase factor $e^{i\theta_{n_C}(\delta x)}$ arises from $d\delta x_m$ along thimble

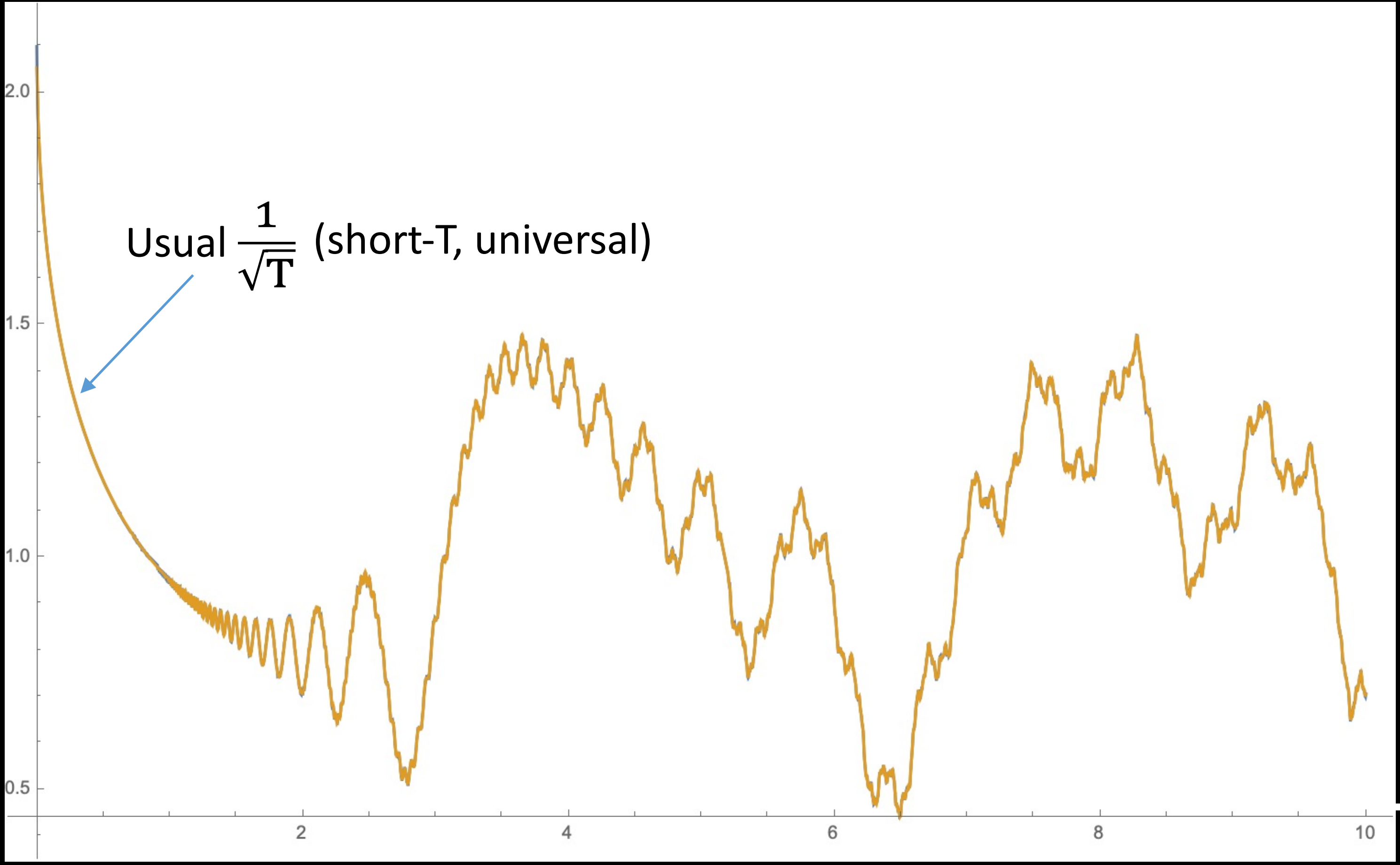
$$\sum_{n_C} e^{i\frac{S[x_C]}{\hbar}} \int_{J_{n_C}} e^{i\theta_{n_C}(\delta x)}\, d\mu_{n_C}(\hbar, \delta x)$$

2)Sum over classical solutions generally does *not* converge:
in NRQM the FPI propagator is in general only a distribution

Time smoothed propagator does converge

e.g., quartic oscillator

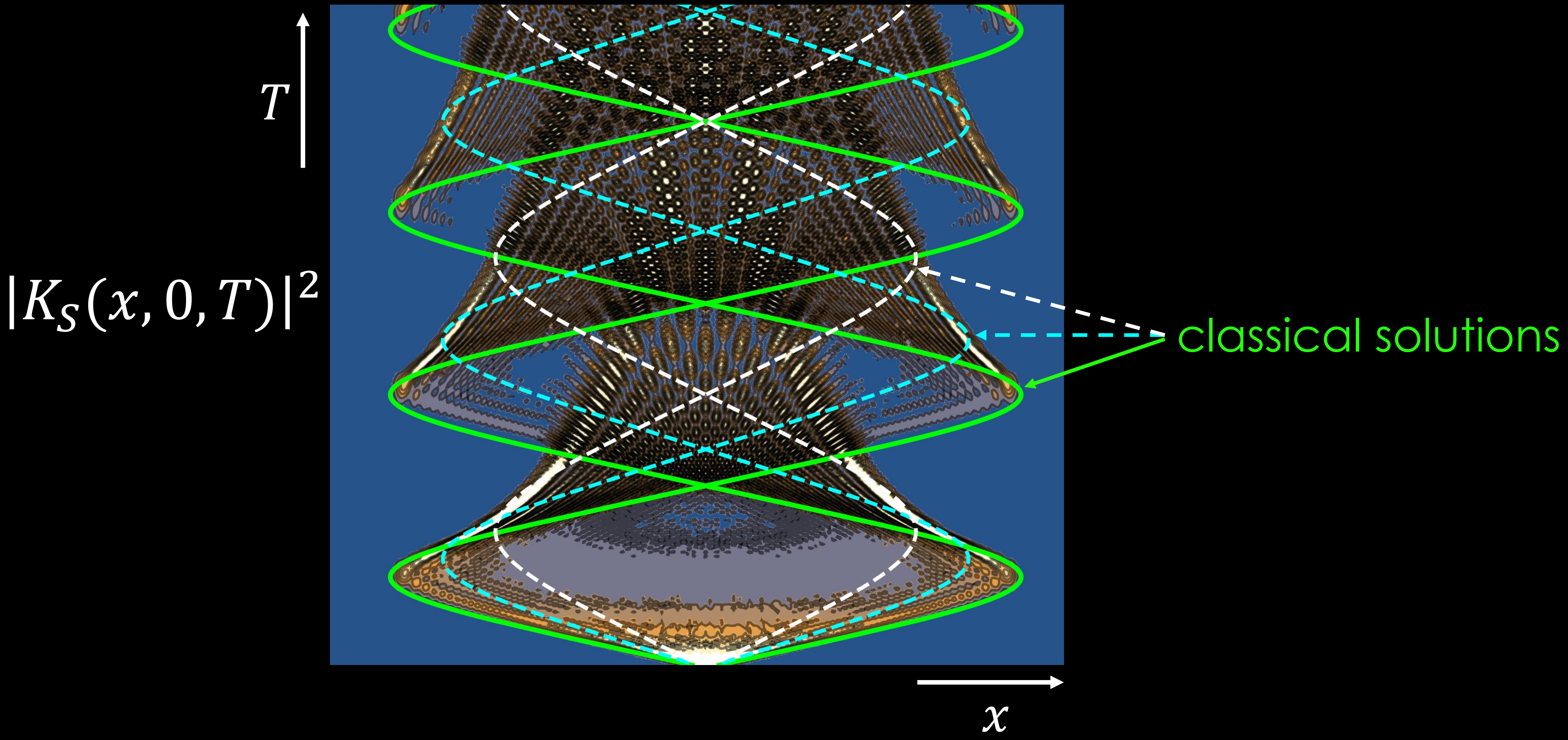
$$\left|K_s(x,0,T)\right|^2$$



T

but the result depends on the smoothing

$$K_s(x,0,T) \equiv \int_0^{\infty} dT' e^{-\frac{T'}{T_S}} K(x,0,T-T')$$


T
$|K_S(x,0,T)|^2$
classical solutions
x

Close interplay between quantum and classical pictures:

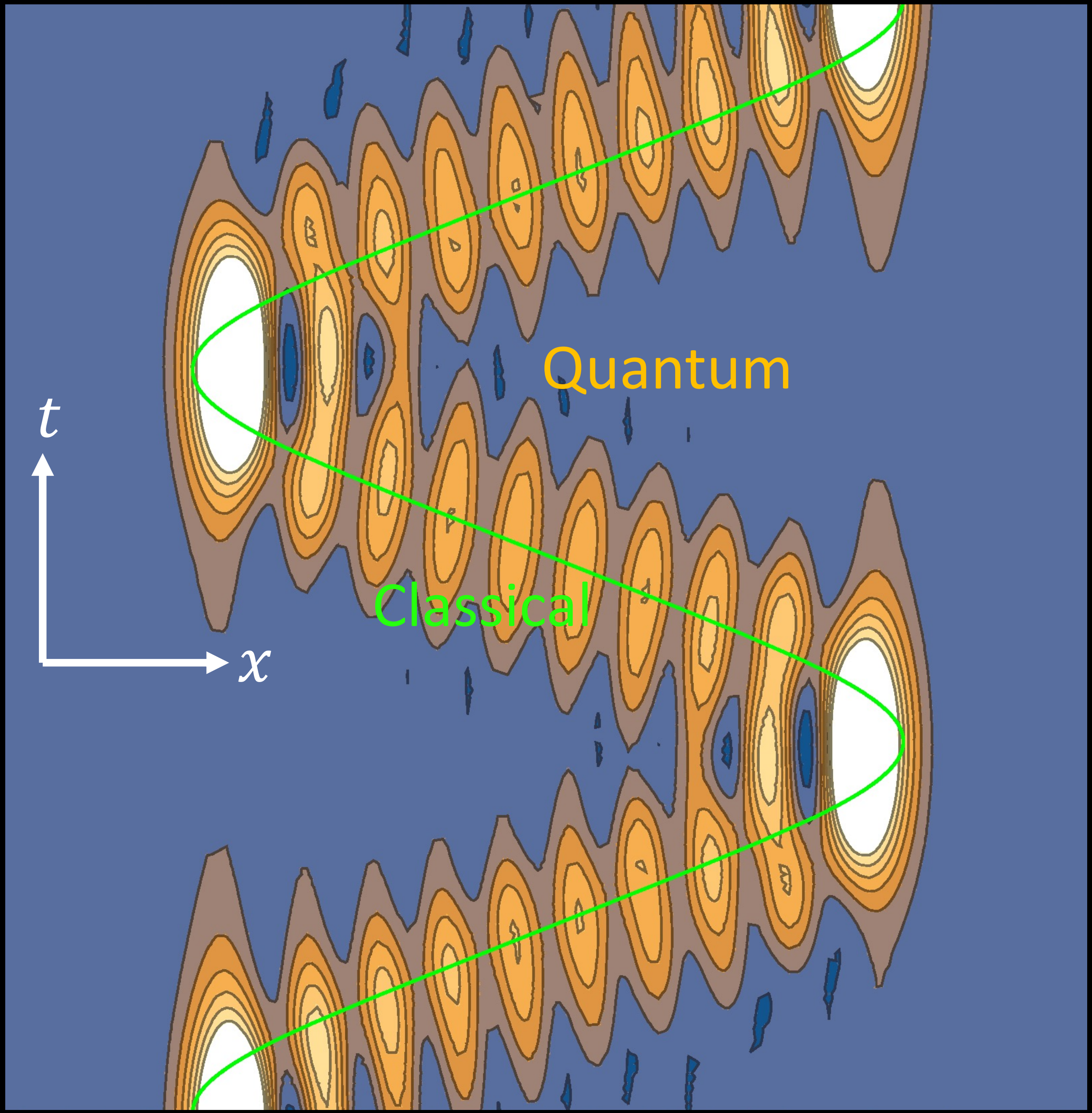
Constructive interference between quantum modes yields the discrete set of classical solutions

Constructive interference between classical solutions yields the semi-classical quantization of Einstein and Keller

$$\frac{\int p\,dq}{\hbar} = 2\,\pi\,(n + \tfrac{1}{4}m)$$

All of this is clarified by our construction

The “weak density”
(*à la* Aharonov *et al.*)
exhibits the influence
of the quantum system on
a weak measurement made in
between state preparation and
strong measurement.



This is how spacetime emerges
in quantum geometrodynamics

Application: gravitational microlensing

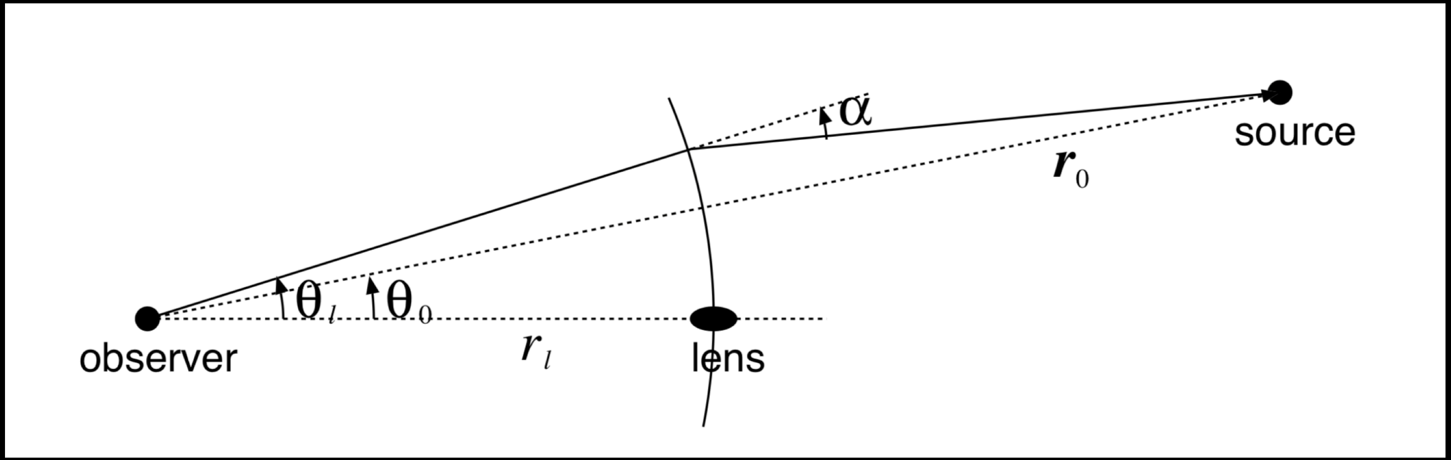
Einstein 1936
Nakamura–Deguchi 1999

$$\Psi(\omega,\vec{\mu}) \sim \omega \int d^2\vec{x} e^{i\omega\left[\frac{1}{2}(\vec{x}-\vec{\mu})^2-\phi(\vec{x})\right]} \text{ where}$$

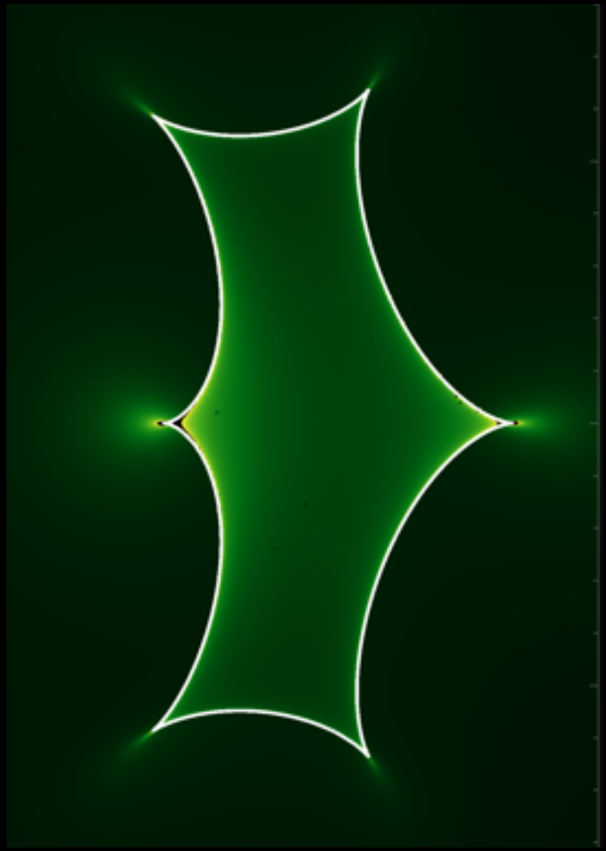
For a point mass in thin lens approx

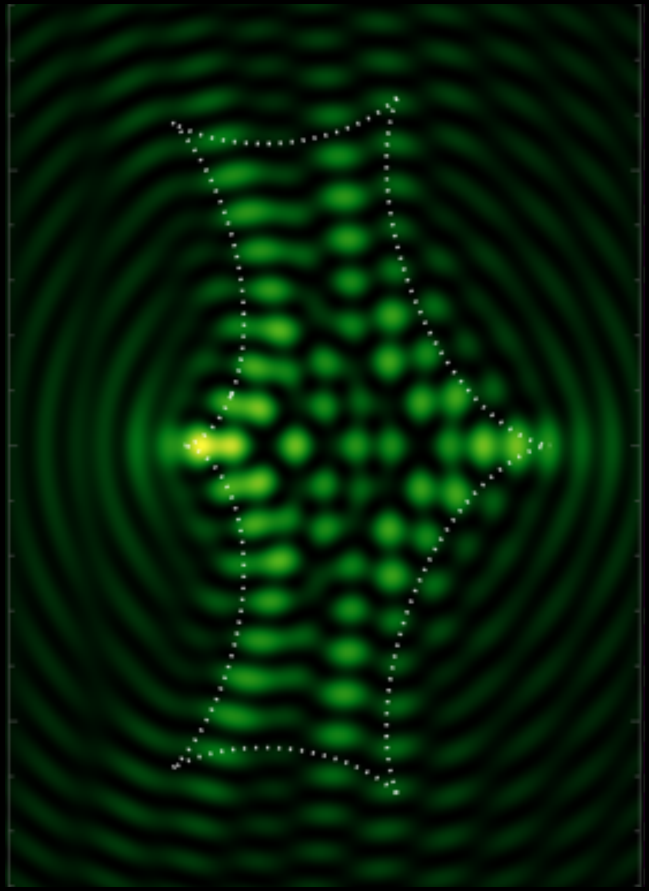
$$\phi=\ln(x),\ \omega \text{ is frequency in units of } r\theta_*^2,$$

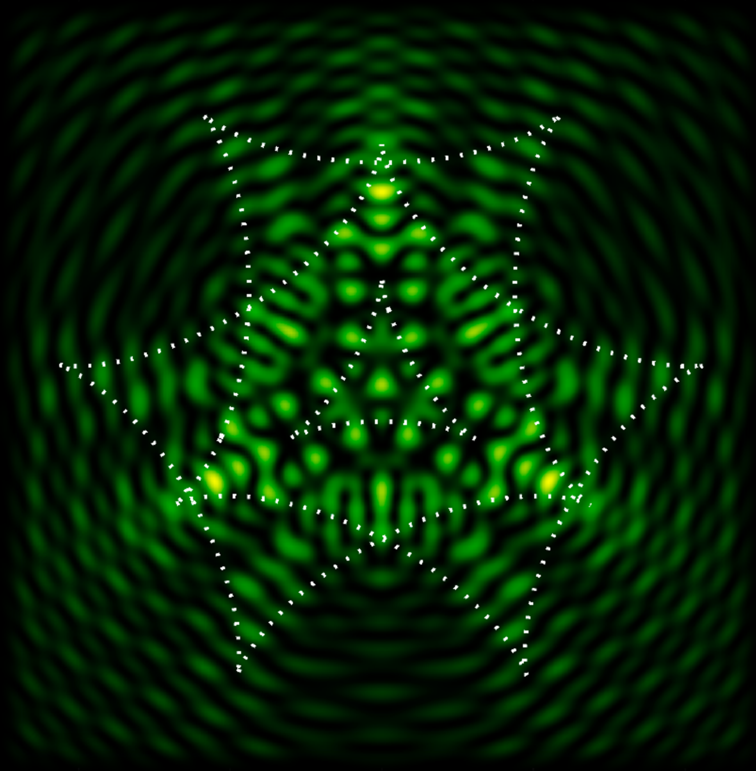
$$\theta_* \text{ is Einstein angle, } \omega=10^5 \frac{M}{M_\odot}\frac{\nu}{GHz}$$



Wave optics effects will be observable in the future: contain much more information







Lensing of a binary system
w/ J. Feldbrugge (1909.04632; 2008.01154)

Спасибо!

Thank you!