ModMax and Others

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Motivation and Outlook

- To present a recently discovered *unique* non-linear modification of free Maxwell theory that preserves all its symmetry properties (dubbed ModMax)
  - Lorentz invariance, U(1) gauge symmetry
  - but in addition, also electric-magnetic duality and conformal invariance

  (play an important role in various areas of theor. physics, from condensed matter to the theory of fundamental interactions, in particular in String Theory)

- Models of non-linear electrodynamics have been extensively studied as possible guides to catch new physics: e.g. for tackling fundamental cosmological problems (such as inflation and dark matter), for an effective description of properties of certain insulator materials and optical media.
- Notable examples:
  - Born-Infeld electrodynamics (1934) - invented to ensure finite electric field self-energy of charged particles.
    An important ingredient of Modern String Theory.
  - Euler-Heisenberg effective theory (1936) of Quantum Electrodynamics
- Most of the non-linear electrodynamics models in the modern theoretical "market" has been constructed in a heuristic way aimed at addressing specific problems, and not using basic theoretical principles, like fundamental symmetries. (Conformal non-linear electrodynamics has been studied since 2000: for a review see Denisova, Garmaev & Sokolov, arXiv:1901.05318)
- A natural question is what is the form of the non-linear electrodynamics, which has all the symmetries of Maxwell's theory?

Conformal and duality invariance of Maxwell theory

- Free Maxwell action

$$\mathcal{S} = -\frac{1}{4}\int d^4x\, F_{\mu\nu}F^{\mu\nu} = \frac{1}{2}\int d^4x\, (\mathbf{E}^2 - \mathbf{B}^2)$$

$$F_{\mu\nu}(x) = \partial_\mu A_\nu(x) - \partial_\nu A_\mu(x), \qquad E_i = \partial_0 A_i - \partial_i A_0, \qquad B^i = \frac{1}{2}\varepsilon^{ijk}F_{jk}$$

$$\mu, \nu = 0, 1, 2, 3; \qquad i, j, k = 1, 2, 3$$

- Free Maxwell equations

$$\left.\begin{aligned} &\partial_\mu F^{\mu\nu} = 0 \\ \text{Bianchi identities:}\quad &\partial_\mu \tilde{F}^{\mu\nu} \equiv \partial_\mu\left(\tfrac{1}{2}\varepsilon^{\mu\nu\rho\sigma}F_{\rho\sigma}\right) = 0 \end{aligned}\right\} \Longrightarrow \begin{aligned} &\frac{\partial \mathbf{E}}{\partial t} = \nabla \times \mathbf{B}, \qquad \nabla\cdot\mathbf{E} = 0, \\ &\frac{\partial \mathbf{B}}{\partial t} = -\nabla \times \mathbf{E}, \qquad \nabla\cdot\mathbf{B} = 0 \end{aligned}$$

- Conformal invariance (e.g. rescaling): $x^\mu \to a\, x^\mu, \quad A_\mu \to a^{-1}A_\mu, \quad F_{\mu\nu} \to a^{-2}F_{\mu\nu}$

- Invariance under SO(2) duality rotation: (only equations of motion)

$$\begin{pmatrix} F'^{\mu\nu} \\ \tilde{F}'^{\mu\nu} \end{pmatrix} = \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} F^{\mu\nu} \\ \tilde{F}^{\mu\nu} \end{pmatrix}$$

Non-linear electrodynamics (NED)

- Generic NED action: $$S_{NED} = \int d^4x\, \mathcal{L}\left(F_{\mu\nu}F^{\mu\nu}, F_{\mu\nu}\tilde{F}^{\mu\nu}\right), \qquad \tilde{F}^{\mu\nu} = \frac{1}{2}\varepsilon^{\mu\nu\rho\lambda}F_{\rho\lambda}$$

(a gauge-invariant non-linear functional of two Lorentz invariants)

In general NED, the electric-magnetic duality and conformal invariance are lost

- Requirement of conformal invariance: $$\mathcal{L}(aFF, aF\tilde{F}) = a\,\mathcal{L}(FF, F\tilde{F}) \quad \to \quad \mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}\,\mathcal{V}\left(\frac{F\tilde{F}}{FF}\right)$$

(homogeneous function of degree 1)

- Requirement of duality invariance (Gaillard & Zumino ‘81,’96; Gibbons & Rasheed ’95,…):

$$\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial F_{\mu\nu}}\right) = 0 \qquad \partial_\mu\tilde{F}^{\mu\nu} = 0$$

$$\begin{pmatrix} \frac{\partial\mathcal{L}(F')}{\partial F'_{mn}} \\ \tilde{F}'^{\mu\nu} \end{pmatrix} = \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} \frac{\partial\mathcal{L}}{\partial F_{\mu\nu}} \\ \tilde{F}^{\mu\nu} \end{pmatrix} \quad \Rightarrow \quad \boxed{F_{\mu\nu}\tilde{F}^{\mu\nu} - 2\varepsilon^{\mu\nu\lambda\rho}\frac{\partial\mathcal{L}}{\partial F^{\mu\nu}}\frac{\partial\mathcal{L}}{\partial F^{\lambda\rho}} = 0}$$

- **Born-Infeld theory:** $$\mathcal{L}_{BI} = T - T\sqrt{-\det(\eta_{\mu\nu} + T^{-\frac{1}{2}}F_{\mu\nu})} = T - \sqrt{T^2 + \frac{T}{2}F_{\mu\nu}F^{\mu\nu} - \frac{1}{16}(F_{\mu\nu}\tilde{F}^{\mu\nu})^2}$$

(duality-invariant but not conformal)

T – coupling parameter of dimension of energy density

at $T \to \infty \quad \mathcal{L}_{BI} \to \mathcal{L}_{Maxwell}$

(week field limit)

Modified Maxwell theory (ModMax)

Bandos, Lechner, D.S. and Townsend, 2020

- Unique non-linear electrodynamics which is simultaneously duality-invariant and conformal, and reduces to Maxwell's ED in the zero-coupling limit

$$\mathcal{L}_{ModMax} = -\frac{\cosh\gamma}{4} F_{\mu\nu}F^{\mu\nu} + \frac{\sinh\gamma}{4}\sqrt{(F_{\mu\nu}F^{\mu\nu})^2 + (F_{\mu\nu}\tilde{F}^{\mu\nu})^2}$$

$$= \frac{\cosh\gamma}{2}(\mathbf{E}^2 - \mathbf{B}^2) + \frac{\sinh\gamma}{2}\sqrt{(\mathbf{E}^2 - \mathbf{B}^2)^2 + 4(\mathbf{E}\cdot\mathbf{B})^2}$$

$\cosh\gamma = \frac{1}{2}(e^{\gamma} + e^{-\gamma}), \quad \sinh\gamma = \frac{1}{2}(e^{\gamma} - e^{-\gamma})$ - dimensionless coupling parameter

$$\text{at } \gamma = 0 \quad \mathcal{L}_{ModMax} = \mathcal{L}_{Maxwell}$$

- ModMax energy-momentum tensor is traceless due to conformal invariance

$$T^{\mu\nu} = \underbrace{\left(F^{\mu}{}_{\rho}F^{\nu\rho} - \frac{1}{4}\eta^{\mu\nu}(F_{\rho\lambda}F^{\rho\lambda})\right)}_{\text{Maxwell energy-momentum}}\left(\cosh\gamma + \sinh\gamma\frac{FF}{\sqrt{(FF)^2 + (F\tilde{F})^2}}\right) \qquad \boxed{T^{00} \geq 0}$$

ModMax Hamiltonian

ModMax energy density: $T^{00} = \frac{1}{2}(\mathbf{E}^2 + \mathbf{B}^2)\left(\cosh\gamma + \sinh\gamma \frac{\mathbf{E}^2 - \mathbf{B}^2}{\sqrt{(\mathbf{E}^2 - \mathbf{B}^2)^2 + 4(\mathbf{E}\cdot\mathbf{B})^2}}\right) \geq 0$

not well defined for $\mathbf{E}^2 - \mathbf{B}^2 = \mathbf{E}\cdot\mathbf{B} = 0$ which are e.g. satisfied by electromagnetic plane waves

- This 0/0 ambiguity gets resolved in the Hamiltonian formulation of ModMax obtained via Legendre transform

$\mathcal{H}(\mathbf{D},\mathbf{B}) = \mathbf{E}\cdot\mathbf{D} - \mathcal{L}(\mathbf{E},\mathbf{B})\,,$ $\mathbf{D} = \frac{\partial\mathcal{L}(\mathbf{E},\mathbf{B})}{\partial\mathbf{E}}$ - electric displacement vector, while $\mathbf{E} = \frac{\partial\mathcal{H}(\mathbf{D},\mathbf{B})}{\partial\mathbf{D}}$

ModMax Hamiltonian density: $\mathcal{H}_{ModMax} = \frac{1}{2}\left(\cosh\gamma\,(\mathbf{D}^2 + \mathbf{B}^2) - \sinh\gamma\sqrt{(\mathbf{D}^2 + \mathbf{B}^2)^2 - 4(\mathbf{D}\times\mathbf{B})^2}\right)$

Note that the Hamiltionian is manifestly duality invariant $\begin{pmatrix}\mathbf{D}'\\ \mathbf{B}'\end{pmatrix} = \begin{pmatrix}\cos\alpha & \sin\alpha\\ -\sin\alpha & \cos\alpha\end{pmatrix}\begin{pmatrix}\mathbf{D}\\ \mathbf{B}\end{pmatrix}$

ModMax Hamiltonian equations and plane waves

$$\mathcal{H}_{ModMax} = \frac{1}{2}\left(\cosh\gamma\,(\mathbf{D}^2+\mathbf{B}^2) - \sinh\gamma\sqrt{(\mathbf{D}^2+\mathbf{B}^2)^2 - 4(\mathbf{D}\times\mathbf{B})^2}\right)$$

$$\boxed{\begin{array}{rcl} \dot{\mathbf{B}} &=& -\boldsymbol{\nabla}\times\mathbf{E}\,, \qquad \boldsymbol{\nabla}\cdot\mathbf{B}=0 \\ \dot{\mathbf{D}} &=& \boldsymbol{\nabla}\times\mathbf{H}\,, \qquad \boldsymbol{\nabla}\cdot\mathbf{D}=0 \end{array}}$$

$$\mathbf{E} = \partial\mathcal{H}/\partial\mathbf{D}\,, \qquad \mathbf{H} = \partial\mathcal{H}/\partial\mathbf{B}$$

constituent relations

- Plane waves: $\mathbf{E}^2 - \mathbf{B}^2 = \mathbf{E}\cdot\mathbf{B} = 0 \quad \rightarrow \quad \mathbf{D}^2 + \mathbf{B}^2 = 2\cosh\gamma\,|\mathbf{D}\times\mathbf{B}|, \qquad \mathcal{H}_{pw} = |\mathbf{D}\times\mathbf{B}|$

  has solutions only for $\gamma \geq 0$

$$\mathbf{D}+i\mathbf{B} = \mathrm{Re}\left[\mathbf{D_0}e^{i(\mathbf{k}\cdot\mathbf{x}-|\mathbf{k}|t)}\right] + i\,\mathrm{Re}\left[\mathbf{B_0}e^{i(\mathbf{k}\cdot\mathbf{x}-|\mathbf{k}|t)}\right], \qquad \mathbf{D_0},\ \mathbf{B_0}\ -\ \text{constant amplitudes}$$

k - pw momentum

- a linearly polarized pw solution: $\mathbf{D}\cdot\mathbf{B} = 0, \quad \mathbf{E} = e^{-\gamma}\,\mathbf{D} = \mathbf{E}_0\cos(|\mathbf{k}|t - \mathbf{k}\cdot\mathbf{x}) = -\frac{\mathbf{k}}{|\mathbf{k}|}\times\mathbf{B}$

Birefringence in ModMax

- in uniform electric/magnetic backgrounds **ModMax exhibits the effect of birefringence** - phenomenon of double refraction whereby a ray of light is split by polarization (with respect to the external EM background) into two rays taking slightly different geodesic paths:

$$(i) \quad \omega^2 = |\mathbf{k}|\,\lambda_+\,, \qquad (ii) \quad \omega^2 = |\mathbf{k}|\,\lambda_- \qquad \lambda_\pm(\mathbf{k}, \mathbf{B}_{ext}) \text{ depend on properties of NED and background}$$

- For ModMax, dispersion relations of two refracted light rays in a magnetic background $\mathbf{B}$ are:

$$(i) \quad \omega^2 = |\mathbf{k}|^{\mathbf{2}} \quad - \quad \text{standar light} - \text{cone}$$

$$(ii) \quad \omega^2 = |\mathbf{k}|^2(\cos^2\varphi + e^{-2\gamma}\sin^2\varphi), \quad \varphi - \text{angle between } \mathbf{k} \text{ and } \mathbf{B_{ext}}$$

- The effect is characterized by the birefringence index $\nabla n = \lambda_+ - \lambda_-$:

$$\Delta n_{Max} = 0 = \Delta n_{BI}, \qquad \Delta n_{QED} = 3.96\times10^{-24} B^2_{ext}, \qquad \Delta n_{ModMax} = 1 - e^{-2\gamma} \quad (\text{for } \mathbf{k} \perp \mathbf{B}_{ext})$$

PVLAS experiment bounds: $\Delta n_{exp} \leq (12 \pm 17) \times 10^{-23}$ for $B_{ext} = 2.5$ Tesla

(arXiv:2005.12913)

ModMax effects on black holes

(arxiv:2011.10836, 2011.13398, 2012.03416, 2012.07443, 2102.06213, 2102.13138)

- Einstein's equations sourced by ModMax: $R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R = 8\pi \, T^{\mu\nu}_{ModMax}$

Example: electrically charged Reissner-Nordstrom black hole

BH mass and charge

$$ds^2_{BH} = -f(r)\, d^2t + \frac{1}{f(r)}\, d^2r + r^2\, d\Omega \,, \qquad f(r) = 1 - \frac{2M}{r} + \frac{Q_e^2\, e^{-\gamma}}{r^2}$$

RN-BH Horizons: $f(r) = 0 \quad \rightarrow \quad r_\pm = M \pm \sqrt{M^2 - Q_e^2 e^{-\gamma}} \quad \rightarrow \quad M^2 \geq Q_e^2 e^{-\gamma}$ (screening effect)

Generalizations of ModMax

- Generalized Born-Infeld (duality-invariant but not conformal):

$$\mathcal{L}_{(\gamma BI)} = T - \sqrt{T^2 + \frac{T}{2}\left((\cosh\gamma)F_{\mu\nu}F^{\mu\nu} - (\sinh\gamma)\sqrt{(F_{\mu\nu}F^{\mu\nu})^2 + (F_{\mu\nu}\tilde{F}^{\mu\nu})^2}\right) - \frac{1}{16}(F_{\mu\nu}\tilde{F}^{\mu\nu})^2}\,,$$

$$\text{at } \gamma = 0 \quad \to \quad \mathcal{L}_{BI}; \qquad \text{at } T \to \infty \qquad \mathcal{L}_{\gamma BI} \to \mathcal{L}_{ModMax}$$

- Selfdual and conformal interacting chiral 2-form theory in 6D (PST formulation):

$$S = \frac{1}{4}\int d^6x \left(\tilde{H}^{MN}H_{MNP}\,v^P + (\cosh\gamma)\tilde{H}^{MN}\tilde{H}_{MN} - (\sinh\gamma)\sqrt{(\tilde{H}^{MN}\tilde{H}_{MN})^2 + \omega^M\omega_M}\right)\,,$$

$$H_{MNP} = 3\partial_{[L}A_{MN]}\,, \qquad v_L = \frac{\partial_L\, a(x)}{\sqrt{(\partial a)^2}}\,, \qquad \tilde{H}^{MN} = \frac{1}{6}\varepsilon^{MNLPQR}H_{PQR}\,v_L\,,$$

$$\omega^M = -\frac{1}{2}\varepsilon^{MNLPQR}\tilde{H}_{NL}\tilde{H}_{PQ}\,v_R\,.$$

Conclusion and Outlook

- ModMax is the unique non-linear extension of Maxwell theory that possesses all its symmetries (including duality and conformal invariance). In 6D there is the unique chiral 2-form counterpart of ModMax.
- ModMax has a dimensionless parameter $\gamma \geq 0$ characterizing the strength of EM self-interaction
- has plane-wave solutions, but the superposition principle is violated due to non-linearity
- has exact topologically non-trivial Hopfion knot solutions (*Dassy & Govaerts*, arXiv: 2105.05802)
- Birefringence (if discovered), may be used as an experimental test of ModMax

- Further developments
  - Detailed study of ModMax properties when coupled to charged sources
  - ModMax quantization
  - Embedding into (Modified) Gravity Theories for further applications to BHs and Cosmology
  - Looking for applications in Condensed Matter and Optical Physics (in particular, via gravity/CMT holography)
  - Supersymmetrization to appear (*Bandos, Lechner, D.S. Townsend*; *Kuzenko*)