Integrability in defect CFT

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Heisenberg anti-ferromagnet

$$H = \sum_{l=1}^{L} \vec{\sigma}_l \cdot \vec{\sigma}_{l+1}$$

Néel "ground state":

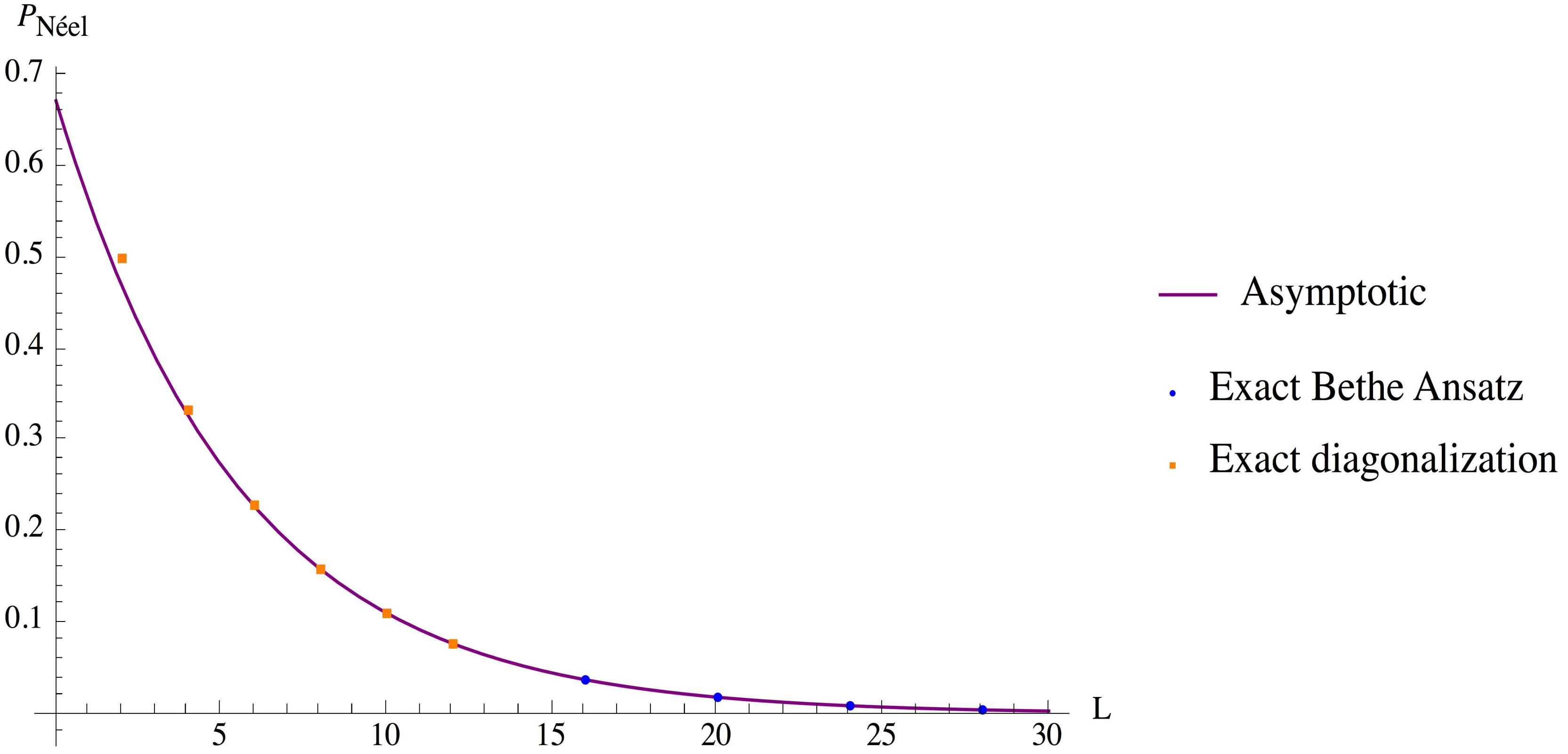
$$|\text{Néel}\rangle = |\uparrow\downarrow\uparrow\downarrow \ldots \uparrow\downarrow\rangle$$

Probality to find AF in this state:

$$P_{\text{Néel}} = \frac{\langle \text{Néel} | 0 \rangle^2}{\langle 0 | 0 \rangle} = ?$$

$$P_{\text{Néel}} \simeq C\, e^{-\nu L}$$

$$\nu = \ln 2 - \frac{1}{2}\int_{-\infty}^{+\infty} \frac{du}{\cosh \pi u} \ln \frac{u^2+\frac{1}{4}}{u^2} \approx 0.181$$



Two types of boundary states

I) Valence-Bond States:

$$|VBS\rangle = |K\rangle^{\otimes \frac{L}{2}} \qquad\qquad |K\rangle = \sum_{s_1 s_2} K_{s_1 s_2} |s_1 s_2\rangle$$

XXZ: K – any 2×2 matrix

Higher rank: $K \equiv K(i/2)$ where $K(u)$ solves reflection eqn. $RKRK = KRKR$

II) Matrix Product States:

de Leeuw, Kristjansen, Z. '15
Piroli, Pozsgay, Vernier '15,18

$$|MPS\rangle = \sum_{\{s_i\}} \mathrm{tr}\, t_{s_1} \ldots t_{s_L} |s_1 \ldots s_L\rangle$$

Ex In SU(3) spin chain: $[t_s, t_r] = i\varepsilon_{srp}\, t_p$

$\uparrow$ generators of k-dim rep. of SU(2)

Overlap formulae

$$\frac{\langle B | u \rangle}{\langle u | u \rangle^{1/2}} = \sum_{\alpha=1}^{d_B} \sqrt{\prod_j f_{B,\alpha}(u_j) \frac{\det G^+}{\det G^-}}$$

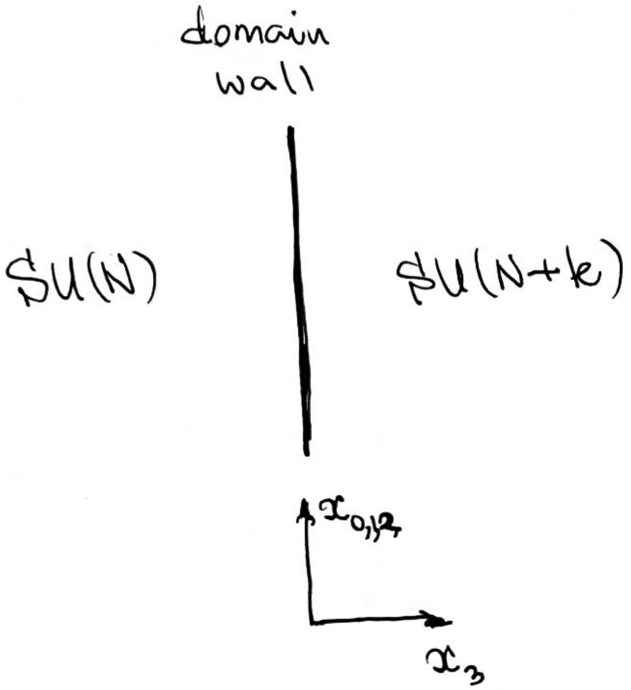
d_B - dim. of twisted Yangian rep. associated with $|B\rangle$.

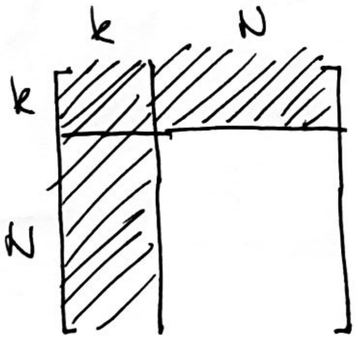
Ex for MPS, d_B is dim of auxiliary space for matrices t_s

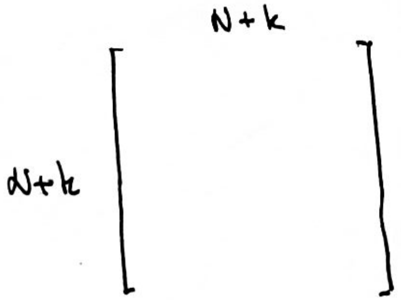
$f_{B,\alpha}(u)$ - eigenvalues of the double-row transfer matrix

- no honest derivations except for XXZ spin-$1/2$

Defect CFT







$\boxed{k \geq 2}$

$$\langle \Phi_i \rangle = \varphi_i \qquad i = 1,2,3$$

$$\varphi_i'' - [\varphi_j, [\varphi_j, \varphi_i]] = 0$$

$$\varphi_i = \frac{t_i}{x_3} \qquad\qquad [t_i, t_j] = i\,\varepsilon_{ijk}\, t_k$$

Constable, Myers, Tafjord '99

$$\begin{array}{cc} & k \\ k & \left[\begin{array}{c|c} t_i & \\ \hline & \\ & \end{array}\right] \end{array}$$

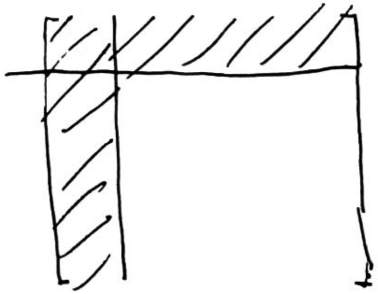
k-dim rep of SU(2)

Higgs mass $\quad m_H^2 \propto \frac{1}{x_3^2}$

$\Rightarrow$ "massive" modes don't propagate beyond $x_3 \doteq 0$

Buhl-Mortensen, de Leeuw, Ipsen, Kristjansen, Wilhelm '16

$\boxed{k=1}$



↳ Dirichlet - Neumann boundary conditions:

$$D_3 \Phi_i + \frac{i}{2} \varepsilon_{ijk} [\Phi_j , \Phi_k] = 0 \qquad i = 1,2,3$$

$$\Phi_i = 0 \qquad i = 4,5,6$$

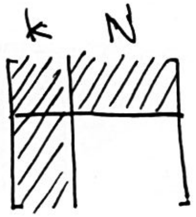
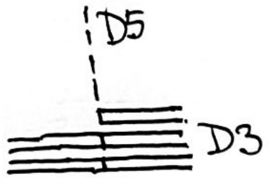
$$F_{\mu 3} = 0$$

$$i \gamma^3 \Psi = \Psi$$

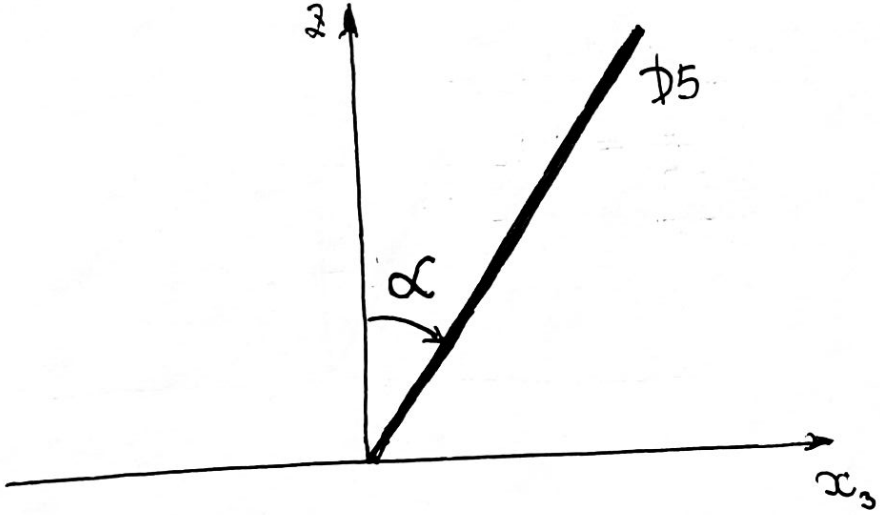
Gaiotto, Witten '08

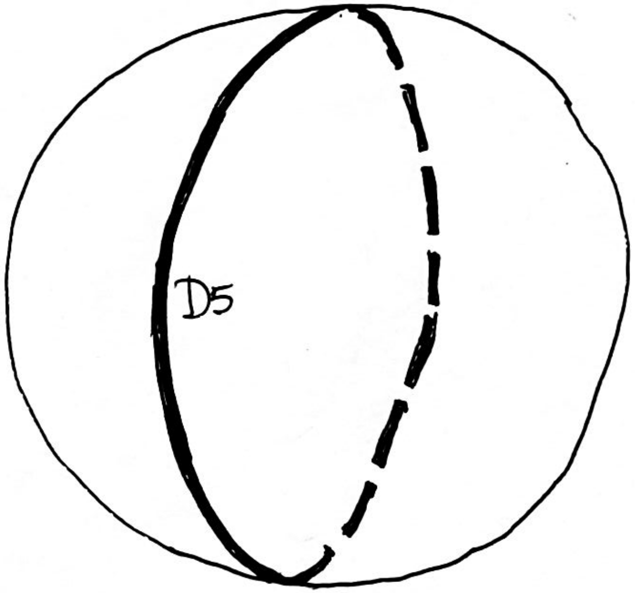
k=0: extra d.o.f. on domain wall
(no symmetry breaking)

dAdS/dCFT



AdS_5 × S^5





$S^2 \subset S^5$

$$\tan\alpha = \frac{k}{\sqrt{\lambda}}$$

$$\lambda = g_{YM}^2 N$$

w. flux:

$$\oint_{S^2} F = k$$

Karch
Randall '01
DeWolf, Freedman, Ooguri '01

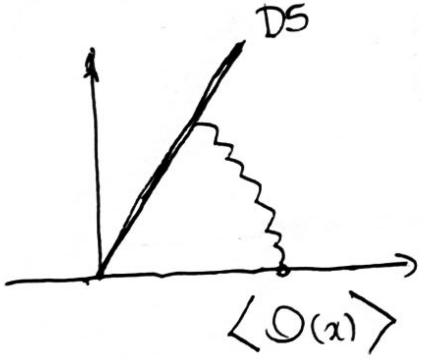
Integrability

- Boundary conditions on D5 preserve integrability

Dekel, OZ'11

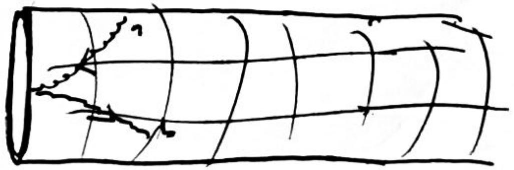
Linardopoulos, Z. '21

1pt functions: $\boxed{\langle \mathcal{O}(x)\rangle = \frac{\mathbb{C}}{x_3^{\Delta}}}$



$\rightsquigarrow$

reflection amplitude: $K_{ab}(p)$



$\xrightarrow{r\to\infty} \mathcal{O}(x)$

Nagasaki, Yamaguchi '12

$\langle \mathcal{O}(x)\rangle =$ world-sheet g-function

Jiang, Komatsu, Vescovi '19

Komatsu, Wang '20

Weak coupling: $k \geq 2$

$$\mathcal{O} = \underset{\substack{\uparrow \\ \text{wave function}}}{\Psi^{i_1 \dots i_L}} \operatorname{tr} \Phi_{i_1} \dots \Phi_{i_L}$$

SO(6) spin chain

$$\Phi_i \longrightarrow \langle \Phi_i \rangle = \frac{t_i}{x_3}$$

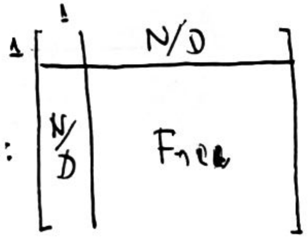
$$\langle \mathcal{O} \rangle = \frac{1}{x_3^L} \Psi^{i_1 \dots i_L} \operatorname{tr} t_{i_1} \dots t_{i_L}$$

$$\boxed{\langle \mathcal{O} \rangle = \left(\frac{4\pi^2}{\lambda} \right)^{\frac{L}{2}} \frac{1}{L^{1/2}} \frac{1}{x_3^L} \frac{\langle \mathrm{MPS} | \Psi \rangle}{\langle \Psi | \Psi \rangle^{1/2}}}$$

$$\mathrm{MPS}_{i_1 \dots i_L} = \operatorname{tr} t_{i_1} \dots t_{i_L}$$

Weak coupling : k = 1

b.c.'s:



$$Z_\uparrow = \Phi_1 + i\,\Phi_4$$

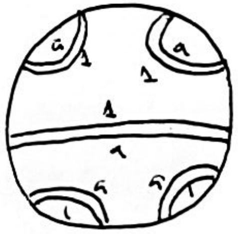
$$Z_\downarrow = \Phi_2 + i\,\Phi_5$$

$$\langle Z_s^{ab}(x)\, Z_{s'}^{cd}(y)\rangle = 0 \qquad \langle Z_s^{1a}(x)\, Z_{s'}^{b1}(y)\rangle = \frac{\lambda\,\delta^{ab}\,\delta_{ss'}}{4\pi^2 N}\,\frac{1}{|\bar{x}-y|^2} \qquad \bar{x} = (x_0, x_1, x_2, -x_3)$$

$$\mathcal{O} = \operatorname{tr} Z_{s_1} \dots Z_{s_L} \; : \; \underset{s_1}{\uparrow}\,\underset{s_2}{\uparrow}\downarrow\uparrow\downarrow\underset{\dots}{\dots}\underset{s_L}{\downarrow}$$

$\sim N^5$

✓

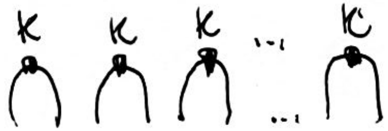


$\propto N^3$

X

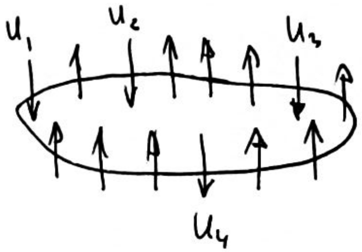
$$\langle \mathcal{O}\rangle = \frac{2^{1-L}L^{-1/2}}{x_3^L}\,\frac{\langle \mathrm{VBS}\,|\,\Psi\rangle}{\langle \Psi|\Psi\rangle^{1/2}}$$

VBS =



$$\langle K| = \langle\uparrow\uparrow| + \langle\downarrow\downarrow|$$

Bethe Ansatz



$$\mathcal{O} = \Phi^{s_1 \dots s_L} \operatorname{tr} Z_{s_1} \dots Z_{s_L}$$

$$e^{i x_j} \equiv \left(\frac{u_j - \frac{i}{2}}{u_j + \frac{i}{2}} \right)^L \prod_k \frac{u_j - u_k + i}{u_j - u_k - i} = -1$$

$$\Delta = L + \frac{\lambda}{16\pi^2} \sum_j \frac{1}{u_j^2 + 1/4} + \mathcal{O}(\lambda^2)$$ /anomalous dimension/

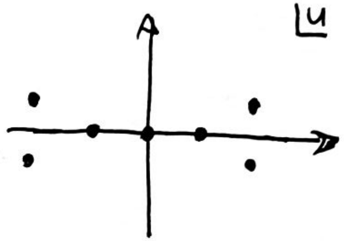
Gaudin matrix:

$$G_{jk} = \frac{\partial x_j}{\partial u_k} = \delta_{jk} \left(\frac{L}{u_j^2 + \frac{1}{4}} - \sum_m K_{jm} \right) + K_{jk}$$

$$K_{jk} = \frac{2}{(u_j - u_k)^2 + 1}$$

Overlaps w. integrable boundary states

Selection rule:



$$\langle B | \{u_j\} \rangle \neq 0 \implies \{u_j\} = \{-u_j\}$$

Ghoshal, Zamolodchikov '94

Piroli, Pozsgay, Vernier '17

$$\boxed{\langle B | \{u_j\} \rangle = \sqrt{\prod_j f_B(u_j)\ \mathrm{Sdet}\, G}}$$

Brockmann, De Nardis, Wouters, Caux '14

...

$$\mathrm{Sdet}\, G = e^{\mathrm{tr}\, \Omega \ln G}$$

$$\det G = \det G^+ \det G^- \qquad \mathrm{Sdet}\, G = \frac{\det G^+}{\det G^-}$$

$$G^\pm_{jk} = \delta_{jk}\left(\frac{L}{u_j^2+\frac{1}{4}} - \sum_m K^+_{jm}\right) + K^\pm_{jk}$$

$$\Omega : u_j \rightarrow -u_j$$

$$K^\pm_{jk} = \frac{2}{(u_j-u_k)^2+1} \pm \frac{2}{(u_j+u_k)^2+1}$$

Valence - bond states:

$$\langle VBS \,|\, \{u_j\}_{j=1\dots M} \rangle = \sqrt{\sin^{L-2M}\beta \;\frac{Q^2\left(\frac{i\nu}{2}\right)}{Q(0)\,Q\left(\frac{i}{2}\right)}\; \mathrm{S}\det G}$$

for $\bigcap\ \bigcap \dots \bigcap$ with $\bigcap = (\cos\beta + \nu)\uparrow\downarrow + (\cos\beta - \nu)\downarrow\uparrow + i\,e^{i\alpha}\sin\beta\uparrow\uparrow + e^{-i\alpha}\sin\beta\downarrow\downarrow$

Ex: Néel ($\beta = 0$, $\nu = 1$)

$$\boxed{Q(u) = \prod_j (u - u_j)}$$ - Baxter polynomial

Matrix - product states:

$$\langle MPS_k \,|\, \{u_j\} \rangle = S_k \sqrt{Q^2\left(\frac{ik}{2}\right) Q\left(\frac{i}{2}\right) Q(0)\; \mathrm{S}\det G}$$

$$S_k = \sum_{a=-\frac{k-1}{2}}^{a=\frac{k-1}{2}} \frac{a^L}{Q\left(\frac{2a+1}{2}i\right) Q\left(\frac{2a-1}{2}i\right)}$$

$\uparrow$ bond dimension

Non-perturbative 1pt functions

Buhl-Mortensen, de Leeuw, Ipsen, Kristjansen, Wilhelm'17

Komatsu, Wang'20

Gombor, Bajnok'20

$$S_k^{\text{all-loop}} = \sum_a \frac{x_a^L \overset{\xleftarrow{\text{dressing phase}}}{\sigma_a}}{Q\left(\frac{2a+1}{2}i\right) Q\left(\frac{2a-1}{2}i\right)}$$

$$2x_a = a + \sqrt{a^2 + \frac{\lambda}{4\pi^2}} = 2a + \mathcal{O}(\lambda) \quad - \text{ Zhukovsky variable}$$

- accurate up to $\mathcal{O}(g^L)$
- Exact result for vacuum overlap (from localization):

$$\text{CPO} = \operatorname{tr} Z^L$$

$$\langle \text{CPO} \rangle = \frac{2^{-L} L^{-1/2}}{x_3^L} \Bigg[\underset{\substack{\uparrow \\ \text{asymptotic}}}{\left(\frac{16\pi^2}{\lambda}\right)^{\frac{L}{2}} \sum_{a=-\frac{k-1}{2}}^{\frac{k-1}{2}}} \quad \underset{\substack{\uparrow \\ \text{½ wrapping}}}{- k\,\delta_{L,2}} \quad + \underset{\substack{\uparrow \\ \text{wrapping}}}{\left(\frac{\lambda}{16\pi^2}\right)^{\frac{L}{2}} \sum_{b\in\mathbb{Z}+\frac{k-1}{2}}' \frac{1}{x_b^L}} \Bigg]$$

Komatsu, Wang'20

Duality transformations

$$\left(\frac{u_j+\frac{i}{2}}{u_j-\frac{i}{2}}\right)^{\frac{L}{2}} = \prod_{j\neq k}\frac{u_j-u_k+i}{u_j-u_k-i} \quad \Longleftrightarrow \quad \underset{\uparrow\,\text{deg } M}{Q^+}\,\overset{\text{deg } L-M+1\,\downarrow}{\tilde{Q}^-} - Q^-\tilde{Q}^+ = i\,(2M-L-1)\,u^L$$

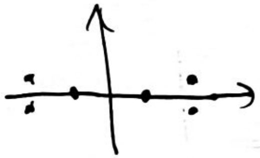
$$Q(u) = \prod_j (u-u_j)$$

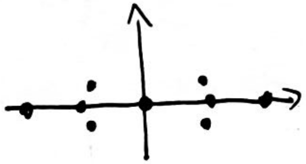
$$\tilde{Q}(u) = \prod_j (u-\underset{\uparrow}{\tilde{u}_j})$$

satisfy same Bethe eqs.

$$f^{\pm}(u) \equiv f(u \pm \tfrac{i}{2})$$

$$Q(u) = Q(-u) \implies \tilde{Q}(u) = -\tilde{Q}(-u)$$





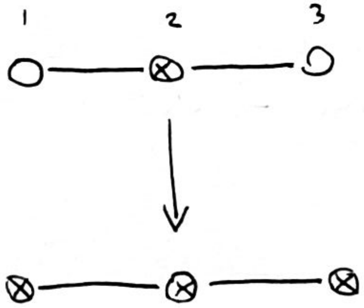
$$\boxed{\mathrm{Sdet}'\tilde{G} = A_{\frac{L}{2}-M}\,\frac{Q(0)\,\tilde{Q}(\frac{i}{2})}{Q(\frac{i}{2})\,\tilde{Q}(0)}\;\mathrm{Sdet}\,G}$$

Kristjansen, Müller, Z' to appear

Sdet with root at zero removed

$$A_n = \frac{2(2^n n!)^4}{(2n)!\,(2n+1)!}$$

Fermionic duality



$$Q_1^- Q_3^+ - Q_1^+ Q_3^- = i\,(M_3 - M_1)\, Q_2 \tilde{Q}_2$$

$$\boxed{\mathrm{Sdet}\,\tilde{G} = \text{const}\, \frac{Q_2(0)\,\tilde{Q}_2(0)}{Q_1\left(\frac{i}{2}\right) Q_3\left(\frac{i}{2}\right)}\ \mathrm{Sdet}\,G}$$

Kristjansen, Müller, Z '20

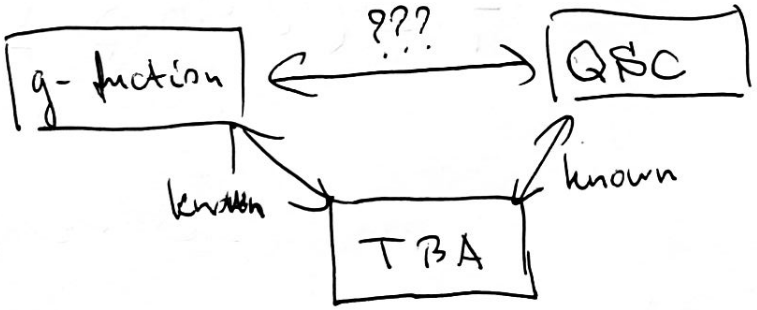
Conclusion

- Exact g-function of dCFT ?

  ↳ will generalize exact 1pt function $\langle O_{CPO} \rangle$ /Komatsu, Wang'20/ to non-protected states

- Relation between g-function and Quantum Spectral Curve:

  Caetano, Komatsu'20



- Other ~~[illegible]~~ integrable dCFTs ?

  - Wilson loops, 't Hooft lines, ...