Integrability and RG flow in 2d sigma models

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Integrable 2d σ-models:

important, in particular, for finding solvable string theories

• no general classification, hard to find:

no general method of finding Lax pair or factorizable S-matrix

• heuristic approach:

look for stability under RG – when flow reduces

to a finite dim "integrable" subspace of couplings

• observed on examples:

classical integrability $\rightarrow$ 1-loop RG stability

[Fateev, Onofri, Zamolodchikov 93; Fateev 96; Lukyanov 13]

• higher loops [Hoare, Levine, AT 19, 20]

RG-stable but may need to add finite counterterms

$$G_{mn} \rightarrow G_{mn} + \hbar G^{(1)}_{mn} + \ldots$$

$\rightarrow$ deformation of target space geometry

RG flow in bosonic 2d sigma model

$S = \frac{1}{4\pi\alpha'} \int d^2z \, G_{mn}(x) \, \partial x^m \partial x^n , \quad \alpha' = \hbar = 1$

∞-coupling theory: $G_{mn}(x) = \delta_{mn} + \sum_r g_{mn;k_1 ... k_r} x^{k_1} ... x^{k_r}$

$\{g_{mn;k_1 ... k_r}\} \leftrightarrow \{R_{mnkl}, D_k R_{mnkl}, ...\}$

RG in ∞-coupling space $\{g_r(\tau)\}$ [Ecker, Honerkamp 71; Friedan 80]

$$\frac{d}{d\tau} G_{mn} = \beta_{mn} + D_{(m} X_{n)}$$

$$\beta_{mn} = R_{mn} + \tfrac{1}{2} R_{mpqr} R_n{}^{pqr} + ...$$

find G_{mn} and X_m when flow is restricted to finite subspace?

e.g. if manifest symmetry (isometry of S^n: 1 coupling)

what if symmetry is hidden?

Conjecture: integrability is related to invariance under RG flow

• Example: deformed S^2

integrable "sausage" model: [Fateev, Onofri, Zamolodchikov 93]

same as YB η-deformation [Klimcik 02; Delduc, Magro, Vicedo 13]

of $\frac{SO(3)}{SO(2)}$ [Hoare, Roiban, AT 14]

$$ds^2 = \frac{h}{1+\varkappa^2 r^2}\left[\frac{dr^2}{1-r^2} + (1-r^2)d\phi^2\right]$$

• solves 1-loop Ricci flow

$$\frac{d}{d\tau}h = (1+\varkappa^2) + \mathcal{O}(h^{-1})$$
$$\frac{d}{d\tau}\varkappa = h^{-1}\varkappa(1+\varkappa^2) + \mathcal{O}(h^{-2})$$
$$\frac{d}{d\tau}\frac{h}{\varkappa} = 0$$

• beyond 1 loop: still 2-coupling $(h, \varkappa)$ flow?

Beyond leading 1-loop: hidden symmetries constrain RG?
classical σ-model non-trivially deformed under RG so that number of running couplings preserved
interpretation: finite counterterms to preserve integrability

- direct 2-loop computation: closed RG for (h, κ) if

$$ds^2 = h\Big[\tfrac{dr^2}{(1-r^2)(1+\varkappa^2 r^2)} + \tfrac{1-r^2}{1+\varkappa^2 r^2} d\phi^2\Big]$$
$$-\Big[(1-\varkappa^2)\tfrac{dr^2}{(1-r^2)(1+\varkappa^2 r^2)} + \big[(1-\varkappa^2)\tfrac{1-r^2}{1+\varkappa^2 r^2} + 2\varkappa^2\big] d\phi^2\Big] + \ldots$$

- heuristic reason: $\exists$ underlying quantum integrable S-matrix generalization of ZZ S-matrix for S^2 [Fateev,Onofri,Zamolodchikov]
RG-invariant parameters (RG scale/coupling in dyn. gen. mass) suggests solution of RG with just one running coupling
- hidden conserved charges expected to constrain RG flow: local counterterms to maintain symmetry/Ward identities

• Example I:

T-duality makes non-abelian symm hidden

yet should remain symmetry of RG equations

$$S^2: \quad ds^2 = h(dx^2 + \sin^2 x \, dy^2) \quad \to$$

$$\tilde{S}^2: \quad \widetilde{ds^2} = h(dx^2 + \frac{1}{\sin^2 x} d\tilde{y}^2)$$

S^2: $SO(3)$ symmetry $\to$ one running coupling h

$\tilde{S}^2$: only $SO(2)$ manifest but $SO(3)$ is hidden [Ricci, AT, Wolf 07]

still only one coupling runs?

• hidden symmetry: corrections to metric are correlated with corrections to T-duality transf. $\to$

only one coupling h runs

T-duality:

$$ds^2 = dx^2 + M(x)dy^2$$

$$M\,\partial_m y \to \epsilon_{mn}\partial^n \tilde{y}, \quad y \to \tilde{y}, \qquad M \to M^{-1}$$

path integral transformation:

$$M(\partial y)^2 \ \to \epsilon^{mn}\partial_m \tilde{y} A_n + M A^m A_m$$

results in non-trivial quantum correction [AT 91; Schwarz, AT 92]

$$\int [dA] \exp\left[i \int d^2x \ \sqrt{g}\, M\, A_+ A_-\right]$$

$$= \exp\left(\frac{i}{4\pi}\int d^2x \ \sqrt{g}\Big[-\tfrac{1}{2}R^{(2)} \ln M - \tfrac{1}{2}(\partial_a \ln M)^2\Big]\right)$$

i.e. in addition to dilaton shift [Buscher 87]

get finite 1-loop correction $\Delta L \sim (\partial_m \log M)^2$

- implies deformation of the T-dual of S^2 metric consistent with 2-loop RG flow with only h running
- same applies to non-abelian duality

Example II:

complex SG model [Pohlmeyer; Lund, Regge]

$$L = \frac{\partial\zeta^*\partial\zeta}{1-\zeta\zeta^*} - m^2\zeta^*\zeta \ \to (\partial\phi)^2 + \tan^2\phi\ (\partial\theta)^2 - m^2\sin^2\phi$$

factorization of 1-loop S-matrix

requires adding counterterm [de Vega, Maillet 8; Bonneau, Delduc 85]

• origin: relation to $SU(2)/U(1)$ gauged WZW [Hoare, AT 10]

leading 1/k deformation of gWZW metric → counterterm

Gauged WZW

example of quantum deformation of effective σ-model

- CFT: geometry probed by point-like string

$$T = \frac{1}{\mathrm{k}+c_G} J_G^2 - \frac{1}{\mathrm{k}+c_H} J_H^2$$ [Dijkgraaf, Verlinde, Verlinde 91]

$$L_0 + \bar{L}_0 = H, \qquad H\Phi \to \Delta\Phi = \frac{1}{e^{-2\phi}\sqrt{G}} \partial_m (e^{-2\phi}\sqrt{G}\, G^{mn}\partial_n)\Phi$$

- alternative derivation of G, ϕ: eff action of gWZW model

[AT 93; Bars, Sfetsos 93]

$$S_{\rm eff} = -\frac{\mathrm{k}+c_G}{4\pi}\mathrm{Tr}\Big[\frac{1}{2}\int d^2x\ g^{-1}\partial_+ g\, g^{-1}\partial_- g - \frac{1}{3}\int d^3x\ (g^{-1}\partial_m g)^3$$
$$+ \int d^2x\ \Big(A_+\partial_- g g^{-1} - A_- g^{-1}\partial_+ g$$
$$-g^{-1}A_+ g A_- - \big[1 - \frac{2(c_G - c_H)}{\mathrm{k}+c_G}\big] A_+ A_-\Big)\Big]$$

- leading $1/k$ correction: from $A_+ M A_- \to -\frac{1}{2}(\partial \log M)^2$

exact $SL(2)/U(1)$ gWZW metric

$$ds^2 = (\mathrm{k}-2)\big(dr^2 + \frac{\tanh^2 r}{1-\frac{2}{\mathrm{k}}\tanh^2 r}\, dy^2\big)$$

deformation of leading-order metric

$$ds^2 = \mathrm{k}\big(dr^2 + \tanh^2 r dy^2\big)$$

• alternative logic: deformation required
to satisfy 2-loop conf. invariance condition [AT 91]

$$\beta^G_{mn} + 2D_m D_n \phi = 0$$

(extends to higher-loop order: [Jack, Jones, Panvel 92])

Back to η-model:
quantum corrections to "sausage" model metric

$$ds^2 = h\Big[\frac{dr^2}{(1-r^2)(1+\varkappa^2 r^2)} + \frac{1-r^2}{1+\varkappa^2 r^2}d\phi^2\Big]$$

- $\varkappa = 0$ is S^2; $\quad \varkappa^2 = -1$ is flat
- $\varkappa^2 \to -1$ with $r^2 \to 1 - (1+\varkappa^2)\sinh^2 r$:

$\frac{SO(1,2)}{SO(2)}$ gWZW: $\qquad ds^2 = h(dr^2 + \tanh^2 r\, d\phi^2)$

- follow analogy with exact gWZW metric: $\quad h = \mathrm{k} - 2$

$$ds^2 = h\Big[dr^2 + \big(\coth^2 r - \frac{2}{\mathrm{k}}\big)^{-1} d\phi^2\Big]$$

• exact solution of RG eqs.: [Hoare, Levine, AT 19]

$$ds^2 = h\Big[\frac{dr^2}{(1-r^2)(1+\varkappa^2 r^2)} + \big(\frac{1+\varkappa^2 r^2}{1-r^2} + \frac{2}{\mathrm{k}}\big)^{-1} d\phi^2\Big]$$

$$\mathrm{k} = \frac{h+1-\varkappa^2}{\varkappa^2}\,, \qquad \mathrm{k}^{-1} = \varkappa^2 h^{-1}\big[1 - h^{-1}(1-\varkappa^2) + ...\big]$$

• no deformation for $\varkappa = 0$ (S^2) and $\varkappa^2 = -1$ (flat space)

• in gWZW limit: $\varkappa^2 \to -1,\ r^2 \to 1-(1+\varkappa^2)\sinh^2 r$:

$$ds^2 = h\Big[dr^2 + \big(\coth^2 r - \frac{2}{h+2}\big)^{-1} dy^2\Big]$$

recover exact gWZW metric with $h = \mathrm{k} - 2$

• $\mathrm{k}\varkappa = \frac{h+1-\varkappa^2}{\varkappa}$=RG invariant

- direct check at 2-loop order:

$$\frac{d}{d\tau}G_{mn} = \beta_{mn} + D_{(m}X_{n)}$$

$$\beta_{mn} = R_{mn} + \tfrac{1}{2}R_{mpqr}R_n{}^{pqr} + ... = (\tfrac{1}{2}R + \tfrac{1}{4}R^2)G_{mn} + ...$$

$$ds^2 = h\Big[\frac{dr^2}{(1-r^2)(1+\varkappa^2 r^2)} + \frac{1-r^2}{1+\varkappa^2 r^2}\Big(1 - 2\varkappa^2 h^{-1}\frac{1-r^2}{1+\varkappa^2 r^2} + ...\Big)d\phi^2\Big]$$

$$\frac{d}{d\tau}h = (1+\varkappa^2) + h^{-1}(1+\varkappa^2)^2 + \mathcal{O}(h^{-2})$$

$$\frac{d}{d\tau}\varkappa = h^{-1}\varkappa(1+\varkappa^2) + \mathcal{O}(h^{-2})$$

- checked also at 3-loop order
- same expected for other YB deformations of σ-models

Quantum deformation of λ-model

$$L = \mathrm{k}\,\mathrm{tr}\big[\tfrac{1}{2}(g^{-1}\partial g)^2 + A_+\partial_- g g^{-1} - A_- g^{-1}\partial_+ g$$
$$- g^{-1}A_+ g A_- + A_+A_- + b^{-2}A_+PA_-\big] + \mathrm{k}\,\mathrm{WZ}(g)$$

$b^{-2} = \lambda^{-1} - 1$ [Sfetsos 13; Hollowood, Miramontes, Schmidt 14]

- $b \to 0$: $PA_\pm = 0 \to$ G/H gWZW model
- $\mathrm{k} \to \infty$, $b \to \infty$, $g = e^{v/\mathrm{k}}$: non-abelian dual of G/H model

$$L = \mathrm{tr}(vF_{+-} + hA_+A_-), \qquad h = \mathrm{k}\,b^{-2}$$

- 1-loop renormalizable [Isios,Sfetsos,Siampos 14; Appadu,Hollowood 15]
- what happens at 2-loop level?

deformation from integral over $A_\pm$ as in gWZW case:

$$L = ... + A_+MA_- \qquad \to \qquad \Delta L = -\tfrac{1}{2}(\partial_m \log \det M)^2$$

- $G = SO(1,2)$, $H = SO(2)$: integrate out $A_\pm$

$$\mathcal{L} = \frac{\mathrm{k}}{p^2 + q^2 - 1}(\kappa\,\partial_+ p \partial_- p + \kappa^{-1}\partial_+ q \partial_- q), \qquad \kappa \equiv (1 + 2b^2)^{-1}$$

- one-loop corrected metric:

$$ds^2 = \mathrm{k}\,\frac{\kappa\, dp^2 + \kappa^{-1} dq^2}{p^2 + q^2 - 1} - \frac{1}{2}\frac{\left[d(p^2 + q^2)\right]^2}{(p^2 + q^2 - 1)^2}$$

- solves 2-loop RG eqs $\qquad \frac{d}{d\tau} G_{mn} = (\frac{1}{2}R + \frac{1}{4}R^2) G_{mn} + ...$

$$\frac{d}{d\tau}\mathrm{k} = (\kappa^{-1} - \kappa)^2 \mathrm{k}^{-1} + \mathcal{O}(\mathrm{k}^{-2})$$
$$\frac{d}{d\tau}\kappa = -(1 - \kappa^2)\mathrm{k}^{-1} + \mathcal{O}(\mathrm{k}^{-3})$$

- RG invariant (as in η-model)

$$\nu = \mathrm{k} - (\kappa^{-1} + \kappa) + \mathcal{O}(\mathrm{k}^{-1}), \qquad \frac{d}{dt}\nu = 0$$

Non-abelian dual (NAD) limit:

$\kappa \to 0, \ \mathrm{k} \to \infty, \quad h \equiv \mathrm{k}\,\kappa$

$p = \kappa X, \quad q = 1 + \kappa^2 Y$:

$$ds^2 = h\,\frac{dX^2 + dY^2}{X^2 + 2Y} - \frac{1}{2}\frac{\left[d(X^2 + 2Y)\right]^2}{(X^2 + 2Y)^2}$$

• 2-loop renormalizable

with same 2-loop β-function as for $H^2 = SO(1,2)/SO(2)$

$$\frac{d}{d\tau}h = -1 + h^{-1} + \mathcal{O}(h^{-2})$$

• NAD consistent with 2-loop RG once metric is deformed: resolves earlier problem at 2 loops – duality indeed remains symmetry of α' pert. theory or string eff. action

• same should be for Poisson-Lie duality at 2-loop level (cf. λ-model and η-model are PL dual [Hoare, AT 15; Vicedo 15])

λ-model formulated on extended space

integrable model associated to G or G/H

• parameters: WZ level k and coupling λ
fields: g in G, 2d vector $A_\pm$ in algebra
$A_+ = h\partial_+ h^{-1}$, $A_- = \bar{h}\partial_- \bar{h}^{-1}$
• σ-model on extended $G \times G \times G$ space $(g, h, \bar{h})$:
renormalizable without deformation, only λ running
• cf. standard σ-model found by integrating out $A_\pm$:
2-loop renormalizability only with finite local counterterms
$\rightarrow$ quantum deformation of target space geometry

$$\mathcal{L}_{G/H}(g,A) = \mathcal{L}_{\rm PCM}(g) + \mathcal{L}_{\rm WZ}(g)$$
$$+{\rm Tr}\big[g^{-1}\partial_+ g A_- - A_+\partial_- g\, g^{-1} + g^{-1}A_+ g A_- - A_+A_-\big]$$

λ-model for G or G/H: effective (current)^2 deformation

$$\mathcal{L} = k\Big[\mathcal{L}_{G/G}(g,A) - b^{-2}{\rm Tr}(A_+A_-)\Big]\ , \qquad b^{-2} = \lambda^{-1} - 1$$

$$\mathcal{L} = k\Big[\mathcal{L}_{G/H}(g,A) - b^{-2}{\rm Tr}(A_+P_{G/H}A_-)\Big]$$

• integrating out $A_\pm$: σ-model on G or G/H config. space then RG flow requires deformation of geometry

• but not needed on $G\times G\times G$ config. space $(g,h,\bar h)$ get sum of WZW model + deformation of $G\times G$ WZW by current-current interaction

• form of action protected under RG flow by chiral gauge symmetries and global symmetries

$$\mathcal{L} = k\Big(\mathcal{L}_G(h^{-1}g\bar{h}) - \mathcal{L}_H(h^{-1}\bar{h})$$
$$-(\lambda^{-1}-1)\big[\mathcal{L}_H(h^{-1}\bar{h}) - \mathcal{L}_H(h^{-1}) - \mathcal{L}_H(\bar{h})\big]\Big)$$

$$\mathcal{L} = k\mathcal{L}_G(\tilde{g}) + \mathcal{L}'(\hat{h},\bar{h})\,, \qquad \tilde{g} \equiv h^{-1}g\bar{h} \in G\,, \qquad \hat{h} \equiv h^{-1}$$

$$\mathcal{L}' = -(k+2c_{_H})\big[\mathcal{L}_H(\hat{h}) + \mathcal{L}_H(\bar{h})\big] + k\lambda^{-1}\mathrm{Tr}[\hat{h}^{-1}\partial_+\hat{h}\,\partial_-\bar{h}\bar{h}^{-1}]$$

- G case: RG equation for λ $\qquad (\frac{d}{d\tau}k = 0)$

$$\frac{d}{d\tau}\lambda = \frac{2c_{_G}}{k}\frac{\lambda^2}{(1+\lambda)^2}\Big[1 - \frac{2c_{_G}}{k}\frac{\lambda^2}{(1+\lambda)^2}\frac{\lambda+\lambda^{-1}-1}{1-\lambda^2}\Big]$$

- G/H case: manifest renormalizability on $G \times H \times H$ space

$$\frac{d}{d\tau}\lambda = \frac{c_{_G}\lambda}{k}\Big[1 + \frac{1}{k}\frac{c_{_H} - (2c_{_G} - c_{_H})\lambda^2}{1-\tilde{\lambda}^2}\Big]$$

- in NAD limit get 2-loop RG eq. in G and G/H σ-models

Integrability–RG connection: new examples [Levine, AT 21]

- RG in integrable $G \times G$ models
- RG in integrable $G \times G/H$ models
- integrable $T^{1,q}$ model

$$S = \frac{1}{4\pi\alpha'} \int d^2\sigma \, \left[G_{\mu\nu}(x) + B_{\mu\nu}(x) \right] \partial_+ x^\mu \partial_- x^\nu$$

special RG scheme for related integrable models:

G, B treated on equal footing (adapted to WZW) [Metsaev, AT 87]

$$\frac{d}{d\tau}(G_{\mu\nu} + B_{\mu\nu}) = \hat{R}_{\mu\nu} + \tfrac{1}{2}\alpha' \left[\hat{R}^{\rho\sigma\kappa}{}_{\nu} \hat{R}_{\mu\rho\sigma\kappa} - \tfrac{1}{2} \hat{R}^{\sigma\kappa\rho}{}_{\nu} \hat{R}_{\mu\rho\sigma\kappa} + \tfrac{1}{2} \hat{R}_{\rho\mu\nu\sigma} (H^2)^{\rho\sigma} \right] + \ldots$$

$$\hat{R} = R(\hat{\Gamma}), \quad \hat{\Gamma} = \Gamma + \tfrac{1}{2} H, \quad H = dB$$

Integrable $G \times ... \times G$ models

how to generate new integrable models from PCM$_k$

$$\mathcal{L}_G = \mathrm{h}\,\mathrm{tr}[J_+ J_-] + k\,\mathcal{L}_{WZ}(g)\,, \qquad J = g^{-1} dg\,, \quad g \in G$$

• one option: integrable deformations

discussed above [Klimčik 02, Delduc, Magro, Vicedo 13] ...

• another: couple PCM$_k$ G-models [Delduc,Lacroix,Magro,Vicedo 18]

$$\mathcal{L}_{G^N} = \rho_{ij}\,\mathrm{tr}[J_+^{(i)} J_-^{(j)}] + k_i\,\mathcal{L}_{WZ}(g^{(i)})$$

$$J^{(i)} = {g^{(i)}}^{-1} dg^{(i)}\,, \qquad (g^{(i)}) \in G^N\,,\ \ i = 1, \ldots, N$$

• integrable if (ρ_{ij}, k_i) satisfy certain polynomial eqs

(from affine Gaudin construction)

$G \times G$ case:

$$\rho_{ij} = h_{(ij)} + b_{[ij]} = \begin{pmatrix} s & t \\ t & u \end{pmatrix} + \begin{pmatrix} 0 & b \\ -b & 0 \end{pmatrix}$$

integrability condition: $\quad f(s,t,u,b,k_1,k_2) = 0$

$$f \equiv -t\,(s+t)\,(t+u) + b^2\,(s+t+u) + t\,k_1\,k_2 + b\,(u\,k_1 - s\,k_2)$$

- $(G_L)^2 \times G_R$ symmetry $\rightarrow$ only ρ_{ij} running

expect integrable model $f(\rho_{ij}, k_i) = 0$ to be RG-stable

$$\frac{d}{d\tau}\rho_{ij} = \beta^{(1)}_{ij} + \alpha' \beta^{(2)}_{ij} + \ldots \,,$$

$$\beta^{(1)}_{ij} = (su - t^2)^{-2}\, H_4(s,t,u,b,k_1,k_2)$$

$$\beta^{(2)}_{ij} = (su - t^2)^{-5}\, H_9(s,t,u,b,k_1,k_2)$$

H_n: long n-order polynomials

check of stability of integrable subspace under RG:

$$\frac{d}{d\tau} f \Big|_{f=0} = \left(\beta_{ij}^{(1)} + \alpha' \beta_{ij}^{(2)} + \ldots\right) \partial_{\rho_{ij}} f \Big|_{f=0} = 0 + 0 + ...$$

1-loop: [Delduc, Lacroix, Sfetsos, Siampos 20]

2-loop: [Levine, AT]

no deformation of the form of the model required:
enough manifest global symmetries

Example: 3-coupling integrable model

$k_1 = k_2 = 0$: integrability condition $\rightarrow \quad b^2 = \frac{t(s+t)(t+u)}{(s+t+u)}$

$$x = s + t + u\,, \quad y = \frac{s}{t}\,, \quad z = \frac{u}{t}$$

$$\frac{d}{d\tau}x = 2c_G + \frac{c_G^2}{2x(yz-1)^3}\left[y^4z^2 + 2y^3z((z-4)z-6) - 16(z+1)^2 \right.$$
$$\left. +y^2(z(z((z-8)z-60)-68)-16) - 4y(z+1)(z+4)(3z+2)\right]$$
$$\frac{d}{d\tau}y = F(x;y,z)\,, \qquad \frac{d}{d\tau}z = F(x;z,y)$$

$$F(x;y,z) \equiv -\frac{y(y+1)(y+2)}{(zy-1)^2}\left(\frac{c_G}{x}\left[zy - 1 + 3(z+1)^2\right]\right.$$
$$-\frac{c_G^2}{2x^2(zy-1)^3}\left[-z^6y^2 + y^5(-y)(z(3y-38)-44) + 2z(y+1)(y(26z+101)+58)\right.$$
$$+20(y+1)^2 - z^4(y(3y((y-14)y-100)-296)-38)$$
$$-z^3(y(y(y((y-4)y-178)-728)-708)-152)$$
$$\left.\left. +2z^2(y+1)(y(y(5y+89)+262)+105)\right]\right)$$

stable under RG

Integrable G^N/H models

standard G/H σ-model: $\quad \mathcal{L} = \mathrm{h}\,\mathrm{tr}\,[J_+ \, P_{G/H} \, J_-]$

gauge G^N model: $g^{(i)} \to g^{(i)} h, \; h(\sigma) \in H_{\text{diag}}$

[Arutyunov, Bassi, Lacroix 20]

$$\mathcal{L} = \rho_{ij} \,\mathrm{tr}[P_+^{(i)} P_-^{(j)}] + r_{ij} \,\mathrm{tr}[B_+^{(i)} B_-^{(j)}] + k_i \, \mathcal{L}_{\mathrm{WZ}}(g^{(i)})$$

$$B^{(i)} = P_H J^{(i)} \,, \quad P^{(i)} = P_{G/H} J^{(i)} \,, \qquad J^{(i)} \equiv {g^{(i)}}^{-1} d g^{(i)}$$

- Integrable $G \times G/H$ model:

$$\mathcal{L} = \rho_{ij} \,\mathrm{tr}[P_+^{(i)} P_-^{(j)}] + r_{ij} \,\mathrm{tr}[B_+^{(i)} B_-^{(j)}] + k_i \, \mathcal{L}_{\mathrm{WZ}}(g^{(i)})$$

- gauge invariant if : $\quad k \equiv k_1 = -k_2 \,, \quad r_{ij} = \begin{pmatrix} r & -r-k \\ -r+k & r \end{pmatrix}$
- free parameters k, r, ρ_{ij}

integrability condition: [Arutyunov, Bassi, Lacroix 20]

$$f_1 \equiv r^2 - k^2 - \rho_{12}\rho_{21} = 0$$

$$f_2 \equiv (r-k)^4\rho_{12} + (r-k)^2(-r+2\rho_{11}+k)(-r+2\rho_{22}+k)\rho_{21} + 2(\rho_{11}+\rho_{22})(-r+k)\rho_{12}\rho_{21}^2 + \rho_{12}^2\rho_{21}^3 + \rho_{12}\rho_{21}^4 = 0$$

e.g. $r = k,\ \rho_{12} = 0$

$G_L \times G_L$ global and H_R gauge symm. $\rightarrow$ only (r, ρ_{ij}) can run

1-loop: $\rho_{ij} \equiv \begin{pmatrix} s & t+b \\ t-b & u \end{pmatrix}$ [Levine, AT 21]

$$\frac{d}{d\tau} r = \frac{c_G - c_H}{(t^2-su)^2}\left(r^2s^2 - 2b^2t^2 - 2r^2t^2 + 2t^4 - 2b^2su - 2st^2u + r^2u^2 + c_H\left(1 - \left(\tfrac{k}{r}\right)^2\right)\right)$$

$$\frac{d}{d\tau} s = \frac{c_G}{r(t^2-su)}\left(b^2s - st^2 + r^2u + 2r(t^2-su) - 2btk + uk^2\right)$$

$$\frac{d}{d\tau} t = \frac{c_G}{r(t^2-su)}\left(-b^2t + b(s+u)k + t(r^2 - su - k^2)\right)$$

$$\frac{d}{d\tau} b = \frac{c_G}{r(t^2-su)}\left(-b(t^2+su) + t(s+u)k\right)$$

$$\frac{d}{d\tau} u = \frac{c_G}{r(t^2-su)}\left(r^2s + b^2u - t^2u + 2r(t^2-su) - 2btk + sk^2\right)$$

2-loop:

$$\beta_r^{(2)} = r^{-1}(su - t^2)^{-4} H_8(k, r, s, t, b, u)$$
$$\beta_{\rho_{ij}}^{(2)} = r^{-2}(su - t^2)^{-3} H_7(k, r, s, t, b, u)$$
$$\frac{d}{d\tau} f_a\Big|_{f_{1,2}=0} = (\beta_{ij}^{(1)} + \alpha' \beta_{ij}^{(2)} + \dots)\ \frac{\partial}{\partial \rho_{ij}} f_a\Big|_{f_{1,2}=0}$$
$$= \ 0 \ + \ \alpha'(non - zero) \quad + \dots\ , \qquad a = 1, 2$$

• integrable $G \times G/H$ model automatically stable at 1-loop but requires finite counterterm (deformation) at 2-loop

• global + gauge symm. $\rightarrow \ \delta \mathcal{L} \sim \delta r,\ \delta \rho_{ij}$

RG-stable surface will be $\ f_a + \alpha' \delta f_a = 0$

• expected: gauging $\rightarrow$

analogy with gWZW and λ-model

• no deformation on extended config. space? cf. λ-model

New integrable $T^{1,q}$ model

if H=abelian: exists another branch of gauge-inv models

e.g. $G\times G/H = SU_2\times SU_2/U_1$

$$\mathcal{L} = \rho_{ij}\,\mathrm{tr}[P_+^{(i)}P_-^{(j)}] + r_{ij}\,\mathrm{tr}[B_+^{(i)}B_-^{(j)}] + k_i\,\mathcal{L}_{\mathrm{WZ}}(g^{(i)})$$

$$\rho_{12}=\rho_{21}=0\,,\quad r_{ij}=\begin{pmatrix} r & q(-r-k_1) \\ q(-r+k_1) & q^2 r\end{pmatrix},\quad q^2\equiv -k_2/k_1$$

$$(g^{(1)},\,g^{(2)})\ \rightarrow\ (g^{(1)}h^q,\,g^{(2)}h)\,,\qquad h(\sigma)\in G$$

extra constraints $\rho_{12}=\rho_{21}=0,$ levels k_1, k_2

• choose $r=k_1$ $(h\equiv\rho_{11},\ \widetilde{h}\equiv\rho_{22},\ k\equiv k_1,\ \widetilde{k}\equiv -k_2)$

$$\mathcal{L} = \mathrm{tr}\big[h\,P_+P_- + \widetilde{h}\,\widetilde{P}_+\widetilde{P}_-\big] + k\,\mathcal{L}_{\mathrm{WZ}}(g) - \widetilde{k}\,\mathcal{L}_{\mathrm{WZ}}(\widetilde{g})$$
$$+\mathrm{tr}\big[k\,B_+B_- + \widetilde{k}\,\widetilde{B}_+\widetilde{B}_- - 2\sqrt{k\widetilde{k}}\,B_+\widetilde{B}_-\big]$$

5d metric of $T^{1,q} \equiv SU_2 \times \widetilde{SU_2}/U_1$ plus B-field

$$ds^2 = r\,(d\psi + \cos\theta_1\, d\phi_1 + q\cos\theta_2\, d\phi_2)^2$$
$$+h\,(d\theta_1^2 + \sin^2\theta_1\, d\phi_1^2) + \widetilde{h}\,(d\theta_2^2 + \sin^2\theta_2\, d\phi_2^2)$$
$$B = k\,(d\psi + \cos\theta_1\, d\phi_1) \wedge (d\psi + q\cos\theta_2\, d\phi_2)\,, \qquad q^2 = \frac{\widetilde{k}}{k}$$

• $k = 0$ and $r = \frac{1}{9}$, $h = \tilde{h} = \frac{1}{6}$, $q = 1$: $T^{1,1}$ Einstein space
(base of 6d Ricci-flat conifold $d\rho^2 + \rho^2 dT^{1,1}$)
non-integrable σ-model [Basu, Pando-Zayas 11]

• $h = k$, $\widetilde{h} = \widetilde{k}$: conformal $T^{1,q}$ [Pando-Zayas, AT 00]
special case of GMM model [Guadagnini, Martellini, Mintchev 87]

• $r = k = \widetilde{k}$ ($q = 1$): integrable $T^{1,1}$ model [Arutyunov, Bassi, Lacroix 20]

• $r = k$ with $\widetilde{k} \neq k$: $T^{1,q}$ model integrable for all q [Levine, AT 21]

5d σ-model is 2-loop RG stable for general $r,\ k$:
[RG stability just from global symmetry,
does not in general imply integrability]

$$\frac{d}{d\tau}r = 2(r^2-k^2)\Big[\frac{1}{h^2}+\frac{\widetilde{k}}{k}\frac{1}{\widetilde{h}^2}+\frac{r^2-3k^2}{r}\Big(\frac{1}{h^4}+\frac{\widetilde{k}^2}{k^2}\frac{1}{\widetilde{h}^4}\Big)\Big]$$

$$\frac{d}{d\tau}h = 4\big[1-\frac{r}{2h}(1+\frac{k^2}{r^2})\big]+\frac{2}{h^3r^2}\Big[8h^2r^2-8hk^2r -12hr^3+4r^2k^2+3k^4+5r^4+\frac{\widetilde{k}}{k}\frac{h^2}{\widetilde{h}^2}(r^2-k^2)(3r^2-k^2)\Big]$$

$$\frac{d}{d\tau}\widetilde{h} = 4\big[1-\frac{\widetilde{k}}{k}\frac{r}{2\widetilde{h}}(1+\frac{k^2}{r^2})\big]+\frac{2}{\widetilde{h}^3r^2}\Big[8\widetilde{h}^2r^2-8\widetilde{h}\widetilde{k}kr-12\frac{\widetilde{k}}{k}\widetilde{h}r^3 +4r^2\widetilde{k}^2+3\widetilde{k}^2k^2+5\frac{\widetilde{k}^2}{k^2}r^4+\frac{\widetilde{k}}{k}\frac{\widetilde{h}^2}{h^2}(r^2-k^2)(3r^2-k^2)\Big]$$

• integrable subspace $r=k$:
fixed subset of RG at 1-loop and 2-loop order

remaining running couplings of integrable $T^{1,q}$ model:

$$\frac{d}{d\tau}h = 4(1-\tfrac{k}{h})\left[1+\tfrac{2}{h}(2-3\tfrac{k}{h})\right]$$

$$\frac{d}{d\tau}\widetilde{h} = 4(1-\tfrac{\widetilde{k}}{\widetilde{h}})\left[1+\tfrac{2}{\widetilde{h}}(2-3\tfrac{\widetilde{k}}{\widetilde{h}})\right], \qquad \frac{d}{d\tau}k = \frac{d}{d\tau}\widetilde{k} = 0$$

- fixed point $h=k,\ \widetilde{h}=\widetilde{k}$:
  conformal GMM model

"explanation" of RG stability of integrable $r = k$ case:
$T^{1,q}$ model is self-dual under T-duality in ψ direction

$$\mathcal{L} = k[(\partial_+\psi + j_+)(\partial_-\psi + j_-) - \tfrac{1}{2}j_+j_-] + \widehat{\mathcal{L}}$$

$$j_+ = 2\cos\theta_1\,\partial_+\phi_1\ , \quad j_- = 2q\cos\theta_2\,\partial_-\phi_2$$

interpolating theory:

$$\mathcal{L} = k[(A_+ + j_+)(A_- + k_-) - \tfrac{1}{2}j_+j_- - \widetilde{\psi}(\partial_+A_- - \partial_-A_+)] + \widehat{\mathcal{L}}$$

integrate out $A_\pm$

$$\widetilde{\mathcal{L}} = k[(\partial_+\widetilde{\psi} + j_+)(\partial_-\widetilde{\psi} - j_-) + \tfrac{1}{2}j_+j_-] + \widehat{\mathcal{L}}\ ,$$

related to original model by $\phi_2 \to -\phi_2$

integration of A_a is trivial – no counterterm $\to$
no 2-loop deformation expected

Summary:

• classically integrable σ-models (admitting Lax)
are 1-loop RG stable with finite set of couplings
• if enough manifest global/local symmetry
RG-stability is preserved at higher loops: coset models,
$G \times ... \times G$ models, λ-model in extended formulation, etc.

• in general symmetry is "hidden"
$\rightarrow$ need finite local counterterms
for RG stability on same space of couplings
$\rightarrow$ quantum deformation of target space geometry
• reason: preservation of hidden symmetries
at quantum level imposes constraints on RG evolution

• existence of local deformation that preserves RG flow
on same space of couplings should be consistent
with existence of underlying quantum integrable S-matrix

• applies to integrable deformations
of standard coset σ-models: YB η-model, λ-model, etc.
• examples of similar deformations:
modification of abelian and non-abelian
2d duality (T-duality) at higher loop orders;
quantum deformation of geometry in gauged WZW models
to preserve conformal invariance
• η-model and λ-model are related by PL duality:
same should apply to PL duality: it commutes with RG
once backgrounds are properly modified